TOKYO J. MATH.
VOL. 24, NO. 1, 2001

Homological Properties of the Module of Logarithmic Forms of Arrangements

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(Communicated by M. Oka)

Abstract. A (central) arrangement is a finite family of one-codimensional subspaces of a vector space V. We study the module of logarithmic forms with poles along the hyperplanes. We use a certain cochain complex and its cohomological groups to prove that cohomological properties of the module are closely related to the explicit structure of the module.

1. Introduction.

In 1991, L. Rose and H. Terao analysed the module of logarithmic forms of generic arrangements. They found that if $\mathcal{A}$ is a generic ℓ-arrangement, the projective dimension of the module of logarithmic p-forms of $\mathcal{A}$ is less than or equal to p for each $0 \leq p \leq \ell$. We consider a cochain complex of the module of logarithmic forms. If the cohomology of the complex of an arrangement $\mathcal{A}$ vanishes up to $\ell - 1$ level, $\mathcal{A}$ is called pure. And if we use the property of projective dimension, we can prove that generic arrangements are pure [5]. Since the module of logarithmic p-forms of free arrangements are free for all p, free arrangements are pure too. Strong Preparation Lemma proves that the module of logarithmic forms of $\mathcal{A} \cup \{H\}$ can be obtained from the module of logarithmic forms of $\mathcal{A}$ under some cohomology conditions. With Strong Preparation Lemma we can prove that k-generic arrangements are k-pure.

2. The setup.

Let K be a field. Let V be an ℓ dimensional vector space over K. A hyperplane H in V is a codimension one subspace of V. An arrangement $\mathcal{A}$ in V is a finite set of hyperplanes in V. Let $S = K[x_1, \cdots, x_\ell]$ be the polynomial algebra. Let $F = K(x_1, x_2, \cdots, x_\ell)$ be the field of quotients of S.

Received October 14, 1999
Revised January 14, 2000
This work was supported by KRF 1997-003-D00041.

Each hyperplane H in V has a defining form

$$\alpha_H = a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell \quad (a_i \in K)$$

unique up to a constant multiple. A *defining polynomial* for $\mathcal{A}$ is given by

$$Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H .$$

Let $L = L(\mathcal{A})$ be the set of all subspaces of V which are intersections of elements of $\mathcal{A}$. We call $L(\mathcal{A})$ *the intersection lattice* of $\mathcal{A}$. A partial order by reverse inclusion will be given to $L(\mathcal{A})$. A hyperplane H in V is called *generic to* $\mathcal{A}$ if $H \cap X \neq X$ for every non zero X in $L(\mathcal{A})$. We call H *k-generic to* $\mathcal{A}$ if $H \cap X \neq X$ for every $X \in L(\mathcal{A})$ with codimension less than k.

DEFINITION 2.1. We call an arrangement $\mathcal{A}$ *generic* if $\mathcal{A}$ has more than or equal to ℓ hyperplanes and every ℓ hyperplanes of $\mathcal{A}$ intersect only at the origin.

DEFINITION 2.2. We call an arrangement $\mathcal{A}$ *k-generic* if $\mathcal{A}$ has more than or equal to k hyperplanes and the codimension of the intersection of every k hyperplanes of $\mathcal{A}$ is k.

We set

$$\Omega^p(V) = \bigoplus_{1 \le i_1 < \cdots < i_p \le \ell} F(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) .$$

We agree that $\Omega^0(V) = F$. We call $\Omega^p(V)$ the module of rational differential p-forms on V. We define a map $d : \Omega^p(V) \to \Omega^{p+1}(V)$ as follows: For $f \in F$, we define

$$df = \sum_{k=1}^{\ell} \frac{\partial f}{\partial x_k} dx_k .$$

For $\omega = \sum f_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, where $1 \le i_1 < \cdots < i_p \le \ell$ and $f_{i_1 \cdots i_p} \in F$, we define

$$d\omega = \sum \sum_{k=1}^{\ell} (\partial f_{i_1 \cdots i_p} / \partial x_k) dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} .$$

The map d is called the *exterior differentiation.* Let

$$\Omega^p[V] = \bigoplus_{1 \le i_1 < \cdots < i_p \le \ell} S(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) .$$

We agree that $\Omega^0[V] = S$, $\Omega^p[V] = 0$ if p is negative. The elements of $\Omega^p[V]$ are called regular differential p-forms on V.

