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A Class of Semi-Selfsimilar Processes Related to Random Walks in Random Scenery

Takuji ARAI

Keio University

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Abstract. We construct a new class of semi-selfsimilar processes as limiting processes of some random walks in random scenery, which extends the class of selfsimilar processes studied by Kesten and Spitzer (1979).

1. Introduction and Results.

An $\mathbf{R}^d$-valued stochastic process $X = \{X(t), t \geq 0\}$ is said to be semi-selfsimilar if there exist $a \in (0, 1) \cup (1, \infty)$ and $b > 0$ such that

$$\{X(at), t \geq 0\} \stackrel{f.d.}{=} \{bX(t), t \geq 0\}, \tag{1.1}$$

where $\stackrel{f.d.}{=}$ denotes equality of all finite dimensional distributions. This notion of semi-self-similarity has recently introduced by Maejima and Sato [MS99]. They have also shown under mild conditions on $\{X(t)\}$ that there exists a unique $H > 0$ such that $b = a^H$ for any pair (a, b) satisfying (1.1). This H is called the exponent of semi-selfsimilarity of X. In this paper, we construct a new class of semi-selfsimilar processes as limiting processes of random walks in random scenery, which extends the class of selfsimilar processes studied by Kesten and Spitzer [KS79].

Let Z_β be an $\mathbf{R}^d$-valued (non-Gaussian) strictly β-semi-stable random variable, where $0 < \beta < 2$, and let φ be the characteristic function of Z_β. Namely, φ satisfies that for some $r \in (0, 1)$,

$$\varphi(\theta)^r = \varphi\left(r^{\frac{1}{\beta}}\theta\right), \quad \theta \in \mathbf{R}^d. \tag{1.2}$$

Let $\Gamma(\beta) = \{r > 0 \mid (1.2) \text{ holds}\}$. Then it is known that $\Gamma(\beta) = \{r_0^n, n \in \mathbf{Z}\}$ for some $r_0 \in (0, 1)$ or $\Gamma(\beta) = (0, \infty)$. See, for instance, [Me73]. Throughout this paper, we fix one $r \in \Gamma(\beta) \cap (0, 1)$, and write it just r. It is also known that non-Gaussian semi-stable

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characteristic function φ has the following Lévy-Khintchine representation (see [CS95]):

$$\varphi(\theta) = \exp\left\{ i\langle\theta, c\rangle + \int_S \gamma(dx) \int_0^\infty [e^{i\langle\theta, sx\rangle} - 1 - i\langle\theta, sx\rangle I[sx \in D]] d\left(-\frac{H_x(s)}{s^\beta}\right)\right\}, \tag{1.3}$$

where $c \in \mathbf{R}^d$, $S = \{x \in \mathbf{R}^d : \|x\| = 1\}$, $D = \{x \in \mathbf{R}^d : \|x\| \le 1\}$, γ is a finite measure on S, $\langle\cdot,\cdot\rangle$ is an inner product in $\mathbf{R}^d$, $H_x(s)$ is a nonnegative function such that

(1) $H_x(s)/s^\beta$ is nonincreasing in s for each x,
(2) $H_x(s)$ is right-continuous in s for each x and measurable in x for each s,
(3) $H_x(1) = 1$,
(4) $H_x(r^{1/\beta}s) = H_x(s)$.

Next, let $\{\xi(k), k \in \mathbf{Z}\}$ be independent and identically distributed $\mathbf{R}^d$-valued random variables belonging to the domain of partial attraction of Z_β in the sense that

$$r^{\frac{n}{\beta}} \sum_{k=1}^{[r^{-n}]} \xi(k) \xrightarrow{w} Z_\beta, \tag{1.4}$$

where $[a]$ is the integer part of a and $\xrightarrow{w}$ denotes weak convergence. When $\beta = 1$, we assume an additional symmetry condition on $\{\xi(k)\}$ as follows: For some K,

$$|E[\xi(0); |\xi(0)| \le \rho]| \le K < \infty \quad \text{for all } \rho > 0. \tag{1.5}$$

Also let $\{S_n\}_{n=0}^\infty$ be an integer-valued random walk, independent of $\{\xi(k)\}$, such that

$$n^{-\frac{1}{\alpha}} S_n \xrightarrow{w} Z_\alpha,$$

where Z_α is an α-stable random variable with $1 < \alpha \le 2$. Then consider the strongly dependent sequence

$$\xi(S_1),\ \xi(S_2), \cdots,$$

which is a random walk in random scenery, and its partial sum

$$W_n = \sum_{j=1}^n \xi(S_j).$$

We defined W_s for $n < s < n+1$ by the linear interpolation

$$W_s = W_n + (s-n)(W_{n+1} - W_n).$$

Kesten and Spitzer [KS79] studied the case when $d = 1$, Z_β is a β-stable random variable and $\{\xi(k), k \in \mathbf{Z}\}$ satisfies that

$$n^{-\frac{1}{\beta}} \sum_{k=1}^n \xi(k) \xrightarrow{w} Z_\beta.$$

They proved that $n^{-H} W_{nt}$, where $H = 1 - 1/\alpha + 1/\alpha\beta$, converges to a selfsimilar process, which is expressed by an integral of the local time with respect to a stable measure. In this paper, we consider the same problem when Z_β is a β-semi-stable random variable. Then as will be seen, we have a new class of semi-selfsimilar processes as limiting processes of some random walks in random scenery.

