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A Congruence Theorem for Compact Spacelike Surfaces in de Sitter Space

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Dedicated to the author's wife, Carmela

Abstract. In this paper we prove that two compact spacelike surfaces in de Sitter space for which there exists an isometry preserving their mean curvature functions are necessarily congruent. As an application of this, we deduce that there exists no compact spacelike Bonnet surface in de Sitter space.

1. Introduction.

The interest for the study of spacelike surfaces in de Sitter space is motivated by the fact that these surfaces exhibit nice Bernstein-type properties. In 1977 Goddard [6] conjectured that the only complete spacelike hypersurfaces with constant mean curvature in the $(n+1)$-dimensional de Sitter space should be the totally umbilical ones. This conjecture, which turned out to be false in this original statement, motivated the work of an important number of authors who considered the problem of characterizing the totally umbilical spacelike hypersurfaces of de Sitter space in terms of some appropriate geometric assumptions. In particular, for the two-dimensional case ($n = 2$) Akutagawa [1] showed that Goddard's conjecture is true if the constant mean curvature H of the surface satisfies $0 \leq H^2 \leq 1$. As an application of it, he proved that when $n = 2$ Goddard's conjecture is also true under the additional hypothesis of the compactness of the surface (see also [9] for a simultaneous and independent alternative proof of these facts given by Ramanathan).

In this paper we will obtain a congruence theorem for compact spacelike surfaces in de Sitter space which states as follows.

THEOREM. *Let $\psi_1 : \Sigma_1 \to \mathbf{S}_1^3 \subset \mathbf{E}_1^4$ and $\psi_2 : \Sigma_2 \to \mathbf{S}_1^3 \subset \mathbf{E}_1^4$ be two compact spacelike surfaces in de Sitter space. If there exists an isometry between Σ_1 and Σ_2 which preserves their mean curvature functions, then Σ_1 and Σ_2 are congruent; that is, ψ_1 and ψ_2 differ by a rigid motion in $\mathbf{S}_1^3$.*

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In particular, that means that, for the case of compact spacelike surfaces in de Sitter space, the mean curvature function and the metric suffice to determine the surface, up to rigid motions of the ambient space. Since the Gaussian curvature of the surfaces is preserved by isometries, our congruence result remains true if we replace "their mean curvature functions" by "their principal curvatures functions". On the other hand, as another application of this theorem we also conclude that no compact spacelike surface in de Sitter space admits a non-trivial one-parameter family of isometric deformations preserving the mean curvature function. In other words, there exists no compact spacelike Bonnet surface in de Sitter space (see section 4 for the details).

2. Preliminaries.

Let $\mathbf{E}_1^4$ denote the four-dimensional Lorentz-Minkowski space endowed with linear coordinates x_1, x_2, x_3, x_4 and the Lorentzian scalar product $\langle\cdot,\cdot\rangle$ given by $x_1^2+x_2^2+x_3^2-x_4^2$, and let $\mathbf{S}_1^3$ denote the three-dimensional unitary de Sitter space, that is, $\mathbf{S}_1^3 \subset \mathbf{E}_1^4$ is the hyperquadric

$$\mathbf{S}_1^3 = \{x \in \mathbf{E}_1^4 : \langle x, x\rangle = 1\}.$$

As is well-known, the induced metric on $\mathbf{S}_1^3$ from $\mathbf{E}_1^4$ is a Lorentzian metric which makes $\mathbf{S}_1^3$ the standard simply connected Lorentzian 3-space form of positive constant sectional curvature.

A smooth immersion $\psi : \Sigma \to \mathbf{S}_1^3 \subset \mathbf{E}_1^4$ of a two-dimensional connected manifold Σ is said to be a spacelike surface if the induced metric via ψ is a Riemannian metric on Σ, which, as usual, it is also denoted by $\langle\cdot,\cdot\rangle$. Let us recall that every spacelike surface in $\mathbf{S}_1^3$ is orientable. This follows from the fact that $(0,0,0,1) \in \mathbf{E}_1^4$ is a unit timelike vector field globally defined on $\mathbf{E}_1^4$, which determines a time-orientation on $\mathbf{E}_1^4$. Thus we can choose a unique unit normal vector field N on Σ which is a future-directed timelike vector in $\mathbf{E}_1^4$, and hence we may assume that Σ is oriented by N.

