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Some Character Sums and Gauss Sums over $G_2(q)$

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0. Introduction.

Following the work of T. Kondo [11] for $GL_n(q)$, we defined a Gauss sum for a finite reductive group defined over $\mathbf{F}_q$ associated with a modular representation over the same characteristic and a complex ordinary representation in [15]. When the ordinary representation is irreducible, the determination of the value of a Gauss sum is equivalent to that of the trace, τ_W (see (2.1) for details). Fixing the modular representation, we have shown that the values of τ_W are canonically determined for the generalized characters of Deligne-Lusztig and gave the values for finite classical groups with their canonical representation using Kloosterman sums and unitary Kloosterman sums. Moreover as an example we have determined the values of Gauss sums over $Sp(4, q)$, with q odd, associated with every complex irreducible representation and with the canonical modular representation.

On the other hand, in a series of papers starting with [10], D. S. Kim, I. Lee, and K. Park have considered a Gauss sum over a finite reductive group, when the complex representation is one dimensional and factored through the determinant of the modular representation. In particular Lee and Park, [12], have determined the Gauss sum for the Chevalley group of type G_2 over $\mathbf{F}_q$, $G_2(q)$, associated with the irreducible 7-dimensional modular representation and the trivial (ordinary) representation.

The purpose of this paper is to give an explicit expression for Gauss sum over $G_2(q)$ associated with every unipotent character and with the 7-dimensional modular representation, applying the method developed in [15]. In particular we give another proof for the theorem obtained by Lee and Park cited above. On the way we also have the Gauss sums associated with $\varepsilon(\mathbf{T})R_{\mathbf{T},\theta}$, when they are irreducible.

Interestingly enough these sums corresponding to unipotent characters are written by using one character sum. In general, it seems that there is a character sum associated with each geometric conjugacy class $\{(\mathbf{T}, \theta)\}$ (cf. [7]), and that if an irreducible character ζ appears as a component of $R_{\mathbf{T},\theta}$, the corresponding Gauss sum $w(\zeta)$ (cf. (2.1)) can be expressed using the character sum.

In our case, we give five relations for the character sums associated with $\{R_{\mathbf{T},1}\}$, and section 1 is devoted to establish such relations between certain character sums. Applying the

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result in section 1, we explicitly express the Gauss sums associated with unipotent characters in section 2.

1. Some character sums and their relations.

1.1. Notation. Let q be a positive integral power of a prime p and k be a finite field with q elements. Let $\bar{k}$ be an algebraic closure of k and k_m be the finite field with q^m elements in $\bar{k}$. The multiplicative group of k_m is denoted by M_m and if $m = 1$, we simply write k or M instead of k_1 or M_1.

C_m is the cyclic subgroup of order $q^m + 1$ of M_{2m} and $C = C_1$. If n divides m, $\mathrm{N}_{m,n} : k_m \to k_n$ is the norm map defined by $\mathrm{N}_{m,n}(x) = x^{(q^m-1)/(q^n-1)}$ and $\mathrm{Tr}_{m,n} : k_m \to k_n$ is the trace map defined by $\mathrm{Tr}_{m,n}(x) = \sum_{i=0}^{m/n-1} x^{q^{ni}}$. Moreover if m/n is odd, we define a map $\mathrm{N}'_{m,n} : C_m \to C_n$ by $\mathrm{N}'_{m,n}(x) = x^{(q^m+1)/(q^n+1)}$.

For a finite abelian group A, the set of all irreducible characters over $\mathbf{C}$ of A will be denoted by $\hat{A}$. Throughout this paper we fix a nontrivial additive character $\chi \in \hat{k}$. For $a \in M$, $\chi^a \in \hat{k}$ is defined by $\chi^a(x) = \chi(ax)$, and $\chi^{(m)} = \chi \circ \mathrm{Tr}_{m,1}$ is the canonical lift of χ to k_m. Similarly for $\theta \in \hat{M}$, $\theta^{(n)} = \theta \circ \mathrm{N}_{n,1}$, and moreover if n is odd, $\varphi^{(n)} = \varphi \circ \mathrm{N}'_{n,1}$ for $\varphi \in \hat{C}$.

For a finite set X of $\bar{k}$, let $L(X)$ be the set of all complex valued functions on X and hence $L(X)$ is a complex vector space furnished with the hermitian inner product $\langle\, ,\rangle_X$ defined by

$$\langle f, g\rangle_X = \frac{1}{|X|}\sum_{x\in X} f(x)\overline{g(x)}, \quad \text{for } f, g \in L(X).$$

Notice that if X is a subgroup of $\bar{k}^\times$, and $\pi \in \hat{X}$, then $\overline{\pi(x)} = \pi(x^{-1})$ and we define $\bar{\pi} \in \hat{X}$, by $\bar{\pi}(x) = \overline{\pi(x)}$.

1.2. Kloosterman sums and unitary Kloosterman sums. Firstly let us recall the definitions and properties of a Kloosterman sum, $K(\chi, \pi; a)$, where $a \in M$, $\pi \in \hat{M}$ and a unitary Kloosterman sum, $J(\chi, \varphi'; a)$, where $\varphi' \in \hat{M}_2$, (cf. [6]):

$$K(\chi, \pi; a) = \sum_{x\in M} \chi(x + ax^{-1})\pi(x),$$

$$J(\chi, \varphi'; a) = \sum_{x\in M_2, \mathrm{N}_{2,1}(x)=a} \chi^{(2)}(x)\varphi'(x).$$

The following is a lemma of B. Chang [3]. We include a proof after C. W. Curtis [5], since it gives a prototype for the proofs of other relations.