To describe our theorem, we need more notation. Let $\{Y_\alpha(t), t \geq 0\}$ be an α-stable Lévy process such that the distribution of $Y_\alpha(1)$ is the same as that of Z_α. Here we mean, by Lévy processes, processes which have independent and stationary increments, start at the origin, and whose sample paths are right continuous and have left limits. Since $1 < \alpha \leq 2$, $L_t(x)$, the local time of $\{Y_\alpha(t)\}$ at x, exists, and we can take a version of $L_t(x)$ (denoted by $L_t(x)$ again) which is continuous in (t, x). Let $\{Z_\beta(t), t \in \mathbf{R}\}$ be another Lévy process independent of $\{Y_\alpha(t), t \geq 0\}$ such that the distribution of $Z_\beta(1)$ is the same as that of Z_β. Then, we can define a stochastic integral

$$\Delta(t) = \int_{-\infty}^{\infty} L_t(x) dZ_\beta(x) . \tag{1.6}$$

Our main theorem is the following.

THEOREM 1.1. *Let*

$$H = 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}$$

and

$$D_n(t) = r^{\alpha n H} W_{r^{-\alpha n} t} , \quad t \geq 0 , \quad n = 1, 2, \cdots .$$

Then $\{D_n(t), t \geq 0\}$ converges weakly in $C([0, \infty); \mathbf{R}^d)$ to the process $\{\Delta(t), t \geq 0\}$ defined in (1.6).

REMARK 1.1. By Theorem 3.1 of [MS99], the limiting process $\{\Delta(t), t \geq 0\}$ is semi-selfsimilar and its exponent is H.

We prove the theorem by showing the following two propositions separately.

PROPOSITION 1.2.

$$\{D_n(t), t \geq 0\} \overset{d}{\Rightarrow} \{\Delta(t), t \geq 0\} \quad \text{as } n \to \infty , \tag{1.7}$$

where $\overset{d}{\Rightarrow}$ denotes convergence of all finite dimensional distributions.

PROPOSITION 1.3. *The family $\{D_n(t), t \geq 0\}, n = 1, 2, \cdots$, is tight in $C([0, \infty)$; $\mathbf{R}^d)$.*

Proposition 1.2 will be proved in Sections 2 and 3, and the proof of Proposition 1.3 will be given in Section 4.

2. Proof of Proposition 1.2.

We start with the relation

$$W_n = \sum_{j=1}^{n} \xi(S_j) = \sum_{u \in \mathbf{Z}} N_n(u) \xi(u) , \tag{2.1}$$

where $N_n(u)$ is the number of visits of random walk $\{S_n\}$ to the point u in the time interval $[0, n]$. All that are needed for us about the occupation time $N_n(u)$ of $\{S_n\}$ and the local time $L_t(x)$ can be found in [KS79]. We state some of them as lemmas below.

For $n < s < n+1$ and all $u \in \mathbf{Z}$, define

$$N_s(u) = N_n(u) + (s-n)(N_{n+1}(u) - N_n(u)).$$

For $-\infty < a < b < \infty$, define

$$T_t^n(a,b) = \frac{1}{n} \sum_{a \le n^{-\frac{1}{\alpha}} u < b} N_{nt}(u)$$

and

$$\Gamma_t(a,b) = \int_a^b L_t(u)du.$$

LEMMA 2.1 ([KS79]). *For any $k \geq 1$ and $t_1, t_2, \cdots, t_k > 0$,*

$$\{T_{t_j}^n(a_j, b_j), 1 \le j \le k\} \xrightarrow{w} \{\Gamma_{t_j}(a_j, b_j), 1 \le j \le k\}.$$

LEMMA 2.2 ([KS79]). *For any $p > 1$,*

$$\sup_{u \in \mathbf{Z}} E[N_s(u)^p] = O\left(s^{p\left(1-\frac{1}{\alpha}\right)}\right)$$

and

$$P\left\{N_s(u) > 0 \text{ for some } u \text{ with } |u| > R(s+1)^{\frac{1}{\alpha}}\right\} \le \varepsilon(R) \quad \text{for any } s > 0,$$

where $\varepsilon(R) \to 0$ as $R \to \infty$ and $\varepsilon(R)$ is independent of s.

LEMMA 2.3 ([KS79]).

$$\lim_{s \to \infty} \sup_{u \in \mathbf{Z}} N_s(u) s^{-H} = 0 \quad \text{in probability}.$$

We have, with the replacement of s by $r^{-\alpha n}$ in Lemma 2.3,

$$\lim_{n \to \infty} \sup_{u \in \mathbf{Z}} N_{r^{-\alpha n}}(u) r^{\alpha n H} = 0 \quad \text{in probability}. \tag{2.2}$$

We need more lemmas.

LEMMA 2.4 (The joint distributions of $\Delta(t)$). *For any $k \geq 1$, $t_1, \cdots, t_k > 0$ and $\theta_1, \cdots, \theta_k \in \mathbf{R}^d$,*

$$E\left[\exp\left\{i\sum_{j=1}^k \theta_j \Delta(t_j)\right\}\right] = E\left[\exp\left\{\int_{-\infty}^{\infty} f\left(\sum_{j=1}^k L_{t_j}(u)\theta_j\right) du\right\}\right],$$

where $f = \log \varphi$ and φ is the characteristic function of Z_β.

Since $\{Z_\beta(t)\}$ is a Lévy process, the proof of Lemma 2.4 can be carried out exactly in the same way as in Lemma 5 in [KS79]. We thus omit it.

LEMMA 2.5. *For any* $k \geq 1$, $t_1, \cdots, t_k > 0$ *and* $\theta_1, \cdots, \theta_k \in \mathbf{R}^d$,

$$\sum_{u \in \mathbf{Z}} f\left(r^{\alpha n H} \sum_{j=1}^{k} N_{r^{-\alpha n} t_j}(u)\theta_j\right) \xrightarrow{w} \int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u)\theta_j\right) du .$$

Denote the characteristic function of $\xi(u)$ by

$$\lambda(\theta) = E[e^{i\langle\theta, \xi(u)\rangle}], \quad \theta \in \mathbf{R}^d . \tag{2.3}$$

Then we have the following.