Since Σ is orientable, the induced Riemannian metric of Σ determines a conformal structure on Σ. In what folllows, it will be useful to treat Σ as a Riemann surface with respect to this conformal structure and ψ as a conformal immersion. If $z = u + iv$ is a local complex coordinate on Σ, then the induced metric on Σ can be expressed by

$$ds^2 = e^\rho |dz|^2$$

for a smooth function $\rho = \rho(z)$, so that

$$\langle\psi_z, \psi_z\rangle = \langle\psi_{\bar z}, \psi_{\bar z}\rangle = 0, \quad \langle\psi_z, \psi_{\bar z}\rangle = \frac{e^\rho}{2},$$

where the subscripts denote partial derivatives,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

The tangent vector fields ψ_z, $\psi_{\bar{z}}$ as well as the future-directed normal N define a moving frame on the surface which satisfies the following structural equations for the immersion

$$\psi_{zz} = \rho_z \psi_z - \frac{1}{2}\phi N\,,$$
$$\psi_{z\bar{z}} = \frac{e^\rho}{2} H N - \frac{e^\rho}{2}\psi\,,$$
$$N_z = H\psi_z - \phi e^{-\rho}\psi_{\bar{z}}\,.$$

Here $H = -(1/2)\text{trace}(-dN) = -2e^\rho\langle\psi_{z\bar{z}}, N\rangle$ is the mean curvature function of the spacelike surface and $\phi = 2\langle\psi_{zz}, N\rangle$. Let us remark that ϕdz^2 defines a global quadratic differential on Σ which is called its Hopf differential. The integrability conditions for this system are the Gauss equation

$$2\rho_{z\bar{z}} = e^\rho(H^2 - 1) - e^{-\rho}|\phi|^2$$

and the Codazzi equation

$$\phi_{\bar{z}} = -e^\rho H_z\,. \tag{1}$$

We refer the reader to [8] for the details, remembering the opposite sign of our mean curvature function. The reason for our sign choice is that, with our convention, the mean curvature timelike vector field $\vec{H}$ is given by $\vec{H} = HN$, so that $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\vec{H}(p)$ is future-directed.

3. Proof of the Theorem.

First of all, let us see that every compact spacelike surface in de Sitter space is a topological 2-sphere. Indeed, if $\psi : \Sigma \to \mathbf{S}^3_1 \subset \mathbf{E}^4_1$ is a spacelike immersion of a compact surface, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$, then we may consider the map $\Pi : \Sigma \to \mathbf{S}^2$ given by

$$\Pi = \frac{1}{\sqrt{1+\psi_4^2}}(\psi_1, \psi_2, \psi_3)\,.$$

Observe that

$$\Pi^*(\langle\cdot,\cdot\rangle_{\mathbf{S}^2}) = \frac{1}{1+\psi_4^2}\langle\cdot,\cdot\rangle + \frac{\langle\nabla\psi_4,\cdot\rangle\langle\nabla\psi_4,\cdot\rangle}{(1+\psi_4^2)^2} \geq \frac{1}{1+\psi_4^2}\langle\cdot,\cdot\rangle\,,$$

where $\nabla\psi_4$ is the gradient of ψ_4 and $\langle\cdot,\cdot\rangle_{\mathbf{S}^2}$ denotes the canonical metric on $\mathbf{S}^2$. From here, it follows that Π is a local diffeomorphism, and the compactness of Σ implies that it is a covering map. Since $\mathbf{S}^2$ is simply connected, then Π is in fact a global diffeomorphism between Σ and $\mathbf{S}^2$.

Therefore, since there exists an isometry between Σ_1 and Σ_2 which preserves their mean curvature functions, we may assume that $\Sigma_1 = \Sigma_2 = \Sigma$ is a topological 2-sphere and we may choose a common local complex coordinate z on Σ satisfying

$$\rho_1 = \rho_2 = \rho\,, \quad \text{and} \quad H_1 = H_2 = H\,.$$

From the Codazzi equation (1) we deduce that $(\phi_1 - \phi_2)_{\bar{z}} = 0$, so that the global quadratic differential on Σ locally given by $(\phi_1 - \phi_2)dz^2$ is holomorphic. But being Σ a topological sphere, that gives $(\phi_1 - \phi_2)dz^2 \equiv 0$, which is $\phi_1 = \phi_2 = \phi$, but this means that the surfaces are congruent. Actually, if N_1 and N_2 denote the cooresponding future-directed normal fields, then $\{(\psi_1)_z, (\psi_1)_{\bar{z}}, N_1, \psi_1\}$ and $\{(\psi_2)_z, (\psi_2)_{\bar{z}}, N_2, \psi_2\}$ are two adapted moving frames on Σ for which there exist an $\mathbf{SO}_1(4)$-valued function $A = A(z)$ satisfying

$$A((\psi_1)_z) = (\psi_2)_z\,, \tag{2}$$

$$A((\psi_1)_{\bar{z}}) = (\psi_2)_{\bar{z}}\,, \tag{3}$$

$$A(N_1) = N_2\,, \tag{4}$$

$$A(\psi_1) = \psi_2\,. \tag{5}$$

Here $\mathbf{SO}_1(4)$ denotes the group of (positive) rigid motions in $\mathbf{S}^3_1$. It remains to see that A is constant or, equivalently, that $A_z = A_{\bar{z}} = 0$. By derivating (5) with respect to z and using (2) we see that $A_z(\psi_1) = 0$. By derivating now (4) with respect to z and using the third structural equation for the immersion ψ_1 we obtain that

