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Generalized Hölder's Theorem for Vignéras' Multiple Gamma Function

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Abstract. We prove that Vignéras' multiple gamma function does not satisfy any algebraic differential equation over $\mathbf{C}(z)$ by using a relation between logarithmic derivative of this function and the psi-function. Furthermore, we apply this relation to calculation of convergent factors in the Weierstrass product representation of Vignéras' multiple gamma function.

1. Introduction.

In 1974, Vignéras [13] showed the existence and the uniqueness of the hierarchy of functions satisfying the following conditions (i)–(iv):

(i) $G_r(z+1) = G_{r-1}(z)G_r(z)$,

(ii) $G_r(1) = 1$,

(iii) $\dfrac{d^{r+1}}{dz^{r+1}} \log G_r(z+1) \geq 0 \quad \text{for} \quad z \geq 0$,

(iv) $G_0(z) = z$.

This hierarchy is a special case of the one of Barnes' multiple gamma functions [2] and includes gamma function $\Gamma(z) = G_1(z)$. The function $G_r(z)$ is called *Vignéras' multiple gamma function* (For simplicity, we often call this function only "*multiple gamma function*" in this paper.). This function appears in various topics. The following are remarkable examples: Representation of determinant of Laplacian and of the Selberg zeta function [7], [8], [12], [13], [14], mean values of L-function [4], Jost functions of some quantum integrable systems [5] and asymptotic behavior of τ-function of the Ising model [10]. It is expected that this function would play important roles in number theory, in spectral geometry and in quantum integrable systems.

In this paper, we prove that the multiple gamma function $G_r(z)$ does not satisfy any algebraic differential equations over the rational function field $\mathbf{C}(z)$. This was considered by

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Barnes [3] as a special case of more general problem. He showed that if there is differential relations between logarithmic derivative of the function, then we can obtain a such good relation that it can be applied to the proof. This is proved through an abstract procedure and the relation is not always represented explicitly if it exists. However, as concerns Vignéras' multiple gamma function, we can find the following relation:

$$\psi_{r+1}(z) = \binom{z}{r}\psi_1(z) + P_r(z)\,, \tag{1}$$

where $\psi_r(z)$ is the logarithmic derivative of $G_r(z+1)$ and $P_r(z)$ is a polynomial of degree r. By using this relation, an algebraic differential equation for $\psi_r(z)$ is easily reduced to an algebraic differential equation for $\psi_1(z)$. From Hölder's theorem for gamma function (cf. Komatsu [6]), it follows that there is not such differential equation.

Relation (1) is applicable to calculation of a convergent factor of the Weierstrass product representation for $G_r(z+1)$. The Weierstrass product representation was derived by Ueno-Nishizawa [11].

PROPOSITION 1. (Ueno-Nishizawa). *$G_r(z+1)$ has a infinite product representation*

$$G_r(z+1) = \exp(F_r(z))\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^{-\binom{-k}{n-1}}\exp(\Phi_r(z,k))\right\}, \tag{2}$$

where $F_r(z)$ is a polynomial of degree r and $\Phi_r(z,k)$ is a convergent factor which is a polynomial of z and a Laurent polynomial of k. Its degree of k is larger equal than -1. □

After slightly complicated calculation, they derived explicit form of $\Phi_r(z,k)$. We give an easier procedure for computation of $\Phi_r(z,k)$ by using relation (1) in Section 4.

2. Logarithmic derivative of multiple gamma function.

In this paper, we denote $\mathbf{C}(z)$ by the rational function field generated by z over the complex number field $\mathbf{C}$. We define a function $\psi_r(z)$ by the logarithmic derivative of $G_r(z+1)$,

$$\begin{aligned}\psi_r(z) &:= \frac{d}{dz}\log G_r(z+1)\\ &= \frac{d}{dz}F_r(z) + \sum_{k=1}^{\infty}\left\{-\frac{1}{z+k}\binom{-k}{r-1} + \phi_r(z,k)\right\},\end{aligned}$$

where $\phi_r(z,k) := \frac{d}{dz}\Phi_r(z,k)$ and $F_r(z)$ is a polynomial of degree r (See Ueno-Nishizawa [11]).

We don't give an explicit form of $F_r(z)$ here because it is no need for the argument in this paper. Let $\widetilde{\psi}_r(z)$ be the infinite sum part of $\psi_r(z)$

$$\widetilde{\psi}_r(z) := \sum_{k=1}^{\infty}\left\{-\frac{1}{z+k}\binom{-k}{r-1} + \phi_r(z,k)\right\}.$$

Then, we can see a relation between $\widetilde{\psi}_{r+1}(z)$ and $\widetilde{\psi}_r(z)$.

