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Norms Related to the Lie Norm

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Abstract. We give a complete proof of Theorem 2.1 related to the Lie norm conjectured by Prof. Mitsuo Morimoto. His conjecture is a special case of our new theorem on convex functions on $\mathbf{R}^n$.

1. Introduction.

For $z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n$ and $w = (w_1, w_2, \cdots, w_n) \in \mathbf{C}^n$, we put $\langle z, w\rangle = z_1w_1 + z_2w_2 + \cdots + z_nw_n$ and $z^2 = \langle z, z\rangle = z_1^2 + z_2^2 + \cdots + z_n^2$. $\|z\| = \langle z, \bar{z}\rangle^{1/2}$ is the Euclidean norm of z. For $x \in \mathbf{R}$, we have $\|x\| = (x^2)^{1/2}$.

The Lie norm $L(z)$ on $\mathbf{C}^n$ is defined by the formula (see [3], [4], or [5]):

$$L(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}}\,.$$

By the simple calculations, we have

$$\frac{|z^2|}{L(z)} = \sqrt{\|z\|^2 - \sqrt{\|z\|^4 - |z^2|^2}}\,,$$

where $z = x + iy$, $x, y \in \mathbf{R}^n$. Let $L^*(z)$ be *the dual Lie norm* defined by

$$L^*(z) = \sup\{|z\cdot\zeta| : L(\zeta) \le 1\} = \sqrt{\frac{\|z\|^2 + |z^2|}{2}}\,.$$

2. Main results.

December 1999 at a symposium held at RIMS (Research Institute for Mathematical Sciences) in Kyoto, Prof. Mitsuo Morimoto conjectured the following theorem.

THEOREM 2.1. *For $z \in \mathbf{C}^n$ we put*

$$N_p(z) = \left(\frac{1}{2}\left(L(z)^p + \left(\frac{|z^2|}{L(z)}\right)^p\right)\right)^{\frac{1}{p}}, \quad p \ge 1\,.$$

Then $N_p(z)$ is a norm on $\mathbf{C}^n$, that is,

(1) $N_p(z) \ge 0$; $N_p(z) = 0 \Leftrightarrow z = 0$,

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(2) $N_p(\lambda z) = |\lambda| N_p(z)$ *for* $\lambda \in \mathbf{C}$,
(3) $N_p(z+w) \le N_p(z) + N_p(w)$ *for any* $z, w \in \mathbf{C}^n$.

REMARK 2.2. Note that $N_1(z) = L^*(z)$, $N_2(z) = \|z\|$ and $N_\infty(z) = L(z)$. If $n = 2$, $N_p(z)$ are norms because we have $N_p(z) = (|z_1 + iz_2|^p + |z_1 - iz_2|^p)^{1/p}$, where $z = (z_1, z_2) \in \mathbf{C}^2$.

PROOF. (1) and (2) are trivial. Because $L(z)$ and $L(z) + |z^2|/L(z) = 2L^*(z)$ are convex, and $L(z) \ge |z^2|/L(z) \ge 0$, Theorem 2.3 below implies $N_p(z)$ is convex. Therefore, we have (3). □

THEOREM 2.3. *Let f and $f + g$ be a real-valued convex function on $\mathbf{R}^n$, and $f \ge g \ge 0$. Then the function $h : \mathbf{R}^n \to \mathbf{R}$ defined by*

$$h(x) = (f(x)^p + g(x)^p)^{\frac{1}{p}}, \quad p \ge 1,$$

is convex on $\mathbf{R}^n$.

PROOF. First we suppose that f and g are both of C^2 class. We denote partial derivatives by $f_{x_i} = \partial f/\partial x_i$, $f_{x_i x_j} = \partial^2 f/\partial x_j \partial x_i$, etc. Then we have

$$\begin{aligned} h_{x_i x_i} &= (f^p + g^p)^{\frac{1}{p}-2}\{(p-1) f^{p-2} g^{p-2} (f g_{x_i} - g f_{x_i})^2 \\ &\quad + (f^p + g^p)(f^{p-1} f_{x_i x_i} + g^{p-1} g_{x_i x_i})\} \ge 0, \\ h_{x_i x_j} &= (f^p + g^p)^{\frac{1}{p}-2}\{(p-1) f^{p-2} g^{p-2} (f g_{x_i} - g f_{x_i})(f g_{x_j} - g f_{x_j}) \\ &\quad + (f^p + g^p)(f^{p-1} f_{x_i x_j} + g^{p-1} g_{x_i x_j})\}. \end{aligned}$$

We consider the following quadratic form.