LEMMA 2.6. $\log \lambda(\theta) \sim \log \varphi(\theta)$ as $\theta \to 0$.

We postpone the proofs of Lemmas 2.5 and 2.6 to the next section and proceed to the proof of Proposition 1.2. We have, by (2.1)–(2.3) and Lemmas 2.4–2.6,

$$\begin{aligned} I_n := & E\left[\exp\left\{i \sum_{j=1}^{k} \langle \theta_j, r^{\alpha n H} W_{r^{-\alpha n} t_j} \rangle\right\}\right] \\ = & E\left[\exp\left\{i \sum_{j=1}^{k} \left\langle \theta_j, r^{\alpha n H} \sum_{u \in \mathbf{Z}} N_{r^{-\alpha n} t_j}(u)\xi(u)\right\rangle\right\}\right] \\ = & E\left[\sum_{u \in \mathbf{Z}} \lambda\left(r^{\alpha n H} \sum_{j=1}^{k} N_{r^{-\alpha n} t_j}(u)\theta_j\right)\right] \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \lim_{n\to\infty} I_n = & \lim_{n\to\infty} E\left[\prod_{u \in \mathbf{Z}} \varphi\left(r^{\alpha n H} \sum_{j=1}^{k} N_{r^{-\alpha n} t_j}(u)\theta_j\right)\right] \\ = & \lim_{n\to\infty} E\left[\exp\left\{\sum_{u \in \mathbf{Z}} f\left(r^{\alpha n H} \sum_{j=1}^{k} N_{r^{-\alpha n} t_j}(u)\theta_j\right)\right\}\right] \\ = & E\left[\exp\left\{\int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u)\theta_j\right) du\right\}\right] \\ = & E\left[\exp\left\{i \sum_{j=1}^{k} \langle \theta_j, \Delta(t_j)\rangle\right\}\right] . \end{aligned}$$

This completes the proof of Proposition 1.2.

3. Proofs of Lemmas 2.5 and 2.6.

In the following, C will denote an absolute positive constant, which may differ from one expression to the other. We need a lemma for proving Lemma 2.5.

LEMMA 3.1. *Let*

$$\begin{cases} \zeta = 1 & \text{if } 1 < \beta < 2 , \\ 0 < \zeta < \beta & \text{if } \beta \leq 1 . \end{cases}$$

Then, for any $\theta_1, \theta_2 \in \mathbf{R}^d$,

$$|f(\theta_1) - f(\theta_2)| \leq C\{\|\theta_1 - \theta_2\|(1 + \|\theta_1\| + \|\theta_2\|) + \|\theta_1 - \theta_2\|^\zeta\}.$$

PROOF. By (1.4) and Lemma 4 of [M96], we have

$$\begin{aligned}
|f(\theta_1) - f(\theta_2)| = \Bigg| & i\langle\theta_1 - \theta_2, c\rangle \\
& + \int_S \gamma(dx) \int_0^1 [e^{i\langle\theta_1, sx\rangle} - e^{i\langle\theta_2, sx\rangle} - i\langle\theta_1 - \theta_2, sx\rangle] d\left(-\frac{H_x(s)}{s^\beta}\right) \\
& + \int_S \gamma(dx) \int_1^\infty [e^{i\langle\theta_1, sx\rangle} - e^{i\langle\theta_2, sx\rangle}] d\left(-\frac{H_x(s)}{s^\beta}\right) \Bigg| \\
\leq\ & c\|\theta_1 - \theta_2\| + 2\|\theta_1 - \theta_2\|(\|\theta_1\| + \|\theta_2\|) \int_S \gamma(dx) \int_0^1 s^2 d\left(-\frac{H_x(s)}{s^\beta}\right) \\
& + 2^{\frac{1-\zeta}{\zeta}} \|\theta_1 - \theta_2\|^\zeta \int_S \gamma(dx) \int_1^\infty s^\zeta d\left(-\frac{H_x(s)}{s^\beta}\right).
\end{aligned}$$

In order to prove Lemma 3.1, it is enough to show that the following.

(i) $\displaystyle\int_S \gamma(dx) \int_0^1 s^2 d\left(-\frac{H_x(s)}{s^\beta}\right) < \infty$,

(ii) $\displaystyle\int_S \gamma(dx) \int_1^\infty s^\zeta d\left(-\frac{H_x(s)}{s^\beta}\right) < \infty$.

Firstly, we prove (i). We put $b = r^{1/\beta}$. Since $H_x(s)/s^\beta$ is nonincreasing, we have

$$\begin{aligned}
\int_S \gamma(dx) \int_0^1 s^2 d\left(-\frac{H_x(s)}{s^\beta}\right) &= \int_S \gamma(dx) \sum_{k=0}^\infty \int_{b^{k+1}}^{b^k} s^2 d\left(-\frac{H_x(s)}{s^\beta}\right) \\
&\leq \int_S \gamma(dx) \sum_{k=0}^\infty b^{2k} \int_{b^{k+1}}^{b^k} d\left(-\frac{H_x(s)}{s^\beta}\right) \\
&= \int_S \gamma(dx) \sum_{k=0}^\infty b^{2k} \left(-\frac{H_x(b^k)}{b^{k\beta}} + \frac{H_x(b^{k+1})}{b^{(k+1)\beta}}\right) \\
&= \sum_{k=0}^\infty b^{2k} \frac{1}{b^{k\beta}} \left(\frac{1}{b^\beta} - 1\right) \gamma(S) \\
&< \infty.
\end{aligned}$$