DEFINITION 2.3. We define *the module of logarithmic p-forms* of $\mathcal{A}$ by

$$\Omega^p(\mathcal{A}) = \{\omega \in \Omega^p(V) \mid Q\omega \in \Omega^p[V] \text{ and } Q(d\omega) \in \Omega^{p+1}[V]\} .$$

Define

$$\Omega^p\langle\mathcal{A}\rangle = \bigwedge^p \left\langle \frac{d\alpha_1}{\alpha_1}, \cdots , \frac{d\alpha_n}{\alpha_n}, dx_1, \ldots , dx_\ell \right\rangle$$

where $\mathcal{A} = \{ker(\alpha_1), \cdots , ker(\alpha_n)\}$.

THEOREM 2.4 ([3, Theorem 1]). *If $\mathcal{A}$ is a k-generic arrangement, then*

$$\Omega^p(\mathcal{A}) = \Omega^p\langle\mathcal{A}\rangle$$

for $p \leq k-2$.

The module D of derivations consists of all k-linear maps $\theta : S \to S$ satisfying

$$\theta(fg) = f\theta(g) + g\theta(f)$$

for any $f, g \in S$.

The module of derivations of $\mathcal{A}$, $D(\mathcal{A})$, is defined by

$$D(\mathcal{A}) = \{\theta \in D \mid Q(\mathcal{A}) \text{ divides } \theta(Q(\mathcal{A}))\}.$$

We say an arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module.

3. Strong Preparation Lemma.

In this section we prove the Strong Preparation Lemma.

Let H be a hyperplane which is not in $\mathcal{A}$. Let $H = ker(f)$. Define a map $\wedge df$; $\Omega^p(\mathcal{A}) \to \Omega^{p+1}(\mathcal{A})$ by $\omega \mapsto \omega \wedge df$. Then we can define a complex

$$0 \to \Omega^0(\mathcal{A}) \to \Omega^1(\mathcal{A}) \to \cdots \to \Omega^\ell(\mathcal{A}) \to 0.$$

We also consider another chain complex

$$0 \to \Omega^0(\mathcal{A}\cup\{H\}) \to \Omega^1(\mathcal{A}\cup\{H\}) \to \cdots \to \Omega^\ell(\mathcal{A}\cup\{H\}) \to 0,$$

with a coboundary map $\wedge df/f$; $\Omega^p(\mathcal{A}\cup\{H\}) \to \Omega^{p+1}(\mathcal{A}\cup\{H\})$ by $\omega \mapsto \omega \wedge df/f$. It is known that the complex $(\Omega^*(\mathcal{A}\cup\{H\}), \wedge df/f)$ is acyclic [4, Prop. 4.86].

LEMMA 3.1 ([6, Lemma 2.2.1]). *For any $\omega \in \Omega^p(\mathcal{A})$, there exist a rational $(p-1)$-form ω' and a rational p-form ω'' such that*

$$\omega = \omega' \wedge (df/f) + \omega'',$$

$Q'\omega'$ and $Q'\omega''$ are both regular.