LEMMA 1.3. *For $a \in M$, $K(\chi, 1_M; a) = -J(\chi, 1_{M_2}; a)$.*

PROOF. Since both sides are functions on M, it is enough to show

$$\langle K(\chi, 1_M; \cdot), \pi\rangle_M = -\langle J(\chi, 1_{M_2}; \cdot), \pi\rangle_M,$$

for all $\pi \in \hat{M}$. Let $G(\chi, \pi)$ be the Gauss sum: $G(\chi, \pi) = \sum_{x\in M} \chi(x)\pi(x)$. Then it is easy to see that the left hand side becomes $(q-1)^{-1}G(\chi, \bar{\pi})^2$, while the right hand side is $-(q-1)^{-1}G(\chi^{(2)},\bar{\pi}^{(2)})$, where $G(\chi^{(2)},\bar{\pi}^{(2)}) = \sum_{x\in M_2}\chi^{(2)}(x)\bar{\pi}^{(2)}(x)$. Therefore if $\pi \neq 1$, the equality holds by the Hasse-Davenport identity, and if $\pi = 1$, it holds trivially. □

1.4. Relations for extension fields. Let $K(\chi,\pi) = K(\chi,\pi;1)$ and

$$J(\chi,\varphi) = \sum_{x\in C}\chi(x+x^{-1})\varphi(x),$$

where $\varphi \in \hat{C}$. We recall the relations with these sums and the sums over extension fields of k. Let

$$K_n(\chi^{(n)},\pi^{(n)}) = \sum_{x\in M_n}\chi^{(n)}(x+x^{-1})\pi^{(n)}(x),$$

$$J_n(\chi^{(n)},\varphi^{(n)}) = \sum_{x\in C_n}\chi^{(n)}(x+x^{-1})\varphi^{(n)}(x), \quad \text{if } n \text{ is odd},$$

and let

$$P_n(u,v) = \sum_{j=0}^{[n/2]}(-1)^{n-j-1}\frac{n}{n-j}\binom{n-j}{j}u^{n-2j}v^j$$

be the Dickson polynomial in the indeterminates u and v. Then

THEOREM 1.5 ([6], Theorems 1 and 2).

1. $P_n(K(\chi,\pi),q\pi(-1)) = K_n(\chi^{(n)},\pi^{(n)})$,
2. $P_n(J(\chi,\varphi),q\varphi(-1)) = \begin{cases} K_n(\chi^{(n)},\varphi^{(n)}) & \text{if } n \text{ is even,} \\ J_n(\chi^{(n)},\varphi^{(n)}) & \text{if } n \text{ is odd,}\end{cases}$

where $\varphi^{(n)}(x) = x^{(q^n-1)/(q+1)}$ (resp. $x^{(q^n+1)/(q+1)}$) if n is even (resp. odd) and $x \in M_n$ (resp. C_n).

1.6. Character sums $K^{(0)}$ and $J^{(0)}$. Now we consider the following character sums:

$$K^{(0)}_{M^n}(a) = \sum_{x_i\in M, x_1,\cdots x_n=a}\chi(x_1+\cdots+x_n+x_1^{-1}+\cdots+x_n^{-1}),$$

$$K^{(0)}_{M_n}(a) = \sum_{x\in M_n, N_{n,1}(x)=a}\chi^{(n)}(x+x^{-1}),$$

$$J^{(0)}_{C^n}(u) = \sum_{x_i\in C, x_1\cdots x_n=u}\chi(x_1+\cdots+x_n+x_1^{-1}+\cdots+x_n^{-1}),$$

$$J^{(0)}_{C_n}(u) = \sum_{x\in C_n, \mathrm{N}'_{n,1}(x)=u}\chi^{(n)}(x+x^{-1}), \quad \text{(for odd } n),$$

where $a \in M$ and $u \in C$. For convenience, we put $K^{(0)}_{M^0} \equiv 1$.

First relation among them is the following.

PROPOSITION 1.7. *Let $c_{n,j}$ be the coefficient of $u^{n-2j}v^j$ in $P_n(u,v)$, i.e. $c_{n,j} = (-1)^{n-j-1}\frac{n}{n-j}\binom{n-j}{j}$. Then*

1. $K^{(0)}_{M_n}(a) = \sum_{j=0}^{[n/2]} c_{n,j} q^j K^{(0)}_{M^{n-2j}}((-1)^j a)$,
2. $J^{(0)}_{C_n}(u) = \sum_{j=0}^{[n/2]} c_{n,j} q^j J^{(0)}_{C^{n-2j}}((-1)^j u)$, *where n is odd.*