LEMMA 2.

$$\frac{z-r+1}{r}\widetilde{\psi}_r(z) = \widetilde{\psi}_{r+1}(z)\,.$$

PROOF. First, we remark the following identity:

$$\frac{z-r+1}{r(z+k)}\binom{k+r-2}{r-1} = -\frac{1}{z+k}\binom{k+r-1}{r} + \frac{1}{r}\binom{k+r-2}{r-1}$$

for $r, k \in \mathbf{Z}_{>0}$. We have

$$\begin{aligned}\frac{z-r+1}{r}\widetilde{\psi}_r(z) &= \frac{z-r+1}{r}\sum_{k=1}^{\infty}\left\{\frac{(-1)^r}{z+k}\binom{k+r-2}{r-1} + \phi_r(z,k)\right\} \\ &= \sum_{k=1}^{\infty}\left\{\frac{-1}{z+k}\binom{-k}{r} + \frac{-1}{r}\binom{-k}{r-1} + \frac{z+r-1}{r}\phi_r(z,k)\right\}.\end{aligned}$$

From absolute convergence of $\widetilde{\psi}_r(z)$, it follows that a sum of the right hand side is absolutely convergent. The convergent factor of this sum have to be equal to $\phi_{r+1}(z,k)$. In fact, the factor is uniquely determined when its degree of k is larger than -2. □

The following formula follows from the above argument:

COROLLARY 3.

$$\phi_{r+1}(z,k) = -\frac{1}{r}\binom{-k}{r-1} + \frac{z-r+1}{r}\phi_r(z,k)\,. \quad \square$$

From Lemma 2, we have relations between logarithmic derivatives.

PROPOSITION 4. (i) *$\psi_r(z)$ satisfies a recurrence formula*

$$\psi_{r+1}(z) = \frac{z-r+1}{r}\psi_r(z) + p_r(z), \tag{3}$$

where $p_r(z)$ is a polynomial of degree less than equal r.

(ii) *$\psi_{r+1}(z)$ is transformed by the following formula:*

$$\psi_{r+1}(z) = \binom{z}{r}\psi_1(z) + P_r(z)\,, \tag{4}$$

where $P_r(z)$ is a polynomial of degree less than equal r.

PROOF. (i) From Lemma 2, it follows that

$$\psi_{r+1}(z) = \frac{z-r+1}{r}\psi_r(z) + \left\{\frac{d}{dz}F_{r+1}(z) - \frac{z-r+1}{r}\frac{d}{dz}F_r(z)\right\}.$$

Thus, the claim holds because $F_r(z)$ is a polynomial of degree r.

(ii) We prove the claim by induction on r. In the case when $r = 1$, it follows from the following result (cf. Barnes [1]):

$$\psi_2(z) = z\psi_1(z) - z + \frac{1}{2}\log\pi - \frac{1}{2}.$$

By using (2) and the assumption of the induction, we have

$$\psi_{r+1}(z) = \binom{z}{r}\psi_1(z) + \frac{z-r+1}{r}P_{r-1}(z) + p_r(z).$$

The claim follows from the facts that $P_r(z)$ and $p_r(z)$ are polynomials and that their degrees are less than equal r. □

3. Generalized Hölder's theorem.

Now, we prove the main theorem.

THEOREM 5. *Vignéras' multiple gamma function does not satisfy any algebraic differential equation over* $\mathbf{C}(z)$. *In other words, for any* $r, n \in \mathbf{Z}_{\geq 0}$, *there is no* $(n+1)$*-variable polynomial*

$$f(t_0, t_1, t_2, \cdots, t_n) \in \mathbf{C}(z)[t_0, t_1, t_2, \cdots, t_n]$$

such that

$$f(G_r(z), G_r^{(1)}(z), \cdots, G_r^{(n)}(z)) = 0 \tag{5}$$

where

$$G_r^{(l)}(z) := \frac{d^l}{dz^l}G_r(z).$$

PROOF. We remark that the logarithmic derivative of a solution for an algebraic differential equation also satisfies some algebraic differential equation (cf. Komatsu [6], Pastro [9]). Therefore, if some polynomial satisfies (5), then there exists an algebraic differential equation for $\psi_r(z)$. However, from Proposition 4 (2), it follows that there exists an algebraic differential equation for $\psi_1(z)$ over $\mathbf{C}(z)$. This contradicts to Hölder's theorem for gamma function (cf. Komatsu [6]). □

4. Representation of a convergent factor.

By means of Corollary 3, we can calculate a convergent factor $\Phi_r(z, k)$ in (2).