$$\begin{aligned} &\left\langle \begin{pmatrix} h_{x_1x_1} & \cdots & h_{x_1x_n} \\ \vdots & \ddots & \vdots \\ h_{x_nx_1} & \cdots & h_{x_nx_n} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\rangle \\ &= (p-1) f^{p-2} g^{p-2} (f^p + g^p)^{\frac{1}{p}-2} \left\langle \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\rangle \\ &\quad + (f^p + g^p)^{\frac{1}{p}-1} \left\langle \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\rangle, \end{aligned}$$

where $a_{ij} = (f g_{x_i} - g f_{x_i})(f g_{x_j} - g f_{x_j})$, $b_{ij} = (f^{p-1} f_{x_i x_j} + g^{p-1} g_{x_i x_j})$. Since $a_{ii} \ge 0$ and any 2×2 minor determinant

$$\begin{vmatrix} a_{ij} & a_{ik} \\ a_{lj} & a_{lk} \end{vmatrix} = 0 \quad \text{for any } i < l, \ j < k,$$

the first quadratic form in the righthand side is non-negative definite. (e.g. [2, X §4]) By convexity of f and $f + g$, the second quadratic form can be calculated as follows:

$$\left\langle \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\rangle$$
$$= f^{p-1} \left\langle \begin{pmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \ddots & \vdots \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\rangle$$
$$+ g^{p-1} \left\langle \begin{pmatrix} g_{x_1x_1} & \cdots & g_{x_1x_n} \\ \vdots & \ddots & \vdots \\ g_{x_nx_1} & \cdots & g_{x_nx_n} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\rangle$$
$$\geq g^{p-1} \left\langle \begin{pmatrix} (f+g)_{x_1x_1} & \cdots & (f+g)_{x_1x_n} \\ \vdots & \ddots & \vdots \\ (f+g)_{x_nx_1} & \cdots & (f+g)_{x_nx_n} \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\rangle \geq 0 .$$

Hence the quadratic form in the lefthand side is non-negative definite and the function h is convex. (e.g. [6, IV 42 Theorem F])

Second, we consider the general case. Let α_ρ be a regularizing function of Friedrichs. (e.g. [1, §1.3]) We put $f_\rho = f * \alpha_\rho$, $g_\rho = g * \alpha_\rho$ and $h_\rho = (f_\rho^p + g_\rho^p)^{1/p}$. Then $f_\rho \geq g_\rho \geq 0$ because

$$f_\rho(x) = \int_{\mathbf{R}^n} f(y)\alpha_\rho(x-y)dy \geq \int_{\mathbf{R}^n} g(y)\alpha_\rho(x-y)dy = g_\rho(x) \geq 0 .$$

The function f_ρ is convex because for any $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$,

$$\begin{aligned} f_\rho(\lambda_1 x_1 + \lambda_2 x_2) &= \int_{\mathbf{R}^n} f(\lambda_1 x_1 + \lambda_2 x_2 - y)\alpha_\rho(y)dy \\ &= \int_{\mathbf{R}^n} f(\lambda_1(x_1 - y) + \lambda_2(x_2 - y))\alpha_\rho(y)dy \\ &\leq \int_{\mathbf{R}^n} (\lambda_1 f(x_1 - y) + \lambda_2 f(x_2 - y))\alpha_\rho(y)dy \\ &= \lambda_1 f_\rho(x_1) + \lambda_2 f_\rho(x_2) , \quad x_1, x_2 \in \mathbf{R}^n . \end{aligned}$$

Similarly, $(f+g)_\rho = f_\rho + g_\rho$ is convex. Therefore, the function h_ρ is convex. Because h is continuous (e.g. [6, IV 41]), h_ρ tends to h pointwisely as $\rho \searrow 0$. So h is also convex. (e.g. [7, Theorem 10.8])

REMARK 2.4. Since f and $f+g$ are subharmonic, $f_\rho \geq f \geq 0$ and $f_\rho + g_\rho \geq f + g \geq 0$. (e.g. [1, Proposition 4.4.16]) If $a_2 \geq a_1 \geq 0$, $b_2 \geq b_1 \geq 0$, $a_1 \leq b_1 \leq 2a_1$ and $a_2 \leq b_2 \leq 2a_2$, then $a_1^p + (b_1 - a_1)^p \leq a_2^p + (b_2 - a_2)^p$. Therefore, h_ρ tends decreasingly to h as $\rho \searrow 0$.

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References

[1] C. BERENSTEIN and R. GAY, *Complex Variables An Introduction*, Springer (1991).
[2] F. R. GANTMACHER, *Matrix theory Vol. 1*, Chelsea (1959).
[3] M. MORIMOTO, Analytic Functionals on the Lie Sphere, Tokyo J. Math. **3** (1980), 1–35.
[4] M. MORIMOTO, *Hyperfunctions on the Sphere*, Sophia Kokyuroku in Math. **12** (1982), Dept. of Math., Sophia Univ. (in Japanese).
[5] M. MORIMOTO, *Analytic Functionals on the Sphere*, Amer. Math. Soc. (1998).
[6] A. W. ROBERTS and D. E. VARBERG, *Convex functions*, Academic Press (1973).
[7] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton Univ. Press (1970).

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