PROOF. (1) For all $\pi \in \hat{M}$, we have

$$\begin{aligned}(q-1)\langle K^{(0)}_{M_n}, \pi\rangle_M &= \sum_{a\in M} K^{(0)}_{M_n}(a)\pi(a^{-1}) \\ &= \sum_a \sum_{x:N(x)=a} \chi^{(n)}(x+x^{-1})\pi(a^{-1}) \\ &= \sum_{x\in M_n} \chi^{(n)}(x+x^{-1})\pi^{(n)}(x^{-1}) \\ &= K_n(\chi^{(n)}, \bar{\pi}^{(n)}) \\ &= P_n(K(\chi,\bar{\pi}), q\bar{\pi}(-1)) \quad \text{(by (1.5.1))} \\ &= \sum_j c_{n,j} K(\chi,\bar{\pi})^{n-2j}(q\bar{\pi}(-1))^j .\end{aligned}$$

Therefore for $a \in M$, we have

$$\begin{aligned}&K^{(0)}_{M_n}(a) \\ &= \sum_{\pi\in\hat{M}} \langle K^{(0)}_{M_n}, \pi\rangle_M \pi(a) \\ &= \frac{1}{q-1}\sum_{j=0}^{[n/2]}\sum_{\pi} c_{n,j} K(\chi,\bar{\pi})^{n-2j}(q\pi(-1))^j\pi(a) \\ &= \frac{1}{q-1}\sum_j\sum_\pi c_{n,j}\left(\sum_{x\in M}\chi(x+x^{-1})\bar{\pi}(x)\right)^{n-2j} q^j\pi((-1)^j a) \\ &= \frac{1}{q-1}\sum_j c_{n,j}q^j \sum_{x_1,\cdots,x_{n-2j}\in M} \chi\left(\sum_{i=1}^{n-2j}(x_i+x_i^{-1})\right)\sum_\pi \bar{\pi}(x_1\cdots x_{n-2j})\pi((-1)^j a) \\ &= \sum_j c_{n,j}q^j \sum_{x_1\cdots x_{n-2j}=(-1)^j a} \chi\left(\sum_{i=1}^{n-2j}(x_i+x_i^{-1})\right) \\ &= \sum_j c_{n,j}q^j K^{(0)}_{M^{n-2j}}((-1)^j a),\end{aligned}$$

which proves (1). The second statement (2) can be proved analogously as functions over C, so we omit the proof. □

The following relation will be used in section 2.

PROPOSITION 1.8. $J^{(0)}_{C^3}(1) = K^{(0)}_{M^3}(1) - 2q\chi(-3)$.

PROOF. This relation is proved by a series of transformations of equations.

$$
\begin{aligned}
&J_{C^3}^{(0)}(1)\\
&= \sum_{z_1, z_2 \in C} \chi(z_1 + z_2 + z_1^{-1} z_2^{-1} + z_1^{-1} + z_2^{-1} + z_1 z_2)\\
&= \sum_{z_1} \chi(z_1 + z_1^{-1}) \sum_{z_2} \chi((1 + z_1) z_2 + (1 + z_1^{-1}) z_2^{-1})\\
&= \sum_{z_1 \neq -1} \chi(z_1 + z_1^{-1}) J\left(\chi, 1; \frac{(1+z_1)^2}{z_1}\right) + \chi(-2) \sum_{z_2} \chi(0)\\
&= \sum_{z_1 \neq -1} \chi(z_1 + z_1^{-1}) \left(-K\left(\chi, 1; \frac{(1+z_1)^2}{z_1}\right)\right) + (q+1)\chi(-2) \quad \text{(by (1.3))}\\
&= - \sum_{z_1 \neq -1, x \in M} \chi(z_1 + z_1^{-1}) \chi\left(x + \frac{(1+z_1)^2}{z_1} x^{-1}\right) + (q+1)\chi(-2)\\
&= - \sum_{x \in C, x \in M} \chi\left(z + z^{-1} + x + \frac{(1+z)^2}{z} x^{-1}\right) + \sum_{x \in M} \chi(-2 + x) + (q+1)\chi(-2)\\
&= - \sum_{x} \chi(x + 2x^{-1}) \sum_{z} \chi((1 + x^{-1})(z + z^{-1})) + q\chi(-2)\\
&= - \sum_{x \neq -1} \chi(x + 2x^{-1}) J(\chi^{1+x^{-1}}, 1; 1) - (q+1)\chi(-3) + q\chi(-2)\\
&= \sum_{x \neq -1} \chi(x + 2x^{-1}) K(\chi^{1+x^{-1}}, 1; 1) - (q+1)\chi(-3) + q\chi(-2) \quad \text{(by (1.3))}\\
&= \sum_{x, y \in M} \chi(x + 2x^{-1} + (1 + x^{-1})(y + y^{-1})) - \sum_{y \in M} \chi(-3) - (q+1)\chi(-3) + q\chi(-2).
\end{aligned}
$$

Now we define $x_1, x_2 \in M$ by $x_1 = y$, $x = x_2(1+y)$, if $y \neq -1$. Notice that $(x, y) \mapsto (x_1, x_2)$ is bijective if $y \neq -1$ and $x + 2x^{-1} + (1 + x^{-1})(y + y^{-1}) = x_1 + x_2 + x_1 x_2 + x_1^{-1} + x_2^{-1} + x_1^{-1} x_2^{-1}$. Thus we have

$$
\begin{aligned}
J_{C^3}^{(0)}(1) &= \sum_{x_1 \neq -1, x_2} \chi(x_1 + x_2 + x_1 x_2 + x_1^{-1} + x_2^{-1} + x_1^{-1} x_2^{-1})\\
&\quad + \sum_{x} \chi(x - 2) - 2q\chi(-3) + q\chi(-2)\\
&= K_{M^3}^{(0)}(1) - \sum_{x_2} \chi(-2) - 2q\chi(-3) + (q-1)\chi(-2)\\
&= K_{M^3}^{(0)}(1) - 2q\chi(-3).
\end{aligned}
$$

□

REMARK 1.8.1. There will exist a relatively simple relation between $J_{C^n}^{(0)}(1)$ and $K_{M^n}^{(0)}(1)$, but we do not know it.

