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Suspension Order of the Suspended Real 6-Projective Space

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1. Introduction.

In this note all spaces and homotopies are based. We denote by ι_X the homotopy class of the identity map of a space X. We denote by ΣX a suspension of X. The self-homotopy set $[\Sigma X, \Sigma X]$ is a group, called a track group ([1]). The order of $\iota_{\Sigma X}$ is called the suspension order of X ([8]). Let P^n be the real n-dimensional projective space. The author proved that the suspension order of $\Sigma^2\mathrm{P}^6$ is 8 ([6]). The purpose of this note is to show the following.

THEOREM 1.1. *The suspension order of* $\Sigma\mathrm{P}^6$ *is* 8.

As a direct consequence of this theorem, we have ([3])

COROLLARY 1.2. *The suspension order of* $\Sigma\mathrm{P}^{2n}$ *is* $2^{\varphi(2n)}$, *where* $\varphi(m)$ *stands for the number of integers* k *satisfying* $1 \le k \le m$ *and* $k \equiv 0, 1, 2$ *or* $4 \mod 8$.

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2. A review of the result of [6].

First we fix the notation. We denote by $i_{k,n} : \mathrm{P}^k \hookrightarrow \mathrm{P}^n$ for $k \le n$ the inclusion map and by $\gamma_n : S^n \to \mathrm{P}^n$ the covering map. We note that

$$i_{m,n} \circ i_{k,m} = i_{k,n} \quad \text{for } k \le m \le n \quad \text{and} \quad i_{k,n} \circ \gamma_k = 0 \quad \text{for } k < n. \tag{1}$$

We often use the same letter for a mapping and its homotopy class. We set $\iota_n = \iota_{S^n}$. Let $\eta_2 \in \pi_3(S^2)$ and $\nu_4 \in \pi_7(S^4)$ be the Hopf maps. We set $\eta_n = \Sigma^{n-2}\eta_2$ $(n \ge 2)$, $\eta_n^2 = \eta_n \circ \eta_{n+1}$ and $\nu_n = \Sigma^{n-4}\nu_4$ $(n \ge 4)$. We recall the following result about the 2-primary components of homotopy groups of spheres ([7]):

$$\pi_n(S^n) = \mathbf{Z}\{\iota_n\} \ (n \ge 1), \quad \pi_3(S^2) = \mathbf{Z}\{\eta_2\}, \quad \pi_{n+1}(S^n) = \mathbf{Z}_2\{\eta_n\} \ (n \ge 3),$$

$$\pi_{n+2}(S^n) = \mathbf{Z}_2\{\eta_n^2\} \ (n \ge 2), \quad \pi_6(S^3) = \mathbf{Z}_4\{\nu'\}, \quad \pi_7(S^4) = \mathbf{Z}\{\nu_4\} \oplus \mathbf{Z}_4\{\Sigma\nu'\},$$

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$$[\iota_4, \iota_4] = 2\nu_4 - \Sigma\nu' \quad \text{and} \quad \pi_{n+3}(S^n) = \mathbf{Z}_8\{\nu_n\} \ (n \geq 5)\,,$$

where $[\iota_4, \iota_4]$ is the Whitehead square of ι_4.

We set $M^n = \Sigma^{n-2}\mathrm{P}^2$ and $i_n = \Sigma^{n-2}i_{1,2} : S^{n-1} \to M^n$ for $n \geq 2$. Here we note that the notation of i_n is different from that of [6]. We denote by $p_n : M^n \to S^n$ the collapsing map. Let $\bar{\eta}_3 \in [M^5, S^3]$ and $\tilde{\eta}_3 \in \pi_5(M^4)$ be an extension and a coextension of η_3 respectively. We set $\bar{\eta}_n = \Sigma^{n-3}\bar{\eta}_3$ for $n \geq 3$. By Lemma 1.5.i) of [6], $[M^5, M^4]$ is stable and

$$[M^5, M^4] = \mathbf{Z}_2\{i_4\bar{\eta}_3\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3 p_5\}\,.$$

We set $\mathrm{P}^6_3 = \mathrm{P}^6/P^2$. Let $q : \mathrm{P}^6 \to \mathrm{P}^6_3$ be the collapsing map and let $\psi : \mathrm{P}^6_3 \to M^3$ be the third map in the cofibration

$$\mathrm{P}^2 \xrightarrow{i_{2,6}} \mathrm{P}^6 \xrightarrow{q} \mathrm{P}^6_3 \xrightarrow{\psi} M^3 \longrightarrow \cdots\,. \tag{2}$$

Then, by §3 of [6], p^6_3 is regarded as a mapping cone of $i_4\bar{\eta}_3 : M^5 \to M^4$ and there exists an extension $\bar{p}_5 \in [\Sigma\mathrm{P}^6_3, S^5]$ of $p_5 : M^5 \to S^5$. This result is improved as follows.