LEMMA 3.2. *Suppose that $\omega \in \Omega^p(\mathcal{A}\cup\{H\})$, $H = Ker(f)$. Then $f\omega$ is in $\Omega^p(\mathcal{A})$.*

PROOF. If $H \in \mathcal{A}$, it is trivial. Let $H \notin \mathcal{A}$. Let Q and Qf be defining polynomials of $\mathcal{A}$ and $\mathcal{A}\cup\{H\}$ respectively. Let $\omega \in \Omega^p(\mathcal{A}\cup\{H\})$. Since $Q(f\omega) = (Qf)\omega \in \Omega^p[V]$, it is enough to show that $Qd(f\omega) = Q(df)\omega + Qfd\omega$. By Lemma 3.1, $\omega = \omega' \wedge (df/f) + \omega''$, so $Q(df) \wedge \omega = Q\omega''$, is regular. $Qf(d\omega)$ is regular, since $\omega \in \Omega^p(\mathcal{A}\cup\{H\})$. So both terms on the right hand side are in $\Omega^{p+1}[V]$. □

COROLLARY 3.3. *Let $\omega \in \Omega^p(\mathcal{A}\cup\{H\})$. Then $\omega \wedge df \in \Omega^{p+1}(\mathcal{A})$.*

PROOF. Since $df/f \in \Omega^1(\mathcal{A}\cup\{H\})$, $\omega \wedge (df/f) \in \Omega^{p+1}(\mathcal{A}\cup\{H\})$. By Lemma 3.2, $\omega \wedge df \in \Omega^{p+1}(\mathcal{A})$. □

LEMMA 3.4. *Let $H = ker(f)$, and $H \notin \mathcal{A}$. Let $\omega \in \Omega^p(\mathcal{A})$ and $\omega \wedge df = 0$. Then $\omega/f \in \Omega^p(\mathcal{A} \cup \{H\})$.*

PROOF. It is enough to show that $Qf \cdot d(\omega/f)$ is regular. But $d(\omega/f) = ((d\omega)f - \omega(df))/f^2 = d\omega/f$. Since $Q(d\omega)$ is regular, it is done. □

LEMMA 3.5 ([7, Lemma 3.6]). *Let $H = ker(f)$, and H is not in $\mathcal{A}$. Let $\omega \in \Omega^p(\mathcal{A})$ and $\omega|_H = 0$. Then $\omega/f \in \Omega^p(\mathcal{A} \cup \{H\})$.*

LEMMA 3.6. *Let $H = ker(f)$. Let $\omega \in \Omega^p(\mathcal{A})$. If $\omega \wedge df = 0$, then $\frac{1}{f}\langle\theta_E, \omega\rangle \in \Omega^{p-1}(\mathcal{A} \cup \{H\})$.*

PROOF. We define an S-linear map $\langle\theta_E, \ \rangle : \Omega^p(\mathcal{A}) \to \Omega^{p-1}(\mathcal{A})$ by

$$\langle\theta_E, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}\rangle = \sum_k (-1)^{k-1} x_{i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_p}\,.$$

Then $\langle\theta_E, \omega\rangle \in \Omega^{p-1}(\mathcal{A})$, and $Qf \cdot \frac{1}{f}\langle\theta_E, \omega\rangle$ is regular. So, it is enough to show that $d(Qf) \cdot \frac{1}{f}\langle\theta_E, \omega\rangle$ is regular. If we use the fact that

$$\langle\theta_E, df\rangle = f\,, \quad df \wedge \omega = 0\,,$$

we get

$$\begin{aligned} d(Qf) \cdot \left(\frac{1}{f}\langle\theta_E, \omega\rangle\right) &= f(dQ)\frac{1}{f}\langle\theta_E, \omega\rangle + Q(df)\frac{1}{f}\langle\theta_E, \omega\rangle \\ &= (dQ) \wedge \langle\theta_E, \omega\rangle + \frac{1}{f}Q(df) \wedge \langle\theta_E, \omega\rangle \\ &= (dQ) \wedge \langle\theta_E, \omega\rangle + \frac{1}{f}Q\langle\theta_E, df\rangle \wedge -\frac{1}{f}Q\langle\theta_E, df \wedge \omega\rangle \\ &= (dQ) \wedge \langle\theta_E, \omega\rangle + Q\omega\,. \end{aligned}$$