2. Gauss sums over $G_2(q)$.

2.1. Gauss sums over a finite reductive group. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $k = \mathbf{F}_q$, with Frobenius map F, and let $G = \mathbf{G}^F$ be the finite group consisting with elements in $\mathbf{G}$ fixed by F. Let R be an ordinary representation of G, $R : G \to GL_n(\mathbf{C})$, and ϕ be a modular representation of G, $\phi : G \to GL_m(k_r)$. We fix ϕ and define a class function ψ on G by

$$\psi(g) = \chi(\mathrm{Tr}_{r,1}(\mathrm{trace}\,\phi(g)))\,.$$

We call the following matrix $W(R)$ a Gauss sum over G associated with R.

$$W(R) = \sum_{g \in G} R(g)\psi(g)\,.$$

If R is irreducible, $W(R) = w(R)I_n$ for some complex number $w(R)$, where I_n is the identity matrix of degree $n = \deg R$. Since $w(R)$ depends only on the equivalent class of R, we shall write $w(\zeta_R)$ instead of $w(R)$, where ζ_R is the character of R, and we shall also call $w(\zeta_R)$ a Gauss sum over G.

Let $\tau_W(\zeta_R)$ be the trace of $W(R)$. Notice that for ordinary representations R_1 and R_2 of G, we have $\tau_W(\zeta_{R_1} + \zeta_{R_2}) = \tau_W(\zeta_{R_1}) + \tau_W(\zeta_{R_2})$. Then by linearity we can extend τ_W to a liner function $\tau_W : cf_{\mathbf{C}}(G) \to \mathbf{C}$, where $cf_{\mathbf{C}}(G)$ is the space of complex valued class functions on G. Thus for $f \in cf_{\mathbf{C}}(G)$ we have

$$\tau_W(f) = \sum_{g \in G} f(g)\psi(g)\,. \tag{1}$$

If R is irreducible, we have $\tau_W(\zeta_R) = w(\zeta_R)\deg R$. Therefore, when R is irreducible, the problem of the determination of $W(R)$ is reduced to the determination of $w(\zeta_R)$ or equivalently to that of $\tau_W(\zeta_R)$.

A first step to determine the value $w(\zeta_R)$ for all irreducible characters ζ_R of G, is the following theorem proved in [15].

THEOREM 2.2. *Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, θ a character of $T = \mathbf{T}^F$ and $R_{\mathbf{T},\theta}$ the generalized character of Deligne-Lusztig corresponding to $\mathbf{T}$ and θ. Then we have*

$$\tau_W(R_{\mathbf{T},\theta}) = |G : T| \sum_{t \in T} \theta(t)\psi(t)\,.$$

2.3. Maximal tori of $G_2(q)$. Let $G = \mathbf{G}^F = G_2(q)$ be the finite Chevalley group of type G_2 over k and ϕ be the 7-dimensional irreducible representation over k. Also let p be the characteristic of k. For the notation relating with the group G, we follow the notation used by B. Chang in [2], and B. Chang-R. Ree in [4]. For the definition of ϕ we follow the one given in [12], by Lee-Park: we also refer to [14]. We use the character tables in [4], if $p \neq 2, 3$, Enomoto [8] if $p = 3$, and Enomoto-Yamada [9] if $p = 2$.

Corresponding to the conjugacy classes of the Weyl group of G, there are six conjugacy classes of F-stable maximal tori and the groups of their rational points over k are isomorphic to the following groups:

$$\begin{aligned}
T_1 &= \{h(z_1, z_2, z_3) \mid z_i \in M, z_1z_2z_3 = 1\},\\
T_2 &= \{h(z_1, z_2, z_3) \mid z_i \in C, z_1z_2z_3 = 1\},\\
T_a &= \{h(z^q, z^{-q+1}, z^{-1}) \mid z \in M_2\},\\
T_b &= \{h(z, z^q, z^{-q-1}) \mid z \in M_2\},\\
T_3 &= \{h(z, z^q, z^{q^2}) \mid z \in Z_{q^2+q+1}\},\\
T_6 &= \{h(z, z^{-q}, z^{q^2}) \mid z \in Z_{q^2-q+1}\},
\end{aligned}$$

where Z_{q^2+q+1} and Z_{q^2-q+1} are the subgroups of $\bar{k}^\times$ of order q^2+q+1 and q^2-q+1 respectively. Notice that the elements $h_\alpha(u)$, $h_\beta(v)$ in [12] can be expressed by $h(z_1, z_2, z_3)$ as follows:

$$h_\alpha(u) = h(u^{-1}, u^2, u^{-1}), \quad h_\beta(v) = h(v, v^{-1}, 1).$$

Therefore, noticing $\psi(g) = \chi(\operatorname{trace} \phi(g))$ in our case, from [12, p. 310] we have

$$\psi(h(z_1, z_2, z_3)) = \chi(1 + z_1 + z_2 + z_3 + z_1^{-1} + z_2^{-1} + z_3^{-1}). \tag{2}$$