$$A_z(N_1) + HA((\psi_1)_z) - \phi e^{-\rho} A((\psi_1)_{\bar{z}}) = (N_2)_z\,,$$

which jointly with (2) and (3), and the third structural equation for ψ_2 implies that $A_z(N_1) = 0$. Similarly, derivating (2) with respect to z and using the first structural equation, as well as (2) and (4), we conclude that $A_z((\psi_1)_z) = 0$; finally derivating (3) with respect to z and using the second structural equation, jointly with (4) and (5), we obtain that $A_z((\psi_1)_{\bar{z}}) = 0$. Summing up, $A_z = 0$. Since A is real, by conjugation $A_{\bar{z}} = 0$. This finishes the proof of the theorem.

REMARK. Let us observe that, in the proof above, once we know that $\phi_1 = \phi_2$, then we could argue that the two immersions ψ_1 and ψ_2 have the same first and second fundamental forms, so that there exists a rigid motion $A \in \mathbf{SO}_1(4)$ such that $\psi_2 = A \circ \psi_1$ (see for example [7]). This is nothing but the uniqueness part (up to isometries) of the fundamental theorem for surfaces. In fact, what we have made in the last part of our proof has been to provide a simple proof of this fact, adapted to our case of spacelike surfaces in de Sitter space.

4. Applications.

First of all, since the Gaussian curvature K of Σ is preserved by isometries, we can rewrite our congruence result as follows

COROLLARY 1. *Let $\psi_1 : \Sigma_1 \to \mathbf{S}^3_1 \subset \mathbf{E}^4_1$ and $\psi_2 : \Sigma_2 \to \mathbf{S}^3_1 \subset \mathbf{E}^4_1$ be two compact spacelike surfaces in de Sitter space. If there exists an isometry between Σ_1 and Σ_2 which preserves their principal curvatures functions, then Σ_1 and Σ_2 are congruent.*

On the other hand, an old problem in classical surface theory first posed by Bonnet [3] is the study of non-trivial one-parameter families of isometric surfaces having the same mean curvature. By non-trivial families here we mean families of surfaces which do not differ by rigid motions. In particular, a surface is said to be a Bonnet surface if it admits such non-trivial isometric deformations. We refer the reader to [2] for a recent comprehensive study of this subject and its relationship to the Painlevé equations. Using this terminology, our congruence result implies the following consequence.

COROLLARY 2. *There exists no compact spacelike Bonnet surface in de Sitter space.*

Chen and Li [4] and Fujioka and Inoguchi [5] have recently considered the local study of spacelike Bonnet surfaces in the three-dimensional Lorentzian space forms, and, in particular, its relationship to isothermic surfaces and to harmonic inverse mean curvature surfaces, respectively. With respect to the global theory, our congruence result has also the following application.

COROLLARY 3. *There exists no compact spacelike Bonnet surface in the three-dimensional Lorentzian space forms.*

To proof this corollary we have simply to observe that the only Lorentzian 3-space form which admits compact spacelike surfaces is the de Sitter space. To see it, let $\mathbf{E}_1^3$ be the three-dimensional Lorentz-Minkowski space, which is the standard model for a Lorentzian 3-space form with zero curvature, and let $\mathbf{H}_1^3$ be the three-dimensional anti de Sitter space, which is the standard model for a Lorentzian 3-space form with negative sectional curvature. Recall that $\mathbf{H}_1^3 \subset \mathbf{E}_2^4$ is the hyperquadric

$$\mathbf{H}_1^3 = \{x \in \mathbf{E}_2^4 : \langle x, x\rangle = x_1^2 + x_2^2 - x_3^2 - x_4^2 = -1\}.$$

Let us assume that $\psi : \Sigma \to \mathbf{E}_1^3$ or $\psi : \Sigma \to \mathbf{H}_1^3 \subset \mathbf{E}_2^4$ is a spacelike surface with timelike Gauss map N. If a is a fixed vector, $a \in \mathbf{E}_1^3$ or $a \in \mathbf{E}_2^4$ respectively, then consider the height function $\langle a, \psi\rangle$ defined on the spacelike surface Σ. The gradient of $\langle a, \psi\rangle$ is

$$\nabla\langle a, \psi\rangle = a^T ,$$

where $a^T \in \chi(\Sigma)$ is the tangent component of a, that is, $a^T = a + \langle a, N\rangle N$ in the case of a spacelike surface in $\mathbf{E}_1^3$, and $a^T = a + \langle a, N\rangle N + \langle a, \psi\rangle\psi$ in the case of a spacelike surface in $\mathbf{H}_1^3$. In particular,

$$|\nabla\langle a, \psi\rangle|^2 \geq \langle a, a\rangle ,$$

so that if we choose a to be spacelike, then the height function has no critical points in Σ. This shows that Σ cannot be compact (here by compact we mean compact without boundary).

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