PROPOSITION 6.

$$\phi_r(z, k) = \frac{1}{(r-1)!}\sum_{i=0}^{r-1}(-1)^i k^{i-1}\left(\sum_{j=0}^{r-1-i} {}_{r-1}S_{j+i} z^j\right)$$

where ${}_kS_j$ is the Stirling number of the 1st kind defined by

$$\binom{x}{n} =: \frac{1}{n!}\sum_{j=0}^{n} {}_nS_j\, x^j$$

for $n \geq 0$.

PROOF. In the case when $r = 1$, $\phi_1(z, k) = 1/k$. It coincides with the well known result about gamma function (cf. Whittaker-Watson [15]). Thus, we show that $\phi_r(z, k)$ satisfies the recurrence relation of Corollary 3. From a relation between the Stirling numbers

$${}_nS_j = {}_{n-1}S_{j-1} - (n-1)\,{}_{n-1}S_j, \qquad \text{for} \quad j = 0, \cdots, n,$$

it follows that

$$\begin{aligned}
&-\frac{1}{r}\binom{-k}{r-1} + \frac{z-r+1}{r}\phi_r(z, k) \\
&= \frac{-1}{r!}\left[\sum_{i=0}^{r-1}(-k)^{i-1}\left\{\sum_{j=0}^{r-1-i} {}_{r-1}S_{j+i}\, z^{j+1} - \sum_{j=0}^{r-1-i}(r-1)_{r-1}S_{j+i}\, z^j\right\}\right. \\
&\quad \left. + \sum_{i=0}^{r-1} {}_{r-1}S_i\,(-k)^{i-1}\right] \\
&= \frac{-1}{r!}\left[\sum_{i=0}^{r-1}(-k)^{i-1}\left\{\sum_{j=1}^{r-i}\left({}_{r-1}S_{j+i-1} - (r-1)_{r-1}S_{j+i}\right) z^j\right\}\right. \\
&\quad \left. + \sum_{i=0}^{r-1} {}_{r-1}S_i\,(-k)^i - (r-1)\sum_{i=0}^{r-1} {}_{r-1}S_i\,(-k)^{i-1}\right] \\
&= \frac{-1}{r!}\left\{\sum_{i=0}^{r-1}(-k)^{i-1}\left(\sum_{j=1}^{r-i} {}_rS_{j+i}\, z^j\right) + \sum_{i=0}^{r} {}_rS_i\,(-k)^{i-1}\right\} \\
&= \phi_{r+1}(z, k),
\end{aligned}$$

for $r \in \mathbf{Z}_{\geq 2}$. We have thus proved the proposition. □

COROLLARY 7. *The convergent factor $\Phi_r(z, k)$ in (2) is represented by the formula*

$$\Phi_r(z, k) = \int_0^z \phi_r(x, k)dx. \quad \square$$

For example, first few $\Phi_r(z,k)$ can be represented by the following formulas:

$$\Phi_1(z,k)=\frac{z}{k}\,,\qquad \Phi_2(z,k)=\frac{z^2}{2k}-z\,,$$

$$\Phi_3(z,k)=\left(\frac{z^3}{6}-\frac{z^2}{4}\right)\frac{1}{k}-\left(\frac{z^2}{4}-\frac{z}{2}\right)+\frac{z}{2}k\,,$$

$$\Phi_4(z,k)=\left(\frac{z^4}{24}-\frac{z^3}{6}+\frac{z^2}{6}\right)\frac{1}{k}-\left(\frac{z^3}{18}-\frac{z^2}{4}+\frac{z}{3}\right)+\left(\frac{z^2}{12}-\frac{z}{2}\right)k-\frac{z}{6}k^2,$$

$$\Phi_5(z,k)=\left(\frac{z^5}{120}-\frac{z^4}{16}+\frac{11}{72}z^3-\frac{z^2}{8}\right)\frac{1}{k}-\left(\frac{z^4}{96}-\frac{z^3}{12}+\frac{11}{48}z^2-\frac{z}{4}\right)$$
$$+\left(\frac{z^3}{72}-\frac{z^2}{8}+\frac{11}{24}z\right)k-\left(\frac{z^2}{48}-\frac{z}{4}\right)k^2+\frac{z}{24}k^3\,.$$

These results coincide with the representations by Ueno-Nishizawa [11], which were derived through power series expansion of $\binom{-k}{r-1}\log(1+z/k)$.

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