The character groups $\widehat{T_\alpha}$ of T_α, $\alpha = 1, 2, a, b, 3$, and 6, are given as follows:

$$\begin{aligned}
\widehat{T_1} &= \{\theta_1 = (\pi_1, \pi_2, \pi_3) \mid \pi_i \in \hat{M}, \theta_1(h(z_1, z_2, z_3)) = \pi_1(z_1)\pi_2(z_2)\pi_3(z_3)\},\\
\widehat{T_2} &= \{\theta_2 = (\varphi_1, \varphi_2, \varphi_3) \mid \varphi_i \in \hat{C}, \theta_2(h(z_1, z_2, z_3)) = \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)\},\\
\widehat{T_a} &= \{\theta_a = (\eta) \mid \eta \in \widehat{M_2}, \theta_a(h(z^q, z^{-q+1}, z^{-1}) = \eta(z)\},\\
\widehat{T_b} &= \{\theta_b = (\eta) \mid \eta \in \widehat{M_2}, \theta_b(h(z, z^q, z^{-q-1}) = \eta(z)\},\\
\widehat{T_3} &= \{\theta_3 = (\xi) \mid \xi \in \widehat{Z_{q^2+q+1}}, \theta_3(h(z, z^q, z^{q^2})) = \xi(z)\},\\
\widehat{T_6} &= \{\theta_6 = (\rho) \mid \rho \in \widehat{Z_{q^2-q+1}}, \theta_6(h(z, z^{-q}, z^{q^2})) = \rho(z)\}.
\end{aligned}$$

With these notation of tori and their characters, we shall simply write the corresponding generalized character of Deligne-Lusztig by $R_{T,\theta}$ instead of $R_{\mathbf{T},\theta}$, where $T = \mathbf{T}^F$.

2.4. Character sums S_α. Corresponding to each maximal torus T_α, we consider the following character sums.

$$S_1(\pi_1,\pi_2,\pi_3) = \sum_{\substack{z_1,z_2,z_3\in M\\ z_1z_2z_3=1}} \pi_1(z_1)\pi_2(z_2)\pi_3(z_3)\chi\left(z_1+z_2+z_3+z_1^{-1}+z_2^{-1}+z_3^{-1}\right),$$

$$S_2(\varphi_1,\varphi_2,\varphi_3) = \sum_{\substack{z_1,z_2,z_3\in C\\ z_1z_2z_3=1}} \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)\chi\left(z_1+z_2+z_3+z_1^{-1}+z_2^{-1}+z_3^{-1}\right),$$

$$S_a(\eta) = \sum_{z\in M_2} \eta(z)\chi\left(z+z^{-q}+z^{-q+1}+z^{-1}+z^{q}+z^{q-1}\right),$$

$$S_b(\eta) = \sum_{z\in M_2} \eta(z)\chi\left(z+z^{q}+z^{q+1}+z^{-1}+z^{-q}+z^{-q-1}\right),$$

$$S_3(\xi) = \sum_{z\in Z_{q^2+q+1}} \xi(z)\chi\left(z+z^{q}+z^{q^2}+z^{-1}+z^{-q}+z^{-q^2}\right),$$

$$S_6(\rho) = \sum_{z\in Z_{q^2-q+1}} \rho(z)\chi\left(z+z^{-q}+z^{q^2}+z^{-1}+z^{q}+z^{-q^2}\right).$$

Also we define the character sums S_α for $\theta = 1$, by $S_1 = S_1(1,1,1)$, $S_2 = S_2(1,1,1)$, $S_a = S_a(1)$, $S_b = S_b(1)$, $S_3 = S_3(1)$, $S_6 = S_6(1)$. Notice that with the notation defined in section 1, we have

$$S_1 = K^{(0)}_{M^3}(1),\quad S_2 = J^{(0)}_{C^3}(1),\quad S_3 = K^{(0)}_{M_3}(1),\quad S_6 = J^{(0)}_{C_3}(1).$$

The following result is a direct consequence of these definitions and (2.2) with the equation (2) in (2.3).

PROPOSITION 2.5. 1. $\tau_W(R_{T_1,\theta_1}) = q^6(q+1)^2(q^4+q^2+1)\chi(1)S_1(\pi_1,\pi_2,\pi_3)$,
2. $\tau_W(R_{T_2,\theta_2}) = q^6(q-1)^2(q^4+q^2+1)\chi(1)S_2(\varphi_1,\varphi_2,\varphi_3)$,
3. $\tau_W(R_{T_a,\theta_a}) = q^6(q^6-1)\chi(1)S_a(\eta)$,
4. $\tau_W(R_{T_b,\theta_b}) = q^6(q^6-1)\chi(1)S_b(\eta)$,
5. $\tau_W(R_{T_3,\theta_3}) = q^6(q^2-1)^2(q^2-q+1)\chi(1)S_3(\xi)$,
6. $\tau_W(R_{T_6,\theta_6}) = q^6(q^2-1)^2(q^2+q+1)\chi(1)S_6(\rho)$.