We next prove (ii) in the same way as in the proof of (i). We have

$$\begin{aligned}
\int_S \gamma(dx) \int_1^\infty s^\zeta d\left(-\frac{H_x(s)}{s^\beta}\right) &\leq \int_S \gamma(dx) \sum_{k=1}^\infty b^{-k\zeta} \int_{b^{-k+1}}^{b^{-k}} d\left(-\frac{H_x(s)}{s^\beta}\right) \\
&= \sum_{k=1}^\infty b^{-k\zeta} \frac{1}{b^{-k\beta}} \left(\frac{1}{b^\beta} - 1\right) \gamma(S) \\
&< \infty.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

PROOF OF LEMMA 2.5. Recall that

$$\varphi(\theta)^r = \varphi\big(r^{\frac{1}{\beta}}\theta\big).$$

Thus, we have by iteration

$$\varphi(\theta)^{r^n} = \varphi\big(r^{\frac{n}{\beta}}\theta\big).$$

We have

$$\sum_{u\in\mathbf{Z}} f\left(r^{\alpha n H}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u)\theta_j\right) = \sum_{u\in\mathbf{Z}} \log\varphi\left(r^{\alpha n\left(1-\frac{1}{\alpha}\right)}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u) r^{\frac{n}{\beta}}\theta_j\right)$$

$$= \sum_{u\in\mathbf{Z}} r^n \log\varphi\left(r^{\alpha n\left(1-\frac{1}{\alpha}\right)}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u)\theta_j\right).$$

Hence it is enough to prove that

$$\sum_{u\in\mathbf{Z}} r^n f\left(r^{n(\alpha-1)}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u)\theta_j\right) \overset{w}{\to} \int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u)\right) du, \tag{3.1}$$

in order to prove Lemma 2.5.

For some small $\tau > 0$ and large M, define

$$A_{n,l} = \{u\in\mathbf{Z} : l\tau r^{-n} \le u < (l+1)\tau r^{-n}\}, \quad l\in\mathbf{Z},$$

$$U(\tau, M, n) = \sum_{|u|>M\tau r^{-n}} r^n f\left(r^{n(\alpha-1)}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u)\theta_j\right)$$

and

$$V(\tau, M, n) = \sum_{-M\le l<M} |A_{n,l}| r^n f\left(r^{n(\alpha-1)}\frac{1}{\tau r^{-n}}\sum_{y\in A_{n,l}}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(y)\theta_j\right),$$

where $|A_{n,l}|$ is the number of integers in $A_{n,l}$. Then

$$I := \sum_{u\in\mathbf{Z}} r^n f\left(r^{n(\alpha-1)}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u)\theta_j\right) - U(\tau, M, n) - V(\tau, M, n)$$

$$= \sum_{-M\le l<M}\sum_{u\in A_{n,l}} r^n \left\{ f\left(r^{n(\alpha-1)}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u)\theta_j\right) - f\left(r^{n(\alpha-1)}\frac{1}{\tau r^{-n}}\sum_{y\in A_{n,l}}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(y)\theta_j\right)\right\}.$$

Let us denote

$$g_j := N_{r^{-\alpha n}t_j}(u) \quad \text{and} \quad h_j := \frac{1}{\tau r^{-n}} \max_{-M\le l<M} \sum_{y\in A_{n,l}} N_{r^{-\alpha n}t_j}(y).$$

By Lemma 3.1, Hölder's inequality and Minkowski's inequality,

$$
\begin{aligned}
E[|I|] \leq C \cdot 2M \max_{-M\leq l<M} |A_{n,l}| r^n \max_{-M\leq l<M} \max_{u\in A_{n,l}} \Bigg\{ & E\left[r^{n(\alpha-1)} \left\| \sum_{j=1}^{k} (g_j - h_j)\theta_j \right\| \right. \\
& \left. \left(1 + r^{n(\alpha-1)} \left\| \sum_{j=1}^{k} g_j\theta_j \right\| + r^{n(\alpha-1)} \left\| \sum_{j=1}^{k} h_j\theta_j \right\| \right) \right] \\
& + E\left[r^{n\zeta(\alpha-1)} \left\| \sum_{j-1}^{k} (g_j - h_j)\theta_j \right\|^{\zeta} \right] \Bigg\} \\
\leq CM\tau \max_{-M\leq l<M} \max_{u\in A_{n,l}} \Bigg\{ & r^{n(\alpha-1)} E\left[\left\| \sum_{j=1}^{k} (g_j - h_j)\theta_j \right\|^2 \right]^{1/2} \\
& E\left[\left(1 + r^{n(\alpha-1)} \left\| \sum_{j=1}^{k} g_j\theta_j \right\| + r^{n(\alpha-1)} \left\| \sum_{j=1}^{k} h_j\theta_j \right\| \right)^2 \right]^{1/2} \\
& + r^{n\zeta(\alpha-1)} \left(E\left[\left\| \sum_{j=1}^{k} (g_j - h_j)\theta_j \right\|^2 \right] \right)^{\zeta/2} \Bigg\} \\
\leq CM\tau \max_{-M\leq l<M} \max_{u\in A_{n,l}} \Bigg\{ & r^{n(\alpha-1)} \sqrt{ E\left[\left\| \sum_{j=1}^{k} (g_j - h_j) \right\|^2 \right] \sum_{j=1}^{k} \|\theta_j\|^2 } \\
& \left(1 + r^{n(\alpha-1)} \sqrt{ \sum_{j=1}^{k} \|\theta_j\|^2 } \left(\sqrt{ E\left[\left\| \sum_{j=1}^{k} g_j \right\|^2 \right] } + \sqrt{ E\left[\left\| \sum_{j=1}^{k} h_j \right\|^2 \right] } \right) \right) \\
& + r^{n\zeta(\alpha-1)} \left(E\left[\left\| \sum_{j=1}^{k} (g_j - h_j) \right\|^2 \right] \sum_{j=1}^{k} \|\theta_j\|^2 \right)^{\zeta/2} \Bigg\}.
\end{aligned} \tag{3.2}
$$