LEMMA 2.1. *There exists an extension* $\bar{p}_4 \in [\mathrm{P}^6_3, S^4]$ *of* $p_4 : M^4 \to S^4$ *satisfying* $\Sigma\bar{p}_4 = \bar{p}_5$.

PROOF. We consider a cofibration

$$M^5 \xrightarrow{i_4\bar{\eta}_3} M^4 \xrightarrow{i'} \mathrm{P}^6_3 \xrightarrow{p'} M^6 \longrightarrow \cdots\,, \tag{3}$$

where i' is the inclusion map and p' is a map pinching M^4 to one point. Since $p_4 \circ i_4\bar{\eta}_3 = 0$, there exists an extension $\bar{p}_4 \in [\mathrm{P}^6_3, S^4]$ of p_4. This completes the proof. □

We recall ([4], [5]) that $\pi_6(M^4) = \mathbf{Z}_4\{\delta\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5\}$, where δ is the attaching map in the Stiefel manifold $V_{5,2} = M^4 \cup e^7$ of 2-frames. By Lemma 2.1, the relation in the third line from the last p. 247 on [6] is written as

$$8\iota_{\Sigma^2\mathrm{P}^6} \equiv 0 \bmod 2(\Sigma^2 i_{2,6})\delta(\Sigma^2\bar{p}_4)(\Sigma^2 q)\,.$$

To prove Theorem 1.1, it suffices to show

$$2(\Sigma^2 i_{2,6})\delta(\Sigma^2\bar{p}_4)(\Sigma^2 q) = 0\,.$$

3. Proof of the theorem.

We set $p'_6 = p_6 \circ p' : \mathrm{P}^6_3 \to S^6$. The set $[\mathrm{P}^6_3, S^3]$ inherits a group structure from S^3. Then we show

LEMMA 3.1. (i) $[M^{n+1}, S^n] = \mathbf{Z}_2\{\eta_n p_{n+1}\}$ $(n \geq 2)$, $[M^{n+2}, S^n] = \mathbf{Z}_4\{\bar{\eta}_n\}$ $(n \geq 3)$, $[M^6, S^3] = \mathbf{Z}_2\{\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{\nu' p_6\}$ *and* $[M^7, S^4] = \mathbf{Z}_2\{\eta_4\bar{\eta}_5\} \oplus \mathbf{Z}_2\{(\Sigma\nu')p_7\} \oplus \mathbf{Z}_2\{\nu_4 p_7\}$.

(ii) *The group* $[\mathrm{P}^6_3, S^3]$ *consists of elements* $\eta_3\bar{p}_4$ *and* $\nu' p'_6$.

(iii) $p_3\psi \equiv \eta_3\bar{p}_4 \bmod \nu' p'_6$ *and* $p_5\Sigma^2\psi = \eta_5\Sigma^2\bar{p}_4$.

PROOF. (i) is immediately obtained. By the exact sequence of groups induced from (3):

$$(*)\quad [M^5, S^3] \xleftarrow{(i_4\bar{\eta}_3)^*} [M^4, S^3] \xleftarrow{(i')^*} [\mathrm{P}^6_3, S^3] \xleftarrow{(p')^*} [M^6, S^3] \xleftarrow{(i_5\bar{\eta}_4)^*} [M^5, S^3]\,,$$

we have $(i')^*(\eta_3\bar{p}_4) = \eta_3 p_4$ and $(i_5\bar{\eta}_4)^*(\bar{\eta}_3) = \eta_3\bar{\eta}_4$. So (i) implies (ii).