COROLLARY 2.6. *If $\theta \in \hat{T}$ is in general position, then $\varepsilon(\mathbf{T})R_{\mathbf{T},\theta}$, where $\varepsilon(\mathbf{T}) = (-1)^{rk(\mathbf{T})}$ and $rk(\mathbf{T})$ is the k-rank of $\mathbf{T}$, is irreducible and we have*
1. $w(R_{T_1,\theta_1}) = q^6\chi(1)S_1(\pi_1,\pi_2,\pi_3)$,
2. $w(R_{T_2,\theta_2}) = q^6\chi(1)S_2(\varphi_1,\varphi_2,\varphi_3)$,
3. $w(-R_{T_a,\theta_a}) = -q^6\chi(1)S_a(\eta)$,
4. $w(-R_{T_b,\theta_b}) = -q^6\chi(1)S_b(\eta)$,
5. $w(R_{T_3,\theta_3}) = q^6\chi(1)S_3(\xi)$,
6. $w(R_{T_6,\theta_6}) = q^6\chi(1)S_6(\rho)$.

2.7. Unipotent characters. Every unipotent character of G can be given in terms of six generalized characters $R_{T_\alpha,1}$ ($\alpha = 1,2,a,b,3$, and 6) of Deligne-Lusztig and four class functions Y_i ($i = 1, 2, 3$, and 4) as follows (see [4], [13] and also (2.7.1) below):

$$X_{11} = \frac{1}{12}R_{T_1,1} + \frac{1}{12}R_{T_2,1} + \frac{1}{4}R_{T_a,1} + \frac{1}{4}R_{T_b,1} + \frac{1}{6}R_{T_3,1} + \frac{1}{6}R_{T_6,1} = 1_G,$$

$$X_{12} = \frac{1}{12}R_{T_1,1} + \frac{1}{12}R_{T_2,1} - \frac{1}{4}R_{T_a,1} - \frac{1}{4}R_{T_b,1} + \frac{1}{6}R_{T_3,1} + \frac{1}{6}R_{T_6,1} = St_G,$$

$$X_{13} = \frac{1}{12}R_{T_1,1} - \frac{1}{12}R_{T_2,1} + \frac{1}{4}R_{T_a,1} - \frac{1}{4}R_{T_b,1} + \frac{1}{3}Y_1 - \frac{1}{3}Y_3 - \frac{1}{3}Y_4,$$

$$X_{14} = \frac{1}{12}R_{T_1,1} - \frac{1}{12}R_{T_2,1} - \frac{1}{4}R_{T_a,1} + \frac{1}{4}R_{T_b,1} + \frac{1}{3}Y_1 - \frac{1}{3}Y_3 - \frac{1}{3}Y_4,$$

$$X_{15} = \frac{1}{6}R_{T_1,1} - \frac{1}{6}R_{T_3,1} - \frac{1}{2}Y_1 - \frac{1}{2}Y_2,$$

$$X_{16} = \frac{1}{6}R_{T_1,1} - \frac{1}{6}R_{T_6,1} + \frac{1}{6}Y_1 + \frac{1}{2}Y_2 + \frac{1}{3}Y_3 + \frac{1}{3}Y_4,$$

$$X_{17} = -\frac{1}{6}R_{T_2,1} + \frac{1}{6}R_{T_6,1} - \frac{1}{2}Y_1 + \frac{1}{2}Y_2,$$

$$X_{18} = -\frac{1}{6}R_{T_2,1} + \frac{1}{6}R_{T_3,1} + \frac{1}{6}Y_1 - \frac{1}{2}Y_2 + \frac{1}{3}Y_3 + \frac{1}{3}Y_4,$$

$$X_{19} = -\frac{1}{6}R_{T_3,1} + \frac{1}{6}R_{T_6,1} + \frac{1}{3}Y_1 + \frac{2}{3}Y_3 - \frac{1}{3}Y_4,$$

$$\bar{X}_{19} = -\frac{1}{6}R_{T_3,1} + \frac{1}{6}R_{T_6,1} + \frac{1}{3}Y_1 - \frac{1}{3}Y_3 + \frac{2}{3}Y_4.$$

REMARK 2.7.1. (1) For $p = 2$ or 3, we regard that the generalized characters Y_i ($i = 1, 2, 3, 4$) are determined by the equations above under suitable identification of unipotent characters. See Tables 1 and 2 in Appendix for the identification with the notation in [8] and [9].

(2) Here we briefly recall how these relations are explained by Lusztig in [13].