On the other hand, we have, by Lemmas 2 and 3 of [KS79],

$$\max_{-M\leq l<M} \max_{u\in A_{n,l}} E|g_j - h_j|^2 \leq C\tau^{\alpha-1} r^{-2n(\alpha-1)}.$$

Moreover, if we let $p = 2$, $s = r^{-\alpha n}$ in the first assertion of Lemma 2.2, then we have

$$\sup_{u\in\mathbf{Z}} E[N_{r^{-\alpha n}}(u)^2] = O(r^{-2n(\alpha-1)}).$$

Hence,

$$\begin{aligned} E[|I|] \le CM\tau\big\{ & r^{n(\alpha-1)}\sqrt{C\tau^{\alpha-1}r^{-2n(\alpha-1)}} \\ & (1 + r^{n(\alpha-1)}\sqrt{r^{-2n(\alpha-1)}} + r^{n(\alpha-1)}\sqrt{r^{-2n(\alpha-1)}}) \\ & + r^{n\zeta(\alpha-1)}C\tau^{\frac{\zeta}{2}(\alpha-1)}r^{-2n(\alpha-1)\frac{\zeta}{2}}\big\} \\ = CM\tau\big(& \tau^{\frac{\alpha-1}{2}} + \tau^{\frac{\zeta(\alpha-1)}{2}}\big). \end{aligned}$$

If we replace s by $r^{-\alpha n}t$ in the second assertion of Lemma 2.2, then we obtain the following estimate:

$$P\big\{N_{r^{-\alpha n}t}(u) > 0 \text{ for some } u \text{ with } |u| > R(r^{-\alpha n}t+1)^{\frac{1}{\alpha}}\big\} \le \varepsilon(R) \quad \text{for any } n \ge 1, \quad t > 0, \tag{3.3}$$

where $\varepsilon(R) \to 0$ as $R \to \infty$, $\varepsilon(R)$ is independent of n and $t > 0$. We thus see that for any $\eta > 0$, we can choose $C_1 > 0$ such that for any $M > 0$ and $\tau > 0$ satisfying $M\tau > C_1$,

$$P\{U(\tau, M, n) \ne 0\} \le \eta .$$

Recall $\alpha > 1$. If we choose τ above so small (and thus M so large) that

$$CM\tau\big(\tau^{\frac{\alpha-1}{2}} + \tau^{\frac{\zeta(\alpha-1)}{2}}\big) \le \eta^2 ,$$

then we have

$$P\left\{\left|\sum_{u\in\mathbf{Z}} f\left(r^{\alpha n H}\sum_{j=1}^{k} N_{r^{-\alpha n}t_j}(u)\theta_j\right) - V(\tau, M, n)\right| > \eta\right\} \le 2\eta . \tag{3.4}$$

Hence, all we have to do in order to prove Lemma 2.5 is to show the convergence of $V(\tau, M, n)$.

Note that Lemma 2.1 remains true, even if we replace $T_t^n(a, b)$ by

$$\tilde{T}_t^n(a, b) = r^{\alpha n} \sum_{ar^{-n} \le u < br^{-n}} N_{r^{-\alpha n}t}(u) .$$

Thus, we have

$$V(\tau, M, n) = \sum_{-M\le l<M} \frac{|A_{n,l}|}{r^{-n}} f\left(\frac{1}{\tau}\sum_{j=1}^{k} \tilde{T}_{t_j}^n(l\tau, (l+1)\tau)\theta_j\right) ,$$

which, as $n \to \infty$, converges weakly to

$$\tau \sum_{-M\le l<M} f\left(\sum_{j=1}^{k} \frac{1}{\tau}\int_{l\tau}^{(l+1)\tau} L_{t_j}(y)dy\theta_j\right) , \tag{3.5}$$

where we have used $|A_{n,l}|/r^{-n} \to \tau$.

Finally, it follows from the continuity and the compact support property of L that (3.5) converges to

$$\int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u)\theta_j\right) du$$

as $\tau \to 0$ and $M \to \infty$. This together with (3.4) shows (3.1) and completes the proof of Lemma 2.5. □

PROOF OF LEMMA 2.6. This is an extension of Lemma 6.1 of [MM94] to the semi-stable case. The idea of the proof is found in [MM94].