Since both terms on the right hand side are regular, it is done. □

PROPOSITION 3.7. *Let $H = ker(f)$ and H is not in $\mathcal{A}$. If $\Omega^{p-2}(\mathcal{A} \cup \{H\}) = \Omega^{p-2}(\mathcal{A}) + \Omega^{p-3}(\mathcal{A}) \wedge (df/f)$, then*

$$H^p(\Omega^*(\mathcal{A}), \wedge df) \cong \frac{\Omega^{p-1}(\mathcal{A} \cup \{H\})}{\Omega^{p-1}(\mathcal{A}) + \Omega^{p-2}(\mathcal{A}) \wedge (df/f)}\,.$$

PROOF. We define two maps γ and δ,

$$\gamma : H^p(\Omega^*(\mathcal{A}), \wedge df) \to \frac{\Omega^{p-1}(\mathcal{A} \cup \{H\})}{\Omega^{p-1}(\mathcal{A}) + \Omega^{p-2}(\mathcal{A}) \wedge (df/f)}$$

$$\delta : \frac{\Omega^{p-1}(\mathcal{A} \cup \{H\})}{\Omega^{p-1}(\mathcal{A}) + \Omega^{p-2}(\mathcal{A}) \wedge (df/f)} \to H^p(\Omega^*(\mathcal{A}), \wedge df)$$

as follows;

$$\gamma([\omega]) = \left[\frac{1}{f}\langle\theta_E, \omega\rangle\right], \quad \delta([\eta]) = [\eta \wedge df]\,.$$

First, we check the well-definedness of γ. Choose $\omega \in \Omega^p(\mathcal{A})$ such that $\omega \wedge df = 0$. Then $\frac{1}{f}\langle\theta_E, \omega\rangle$ is in $\Omega^{p-1}(\mathcal{A} \cup \{H\})$. Let $\omega \in \Omega^{p-1}(\mathcal{A}) \wedge df$. Then for some $\omega^* \in \Omega^{p-1}(\mathcal{A})$,

$$\begin{aligned}\frac{1}{f}\langle\theta_E, \omega\rangle &= \frac{1}{f}\langle\theta_E, \omega^* \wedge df\rangle \\ &= \frac{1}{f}\langle\theta_E, \omega^*\rangle \wedge df + (-1)^{p-1}\frac{1}{f}\omega^* \wedge \langle\theta_E, df\rangle \\ &= \langle\theta_E, \omega^*\rangle \wedge \frac{df}{f} + (-1)^{p-1}\omega^* .\end{aligned}$$

In this case we have $\gamma([\omega]) = 0$. This proves the well-definedness of γ. Second, we check the well-definedness of δ. Choose $\eta \in \Omega^{p-1}(\mathcal{A} \cup \{H\})$. Then $\eta \wedge df \in \Omega^{p-1}(\mathcal{A}) \wedge df$. This proves the well-definedness of δ. Third, we prove $\delta\gamma = \pm$*identity*. Choose $\omega \in \Omega^p(\mathcal{A})$ such that $\omega \wedge df = 0$. Then,

$$\begin{aligned}\delta\gamma([\omega]) = \delta\left(\frac{1}{f}\langle\theta_E, \omega\rangle\right) &= \left[\frac{1}{f}\langle\theta_E, \omega\rangle \wedge df\right] \\ &= \left[\frac{1}{f}\langle\theta_E, \omega \wedge df\rangle + (-1)^p\frac{1}{f}\omega \wedge \langle\theta_E, df\rangle\right] \\ &= \left[(-1)^p\frac{1}{f}\omega f\right] \\ &= [(-1)^p\omega] .\end{aligned}$$

Fourth, we prove $\gamma\delta = \pm$*identity*. Choose $\eta \in \Omega^{p-1}(\mathcal{A} \cup \{H\})$. Then,

$$\begin{aligned}\gamma\delta(\eta) = \gamma([\eta \wedge df]) &= \left[\frac{1}{f}\langle\theta_E, \eta \wedge df\rangle\right] \\ &= \left[\frac{1}{f}\langle\theta_E, \eta\rangle \wedge df + (-1)^{p-1}\frac{1}{f}\eta \wedge \langle\theta_E, df\rangle\right] .\end{aligned}$$

$\langle\theta_E, \eta\rangle$ is in $\Omega^{p-2}(\mathcal{A} \cup \{H\})$. By assumption, $\langle\theta_E, \eta\rangle \in \Omega^{p-2}(\mathcal{A}) + \Omega^{p-3}(\mathcal{A}) \wedge df/f$. This shows that $\gamma\delta([\eta]) = [(-1)^{p-1}\eta]$. □

As a corollary of this, we get the following theorem.