Let $\mathcal{W}$ be the Weyl group of G and thus it is the dihedral group of order 12 having the following presentation: $\mathcal{W} = \langle a, b \mid a^2 = b^2 = (ab)^6 = 1 \rangle$. Let ρ be an irreducible representation of $\mathcal{W}$ and hence ρ is equivalent to one of $1_{\mathcal{W}}, \varepsilon, \varepsilon_1, \varepsilon_2, V, V'$, where $1_{\mathcal{W}}$ is the trivial representation, ε is the sign character, ε_1 (resp. ε_2) is the linear character determined by $\varepsilon_1(a) = 1, \varepsilon_1(b) = -1$, (resp. $\varepsilon_2(a) = -1, \varepsilon_2(b) = 1$), V is the reflection representation and $V' = V \otimes \varepsilon_1$. Let $R_\rho = |\mathcal{W}|^{-1} \sum_{w \in \mathcal{W}} \operatorname{trace} \rho(w) R_{T_w,1}$. Notice that with the notation in [4], $T_2 = T_{(ab)^3}$, $T_3 = T_{(ab)^2}$ and $T_6 = T_{ab}$. Now we have $R_{1_{\mathcal{W}}} = 1_G$, $R_\varepsilon = St =$ the Steinberg representation. Let $\mathfrak{S}_3$ be the symmetric group of degree 3 and $\mathcal{M}(\mathfrak{S}_3)$ be the set consisting of all pairs (x, σ) where x is an element of $\mathfrak{S}_3$ and σ is an irreducible representation of the centralizer of x modulo the equivalence relation determined by the canonical action of $\mathfrak{S}_3$. Let $\mathcal{X} = \{X_{13}, X_{14}, \cdots, X_{19}, \bar{X}_{19}\}$ and $\mathcal{Y} = \{R_{\varepsilon_1}, R_{\varepsilon_2}, R_V, R_{V'}, Y_1, Y_2, Y_3, Y_4\}$. Then the elements of $\mathcal{X}$ and $\mathcal{Y}$ are parametrized by $\mathcal{M}(\mathfrak{S}_3)$ respectively and they are transformed to each other by a pairing over $\mathcal{M}(\mathfrak{S}_3) \times \mathcal{M}(\mathfrak{S}_3)$. For more details see [13]. We only give the parameterization of the elements of $\mathcal{Y}$, using the notation in [13]. For the parameterization of the elements of $\mathcal{X}$, see Table 2 in Appendix.

$(1,1)$	$(g_2,1)$	$(1,r)$	$(g_3,1)$	$(1,\varepsilon)$	(g_2,ε)	(g_3,θ)	(g_3,θ^2)
R_V	$R_{V'}$	R_{ε_1}	R_{ε_2}	Y_1	Y_2	Y_3	Y_4

LEMMA 2.8. *$\tau_W(Y_i) = 0$ for $i = 1, 2, 3$ and 4.*

PROOF. For $p \neq 2, 3$, we can easily obtain the result using the equations (1) in (2.1), (2) in (2.3) and the values of Y_i given in [4]. For $p = 2$ or 3 we use Table 4 or 3 in Appendix respectively. □

PROPOSITION 2.9. *The sums $S_2, \cdots, S_6$ are expressed by S_1 as follows:*
1. $S_2 = S_1 - 2q\chi(-3)$,
2. $S_a = -S_1 + 2q\chi(-3) + 2q\chi(-2)$,
3. $S_b = -S_1 + 2q\chi(-2)$,
4. $S_3 = S_1 - 3q\chi(-2)$,
5. $S_6 = S_1 - 2q\chi(-3) - 3q\chi(-2)$.

PROOF. The identity (1) is proved in (1.8) and (4) (resp. (5)) is obtained from (1.7.1) (resp. (1.7.2)) by putting $n = 3$. By (1), the identity (2) is equivalent to

2$'$. $S_a = -S_2 + 2q\chi(-2)$.

Since (2$'$) and (3) can be proved similarly, we prove only (3). For the proof we extend S_b to a function $\tilde{S}_b$ over M as follows: for $x \in M$, let

$$\tilde{S}_b(x) = \sum_{\substack{y\in M, z\in M_2\\ x=yN_{2,1}(z)}} \chi(y+y^{-1})\chi^{(2)}(z+z^{-1}).$$

Notice that $\tilde{S}_b(1) = S_b$. As before $\tilde{S}_b = \sum_{\pi\in\hat{M}}\langle\tilde{S}_b, \pi\rangle\pi$ and

$$\begin{aligned}(q-1)\langle\tilde{S}_b, \pi\rangle &= \sum_{x\in M}\tilde{S}_b(x)\pi(x^{-1})\\ &= \sum_{y\in M}\chi(y+y^{-1})\pi(y^{-1})\sum_{z\in M_2}\chi^{(2)}(z+z^{-1})\pi^{(2)}(z^{-1})\\ &= K(\chi,\bar{\pi})K_2(\chi^{(2)},\bar{\pi}^{(2)})\\ &= -K(\chi,\bar{\pi})^3 + 2q\pi(-1)K(\chi,\bar{\pi}) \quad \text{(by (1.5.1))}.\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{S}_b(x) &= \frac{1}{q-1}\sum_{\substack{x_1,x_2,x_3\in M\\ \pi\in\hat{M}}}\chi\left(\sum_{i=1}^{3}(x_i+x_i^{-1})\right)\pi((x_1x_2x_3)^{-1}x)\\ &\quad + \frac{2q}{q-1}\sum_{x_1\in M,\pi\in\hat{M}}\chi(x_1+x_1^{-1})\pi(x_1^{-1}(-x))\\ &= -K^{(0)}_{M^3}(x) + 2q\chi(-x-x^{-1}).\end{aligned}$$

Thus putting $x = 1$, we have the required identity. □

REMARK 2.9.1. (1) We can prove similarly

$$S_b(\pi^{(2)}) = -S_1(\pi, \pi, 1) + 2q\pi(-1)\chi(-2),$$
$$S_a(\varphi^{(2)}) = -S_2(\varphi, \varphi, 1) + 2q\varphi(-1)\chi(-2).$$

(2) For the identities (2)–(5), we can give an alternate proof using a subset of polynomials of degree 3 over k, as B. Chang used a subset of polynomials of degree 2 to prove (1.3).