By (1.4), we have

$$\lim_{n\to\infty} \lambda\big(r^{\frac{n}{\beta}}\theta\big)^{[r^{-n}]} = \varphi(\theta) \quad \text{uniformly on any compact set of } \theta \,. \tag{3.6}$$

We first prove

$$\lim_{n\to\infty} \lambda\big(r^{\frac{n}{\beta}}\theta\big)^{r^{-n}} = \varphi(\theta) \quad \text{uniformly for } \theta \text{ with } r^{\frac{1}{\beta}} \le \|\theta\| \le 1 \,. \tag{3.7}$$

For each $n > 0$, let $\varepsilon_n = r^{-n} - [r^{-n}]$. Then

$$\lambda\big(r^{\frac{n}{\beta}}\theta\big)^{r^{-n}} = \lambda\big(r^{\frac{n}{\beta}}\theta\big)^{[r^{-n}]}\lambda\big(r^{\frac{n}{\beta}}\theta\big)^{\varepsilon_n} .$$

Hence, it is enough to show

$$\lim_{n\to\infty} \lambda\big(r^{\frac{n}{\beta}}\theta\big)^{\varepsilon_n} = 1 \quad \text{uniformly for } \theta \text{ with } r^{\frac{1}{\beta}} \le \|\theta\| \le 1 \tag{3.8}$$

in order to show (3.7) by (3.6). For any $\varepsilon > 0$, there exists an $N > 0$ such that if $|z| < \delta := |r^{N/\beta}|$ then $|\lambda(z) - 1| < \varepsilon$. For any $n > N$ and any $\theta \in \mathbf{R}^d$ with $r^{1/\beta} \le \|\theta\| \le 1$,

$$\big\|r^{\frac{n}{\beta}}\theta\big\| \le \big|r^{\frac{n}{\beta}}\big| \le \big|r^{\frac{N}{\beta}}\big| = \delta$$

so that

$$\lambda\big(r^{\frac{n}{\beta}}\theta\big) = 1 + \xi \,, \quad |\xi| < \varepsilon \,.$$

Thus, for any $\varepsilon > 0$, there exists an $N > 0$ such that for any $n > N$ and $\theta \in \mathbf{R}^d$,

$$\lambda\big(r^{\frac{n}{\beta}}\theta\big)^{\varepsilon_n} = (1 + \xi)^{\varepsilon_n} = 1 + \tilde{\xi} \,, \quad |\tilde{\xi}| < \text{const.}\varepsilon \,.$$

Therefore, (3.8) holds, and thus, we obtain (3.7).

By (3.7),

$$\lim_{n\to\infty} \frac{\log \lambda\big(r^{\frac{n}{\beta}}\theta\big)}{r^n \log \varphi(\theta)} = 1 \quad \text{uniformly for } \theta \text{ with } r^{\frac{1}{\beta}} \le \|\theta\| \le 1 \,.$$

Moreover by (1.2),

$$\lim_{n\to\infty} \frac{\log \lambda\big(r^{\frac{n}{\beta}}\theta\big)}{\log \varphi\big(r^{\frac{n}{\beta}}\theta\big)} = 1 \quad \text{uniformly for } \theta \text{ with } r^{\frac{1}{\beta}} \le \|\theta\| \le 1 \,.$$

So that, for all $n > N$, we have

$$\left| \frac{\log \lambda\big(r^{\frac{n}{\beta}}\theta\big)}{\log \varphi\big(r^{\frac{n}{\beta}}\theta\big)} - 1 \right| < \varepsilon \,.$$

Furthermore, let $\delta = |r^{N/\beta}|$ (> 0) then, for any θ satisfying $\|\theta\| < \delta$ (< 1), there exist a θ_0 with $r^{1/\beta} \le \|\theta_0\| \le 1$ and an $n_0 > N$ such that $\theta = r^{n_0/\beta}\theta_0$. Hence, for θ satisfying $\|\theta\| < \delta$, we have

$$\left|\frac{\log\lambda(\theta)}{\log\varphi(\theta)} - 1\right| = \left|\frac{\log\lambda\big(r^{\frac{n_0}{\beta}}\theta\big)}{\log\varphi\big(r^{\frac{n_0}{\beta}}\theta\big)} - 1\right| < \varepsilon,$$

which completes the proof of Lemma 2.6. □

4. Proof of Proposition 1.3.

To show the tightness in $C([0,\infty);\mathbf{R}^d)$, it is enough to show it in $C([0,T];\mathbf{R}^d)$ for every $T > 0$ (see Theorem 6 of [W70]). We are going to show that, for every $T > 0$, $\{D_n(t), 0 \le t \le T\}$, $n = 1, 2, \cdots$, is tight in $C([0,T];\mathbf{R}^d)$. Firstly, we shall prove three lemmas as follows.

LEMMA 4.1. *For any $\varepsilon > 0$, there exists an $A(\varepsilon)$ such that*

$$P(N_{r^{-\alpha n}t}(u) > 0 \quad \textit{for some } |u| > A(\varepsilon)r^{-n} \textit{ and } t \le T) \le \frac{\varepsilon}{4},$$

for all $n \ge 1$.

PROOF. Consider $\varepsilon(R)$ in (3.3). For any $\varepsilon > 0$, there exists an $A(\varepsilon)$ such that $\varepsilon(A(\varepsilon)(T+1)^{-1/\alpha}) < \varepsilon/4$. Thus, by (3.3) with $R = A(\varepsilon)(T+1)^{-1/\alpha}$ and $t = T$,

$$\begin{aligned}
&P(N_{r^{-\alpha n}t}(u) > 0 \text{ for some } |u| > A(\varepsilon)r^{-n} \text{ and } t \le T)\\
&= P(N_{r^{-\alpha n}T}(u) > 0 \text{ for some } |u| > A(\varepsilon)r^{-n})\\
&= P(N_{r^{-\alpha n}T}(u) > 0 \text{ for some } |u| > A(\varepsilon)(T+1)^{-\frac{1}{\alpha}}r^{-n}(T+1)^{\frac{1}{\alpha}})\\
&\le P(N_{r^{-\alpha n}T}(u) > 0 \text{ for some } |u| > A(\varepsilon)(T+1)^{-\frac{1}{\alpha}}(r^{-\alpha n}T+1)^{\frac{1}{\alpha}})\\
&\le \varepsilon\big(A(\varepsilon)(T+1)^{-\frac{1}{\alpha}}\big)\\
&\le \frac{\varepsilon}{4}.
\end{aligned}$$

This completes the proof of Lemma 4.1. □

LEMMA 4.2. *For any $\varepsilon > 0$, there exists a $\rho(\varepsilon)$ such that*

$$(2A(\varepsilon)+1)r^{-n}\big\{1 - P\big(|\xi(0)| \le \rho(\varepsilon)r^{-\frac{n}{\beta}}\big)\big\} \le \frac{\varepsilon}{4}, \tag{4.1}$$

for all $n \in \mathbf{Z}$.