COROLLARY 3.8. *The following two conditions are equivalent*:

1. $H^p(\Omega^*(\mathcal{A}), \wedge df) = 0$, *for* $0 \le p \le k$,
2. $\Omega^p(\mathcal{A} \cup \{H\}) = \Omega^p(\mathcal{A}) + \Omega^{p-1}(\mathcal{A}) \wedge df/f$, *for* $0 \le p \le k-1$.

Under these conditions, we have

$$H^{k+1}(\Omega^*(\mathcal{A}, \wedge df) \cong \frac{\Omega^k(\mathcal{A} \cup \{H\})}{\Omega^k(\mathcal{A}) + \Omega^{k-1}(\mathcal{A}) \wedge (df/f)} .$$

4. k-pure arrangements.

DEFINITION 4.1. An arrangement $\mathcal{A}$ is called *pure* if

$$H^p(\Omega^*(\mathcal{A}), \wedge df) = 0,$$

for $p < \ell$, for every H generic to $\mathcal{A}$.

THEOREM 4.2 ([5, Prop. 5.10]). *Every free arrangement is pure.*

THEOREM 4.3 ([5, Prop. 5.12]). *Every generic arrangement is pure.*

THEOREM 4.4 ([5, Prop. 5.15]). *Every ℓ-arrangement is pure for $\ell \leq 3$.*

The following proposition is a corollary of Corollary 3.8.

PROPOSITION 4.5. *Let $H = ker(f)$. If H is generic to $\mathcal{A}$ and $\mathcal{A}$ is pure, then $\Omega^p(\mathcal{A} \cup \{H\}) = \Omega^p(\mathcal{A}) + \Omega^{p-1}(\mathcal{A}) \wedge df/f$, for $0 \leq p \leq \ell - 2$.*

Now we define k-pure arrangements and get the followings.

DEFINITION 4.6. An arrangement $\mathcal{A}$ is called *k-pure* if

$$H^p(\Omega^*(\mathcal{A}), \wedge df) = 0,$$

for $p < k$, for every H generic to $\mathcal{A}$.

COROLLARY 4.7. *Let $H = Ker(f)$. If H is generic to $\mathcal{A}$, and $\mathcal{A}$ is k-pure, then $\Omega^p(\mathcal{A} \cup \{H\}) = \Omega^p(\mathcal{A}) + \Omega^{p-1}(\mathcal{A}) \wedge df/f$ for $p \leq k - 2$.*

PROPOSITION 4.8. *Every k-generic arrangement is k-pure.*

PROOF. Let $\mathcal{A}$ be a k-generic arrangement. If H is generic to $\mathcal{A}$, then $\mathcal{A} \cup \{H\}$ is a k-generic arrangement. By Theorem 2.4, $\Omega^p(\mathcal{A} \cup \{H\}) = \Omega^p\langle \mathcal{A} \cup \{H\}\rangle$ if $p \leq k - 2$. Also $\Omega^p(\mathcal{A}) = \Omega^p\langle \mathcal{A}\rangle$ if $p \leq k-2$. This implies that $\Omega^p(\mathcal{A} \cup \{H\}) = \Omega^p(\mathcal{A}) + \Omega^{p-1}(\mathcal{A}) \wedge df/f$ if $p \leq k - 2$. By Corollary 3.8, we get $H^p(\Omega^*(\mathcal{A}), \wedge df) = 0$ if $p \leq k - 1$. □

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