THEOREM 2.10. *For the* 10 *unipotent characters* X_{1j} *of* $G_2(q)$, *the values of Gauss sums* $w(X_{1j})$ *are given as follows*:

$$\begin{aligned}
w(X_{11}) &= q^6\chi(1)S_1 + q^7(q^3-1)\chi(-2) + q^7(q^2-1)(q^2+2)\chi(-1),\\
w(X_{12}) &= q^6\chi(1)S_1 - q^4(q^3-1)\chi(-2) - q^3(q^2-1)(2q^2+1)\chi(-1),\\
w(X_{13}) &= q^6\chi(1)S_1 + q^6(q-1)(2q+1)\chi(-2),\\
w(X_{14}) &= q^6\chi(1)S_1 - q^6(q-1)(q+2)\chi(-2),\\
w(X_{15}) &= q^6\chi(1)S_1 + q^6(q-1)^2\chi(-1),\\
w(X_{16}) &= q^6\chi(1)S_1 + 2q^6(q-1)^2\chi(-2) + 3q^6(q-1)^2(q^2+2)\chi(-1),\\
w(X_{17}) &= q^6\chi(1)S_1 - 2q^6\chi(-2) - q^6(q+1)^2\chi(-1),\\
w(X_{18}) &= q^6\chi(1)S_1 + 2q^6(q^2+q+1)\chi(-2) - 3q^6(q+1)^2\chi(-1),\\
w(X_{19}) &= w(\overline{X_{19}})\\
&= q^6\chi(1)S_1 - q^6(q^2+q+1)\chi(-2) - 3q^7\chi(-1).
\end{aligned}$$

PROOF. Since τ_W is additive, it is elementary to calculate $\tau_W(X_{1j})$ using (2.7), (2.8) and (2.9). Thus to obtain $w(X_{1j})$, it is enough to use the relation, $w(X_{1j}) = (\deg X_{1j})^{-1}\tau_W(X_{1j})$.

REMARK 2.10.1. For X_{11} = the trivial representation of $G_2(q)$, $w(X_{11})$ is already obtained by Lee and Park [12].

Appendix.

In this appendix we summarize tables which are necessary to prove (2.8) in the case $p = 2$ or 3. For the notation of Lusztig and correspondence of unipotent characters with the elements of $\mathcal{M}(\mathfrak{S}_3)$ in Table 2, see [13, p. 372]. For the irreducible characters and conjugacy classes we use the notation in [4], [8] or [9] according to $p \neq 2, 3$, $p = 3$ or $p = 2$.

TABLE 1. Correspondence of $R_{T,1}$ in different characteristics

	$R_{T_1,1}$	$R_{T_2,1}$	$R_{T_a,1}$	$R_{T_b,1}$	$R_{T_3,1}$	$R_{T_6,1}$
$p \neq 2, 3$	X_1	X_2	$-X_a$	$-X_b$	X_3	X_6
$p = 3$	$\chi_9(0,0)$	$\chi_{12}(0,0)$	$-\chi_{10}(0)$	$-\chi_{11}(0)$	$\chi_{13}(0)$	$\chi_{14}(0)$
$p = 2$	$\chi_5(0,0)$	$\chi_5'(0,0)$	$-\chi_6(0)$	$-\chi_6'(0)$	$\chi_7(0)$	$\chi_7'(0)$

TABLE 2. Correspondence of unipotent characters

$\mathcal{M}(\mathfrak{S}_3)$	$(1,1)$	$(g_2,1)$	$(1,r)$	$(g_3,1)$	$(1,\varepsilon)$	(g_2,ε)	(g_3,θ)	(g_3,θ^2)
$p \neq 2,3$	X_{16}	X_{15}	X_{13}	X_{14}	X_{18}	X_{17}	X_{19}	$\bar{X}_{19}$
$p=3$	θ_1	θ_2	θ_3	θ_4	θ_{10}	θ_{11}	$\theta_{12}(1)$	$\theta_{12}(-1)$
$p=2$	θ_1	θ_2	θ_3	θ_4	θ'_1	θ'_2	$\theta_9(1)$	$\theta_9(2)$

$(X_{11} = \theta_0(p=3) = \theta_0(p=2) = 1, X_{12} = \theta_5(p=3) = \theta_5(p=2) = St)$

TABLE 3. Values of Y_i, $(p=3)$

	order of centralizer	Y_1	Y_2	Y_3	Y_4
A_{41}	$2q^4$	q^2			
A_{42}	$2q^4$	$-q^2$			
A_{51}	$3q^2$			q	q
A_{52}	$3q^2$			ωq	$\omega^2 q$
A_{53}	$3q^2$			$\omega^2 q$	ωq
B_4	$2q^2$		q		
B_5	$2q^2$		$-q$		

TABLE 4. Values of Y_i, $(p=2, q \equiv \varepsilon \bmod 3)$

	order of centralizer	Y_1	Y_2	Y_3	Y_4
A_{31}	$6q^4$	εq^2			
A_{32}	$2q^4$	$-\varepsilon q^2$			
A_4	$3q^4$	εq^2			
A_{51}	$2q^2$		q		
A_{52}	$2q^2$		$-q$		
$B_2(0)$	$3q^2$			q	q
$B_2(1)$	$3q^2$			ωq	$\omega^2 q$
$B_2(2)$	$3q^2$			$\omega^2 q$	ωq

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