PROOF. For each $y > 0$, we define

$$H(y) := \int_S H_x(y)\gamma(dx). \tag{4.2}$$

By Theorem 4 of Section 25 of [GK54], for all $x > 0$, we have

$$r^{-n}P\big(|\xi(0)| > r^{-\frac{n}{\beta}}x\big) \to H(x)x^{-\beta} \quad \text{as } n \to \infty. \tag{4.3}$$

Since $\lim_{x\to\infty} H(x)x^{-\beta} = 0$, we can choose a large x_0 such that $H(x_0)x_0^{-\beta} \le \varepsilon/(16A(\varepsilon)+8)$. By (4.3), there exists an n_0 such that, for all $n \ge n_0$,

$$\max_{n\ge n_0} r^{-n} P\big(|\xi(0)| > r^{-\frac{n}{\beta}} x_0\big) \le \frac{\varepsilon}{16A(\varepsilon)+8} + H(x_0)x_0^{-\beta}.$$

Namely, we have

$$\max_{n\ge n_0} r^{-n} P\big(|\xi(0)| > r^{-\frac{n}{\beta}} x_0\big) \le \frac{\varepsilon}{8A(\varepsilon)+4}.$$

Note also that there exists an $n_1 < 0$ such that

$$\max_{n\le n_1} r^{-n} P\big(|\xi(0)| > r^{-\frac{n}{\beta}} x_0\big) \le \frac{\varepsilon}{8A(\varepsilon)+4}.$$

Hence, there exists a $\rho(\varepsilon)$ such that

$$\max_{n\in\mathbf{Z}} r^{-n} P\big(|\xi(0)| > r^{-\frac{n}{\beta}} \rho(\varepsilon)\big) \le \frac{\varepsilon}{8A(\varepsilon)+4},$$

and thus (4.1) holds. □

LEMMA 4.3. *There exists a* $C_2 > 0$ *such that*

$$P(|\xi(0)| > y) \le C_2 H(y) y^{-\beta},$$

for all $y > 0$.

PROOF. For each $n \in \mathbf{Z}$ and all $x \in (\rho(1)r^{1/\beta}, \rho(1)]$,

$$\begin{aligned}\frac{r^{-n}P(|\xi(0)| > r^{-n/\beta}x)}{H(x)x^{-\beta}} &\le \frac{r^{-n}P(|\xi(0)| > r^{-\frac{n}{\beta}}\rho(1)r^{\frac{1}{\beta}})}{H(\rho(1))\rho(1)^{-\beta}} \\ &= \frac{r^{-(n-1)}P(|\xi(0)| > \rho(1)r^{-\frac{1}{\beta}(n-1)})}{H(\rho(1))\rho(1)^{-\beta}r} \\ &\le \frac{1}{(8A(1)+4)}\frac{1}{H(\rho(1))\rho(1)^{-\beta}r}. \qquad (4.4)\end{aligned}$$

If we let $y = r^{-n/\beta}x$, then the range of y is $(0, \infty)$. For all $y > 0$, we have, by (4.4),

$$\frac{P(|\xi(0)| > y)}{H(y)y^{-\beta}} \le C_2,$$

where we have used $H(r^{n/\beta}y) = H(y)$, which follows from the assumption that $H_x(r^{1/\beta}s) = H_x(s)$ and (4.2). □

Next we introduce the following notation:

$$\begin{aligned}\xi_n^\varepsilon(u) &:= \xi(u)I[|\xi(u)| \le \rho(\varepsilon)r^{-\frac{n}{\beta}}], \\ E_n^\varepsilon(t); &= r^{\alpha n H}\sum_{u\in\mathbf{Z}} N_{r^{-\alpha n}t}(u)E[\xi_n^\varepsilon(u)], \\ D_n^\varepsilon(t) &:= r^{\alpha n H}\sum_{u\in\mathbf{Z}} N_{r^{-\alpha n}t}(u)\{\xi_n^\varepsilon(u) - E[\xi_n^\varepsilon(u)]\}.\end{aligned}$$

We shall prove four more lemmas.

LEMMA 4.4. *For any $\varepsilon > 0$, there exists a $C_3(\varepsilon) > 0$ such that*

$$E[|D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)|^2] \le C_3(\varepsilon)|t_2 - t_1|^{2-\frac{1}{\alpha}}, \tag{4.5}$$

for all $0 \le t_1 < t_2 \le T$ and $n \ge 1$.