We consider a commutative diagram between cofibrations:

$$\begin{array}{ccccccccc}
\mathrm{P}^2 & \xrightarrow{i_{2,4}} & \mathrm{P}^4 & \xrightarrow{p} & M^4 & \xrightarrow{\overline{\Sigma\gamma_2}} & M^3 & \xrightarrow{\Sigma i_{2,4}} & \Sigma\mathrm{P}^4 \\
\downarrow{=} & & \downarrow{i_{4,6}} & & \downarrow{i'} & & \downarrow{=} & & \downarrow{\Sigma i_{4,6}} \\
\mathrm{P}^2 & \xrightarrow{i_{2,6}} & \mathrm{P}^6 & \xrightarrow{q} & \mathrm{P}^6_3 & \xrightarrow{\psi} & M^3 & \xrightarrow{\Sigma i_{2,6}} & \Sigma\mathrm{P}^6\,,
\end{array}$$

where $\overline{\Sigma\gamma_2}$ is an extension of $\Sigma\gamma_2$ and p is a map pinching P^2 to one point. By (2.1) of [2], $\Sigma^\infty(\overline{\Sigma\gamma_2}) = \Sigma^\infty(\tilde{\eta}_3 p_5) \in \{M^5, M^4\}$. So we have $\Sigma(\overline{\Sigma\gamma_2}) = \tilde{\eta}_3 p_5$ (cf. Lemma 1.1 of [6]) and $\Sigma(p_3 \circ \overline{\Sigma\gamma_2}) = p_4 \circ \tilde{\eta}_3 p_5 = \eta_4 p_5$. By Lemma 3.1.(i), $\Sigma : [M^4, S^3] \to [M^5, S^4]$ is an isomorphism. So we have $p_3 \circ \overline{\Sigma\gamma_2} = \eta_3 p_4$ and

$$p_3 \circ \psi \circ i' = p_3 \circ \overline{\Sigma\gamma_2} = \eta_3 p_4\,. \tag{4}$$

Making use of the exact squence $(*)$, by (ii) and (4), we obtain

$$p_3\psi \equiv \eta_3\bar{p}_4 \bmod \nu' p'_6\,.$$

This leads to the first half of (iii).

By the fact $\Sigma^2\nu' = 2\nu_5$, $\Sigma^2(\nu' p'_6) = \nu_5 \circ 2\Sigma^2 p'_6 = 0$. This leads to the second half of (iii), completing the proof. □

Now we show the following.

LEMMA 3.2. $2(\Sigma^2 i_{2,5})\delta\Sigma^2\bar{p}_4\Sigma^2 q = 0 \in [\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^5]$.

PROOF. We consider the exact sequence induced from (2):

$$[\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4] \xleftarrow{(\Sigma^2 q)^*} [\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^4] \xleftarrow{(\Sigma^2\psi)^*} [M^5, \Sigma^2\mathrm{P}^4]\,.$$

By Lemmas 1.2 and 1.3 of [6], we recall the following:

$$\pi_5(\Sigma^2\mathrm{P}^4) = \mathbf{Z}_8\{\beta\}\,, \quad \pi_6(\Sigma^2\mathrm{P}^4) = \mathbf{Z}_4\{(\Sigma^2 i_{2,4})\delta\} \oplus \mathbf{Z}_2\{\beta\eta_5\}$$

and

$$\beta\eta_5 = \Sigma^2\gamma_4 + 2(\Sigma^2 i_{2,4})\delta\,.$$

Therefore, by Lemma 3.1. (iii), we see that

$$\beta p_5\Sigma^2\psi = \beta\eta_5\Sigma^2\bar{p}_4 = (\Sigma^2\gamma_4)(\Sigma^2\bar{p}_4) + 2(\Sigma^2 i_{2,4})\delta\Sigma^2\bar{p}_4\,.$$

By the relation (1), $(\Sigma^2 i_{4,5}) \circ (\Sigma^2\gamma_4) = 0$ and $(\Sigma^2 i_{4,5}) \circ (\Sigma^2 i_{2,4}) = \Sigma^2 i_{2,5}$. So we have the equality

$$(\Sigma^2 i_{4,5})\beta p_5\Sigma^2\psi = 2(\Sigma^2 i_{2,5})\delta\Sigma^2\bar{p}_4$$

in $[\Sigma^2 P_3^6, \Sigma^2 P^5]$. Thus, by the fact $\psi \circ q = 0$, we obtain

$$2(\Sigma^2 i_{2,5})\delta(\Sigma^2 \bar{p}_4)(\Sigma^2 q) = (\Sigma^2 i_{4,5})\beta p_5(\Sigma^2 \psi)(\Sigma^2 q) = 0\,.$$

This completes the proof. □

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