PROOF. By the definition of $D_n^\varepsilon(t)$ and independence of $\{\xi(u), u \in \mathbf{Z}\}$,

$$\begin{aligned} E[|D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)|^2] &= r^{2\alpha n H} E\left[\sum_{u\in\mathbf{Z}}\{(N_{r^{-\alpha n}t_2}(u) - N_{r^{-\alpha n}t_1}(u))(\xi_n^\varepsilon(u) - E[\xi_n^\varepsilon(u)])\}^2\right] \\ &\le r^{2\alpha n H} E[|\xi_n^\varepsilon(0)|^2] E\left[\sum_{u\in\mathbf{Z}}(N_{r^{-\alpha n}t_2}(u) - N_{r^{-\alpha n}t_1}(u))^2\right]. \end{aligned} \tag{4.6}$$

By [KS79],

$$\begin{aligned} E\left[\sum_{u\in\mathbf{Z}}(N_{r^{-\alpha n}t_2}(u) - N_{r^{-\alpha n}t_1}(u))^2\right] &\le C_4(r^{-\alpha n}(t_2 - t_1))^{2-\frac{1}{\alpha}} \\ &= C_4 r^{-2\alpha n+n}(t_2 - t_1)^{2-\frac{1}{\alpha}}. \end{aligned} \tag{4.7}$$

Moreover,

$$\begin{aligned} E[|\xi_n^\varepsilon(0)|^2] &= E[\xi^2(0); |\xi(0)| < \rho(\varepsilon)r^{-\frac{n}{\beta}}] \\ &\le \int_0^{\rho(\varepsilon)^2 r^{-2n/\beta}} P(|\xi(0)|^2 > x)dx \\ &= \int_0^{\rho(\varepsilon)r^{-n/\beta}} P(|\xi(0)| > y)2y dy \\ &\le \int_0^{\rho(\varepsilon)r^{-n/\beta}} C_2 H(y) y^{-\beta} 2y dy \\ &= 2C_2 \sum_{k=-\infty}^{n-1} \int_{\rho(\varepsilon)r^{-k/\beta}}^{\rho(\varepsilon)r^{-(k+1)/\beta}} H(y) y^{-\beta} y dy \\ &\le C_5(\varepsilon) r^{n(1-2/\beta)}. \end{aligned} \tag{4.8}$$

The estimates (4.6)–(4.8) give us (4.5) with $C_3(\varepsilon) = C_4 C_5(\varepsilon)$. This completes the proof. □

LEMMA 4.5. *For any $\varepsilon > 0$ and $\eta > 0$, we have*

$$\lim_{\delta\to 0} \sup_{n\ge 1} P\left(\sup_{\substack{0\le t_1, t_2\le T \\ |t_2 - t_1|\le\delta}} |D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)| \ge \frac{\eta}{4}\right) = 0.$$

We need a lemma for proving Lemma 4.5.

LEMMA 4.6. *Let $\{X_n(t), 0 \le t \le T\}$ be a sequence of stochastic processes whose sample paths are in $C([0,T];\mathbf{R}^d)$. If there exist $\gamma \ge 0$, $p > 1$ and a nondecreasing, continuous function F on $[0,T]$ such that*

$$P\{|X_n(t_2) - X_n(t_1)| \ge \lambda\} \le \frac{1}{\lambda^\gamma}|F(t_2) - F(t_1)|^p$$

holds for all $0 \le t_1 < t_2 \le T$, $n \ge 1$ and $\lambda > 0$, then for any $\varepsilon_1 > 0$, there exists a $K > 0$ such that

$$\sum_{j<\delta^{-1}T} P\left\{\sup_{j\delta \le s \le (j+1)\delta} |X_n(s) - X_n(j\delta)| \ge \varepsilon_1\right\}$$
$$\le \frac{K}{\varepsilon_1^\gamma}[F(T) - F(0)]\left[\max_{j<\delta^{-1}T}|F((j+1)\delta) - F(j\delta)|\right]^{p-1},$$

if $\delta^{-1}T$ is integer.

PROOF. See the proof of Theorem 12.3 of Billingsley [B68]. □

PROOF OF LEMMA 4.5. By Chebychev's inequality and Lemma 4.4, for $\eta > 0$ we have

$$P\left\{|D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)| \ge \frac{\eta}{4}\right\} \le \frac{16}{\eta^2}E[|D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)|^2]$$
$$\le \frac{16}{\eta^2}C_3(\varepsilon)|t_2 - t_1|^{2-\frac{1}{\alpha}}$$

for all $0 \le t_1 < t_2 \le T$ and $n \ge 1$. Thus $\{D_n^\varepsilon(t), 0 \le t \le T\}, n = 1, 2, \cdots$, satisfy the condition of Lemma 4.6 with $\gamma = 2$, $p = (2\alpha - 1)/\alpha$ and $F(t) = C_3(\varepsilon)^{\alpha/(2\alpha-1)}t$. Hence, by Lemma 4.6 and the Corollary to Theorem 8.3 of [B68], we have

$$P\left\{\sup_{\substack{0\le t_1,t_2\le T\\ |t_2-t_1|\le\delta}} |D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)| \ge \frac{\eta}{4}\right\} \le \sum_{j<\delta^{-1}T} P\left\{\sup_{j\delta\le s\le(j+1)\delta}|D_n^\varepsilon(s) - D_n^\varepsilon(j\delta)| \ge \frac{\eta}{12}\right\}$$
$$\le \frac{144}{\eta^2}KC_3(\varepsilon)T\delta^{1-\frac{1}{\alpha}},$$

if $\delta^{-1}T$ is integer and $\delta < 1$. Since $1 - 1/\alpha > 0$, we have

$$\lim_{\delta\to 0}\sup_{n\ge 1} P\left(\sup_{\substack{0\le t_1,t_2\le T\\ |t_2-t_1|\le\delta}} |D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)| \ge \frac{\eta}{4}\right) = 0,$$

which completes the proof of Lemma 4.5. □

LEMMA 4.7. *For any $\varepsilon > 0$, there exists a $C_6(\varepsilon) > 0$ such that*

$$|E_n^\varepsilon(t_2) - E_n^\varepsilon(t_1)| \le C_6(\varepsilon)|t_2 - t_1|,$$

for all $0 \le t_1 < t_2 \le T$ and $n \ge 1$.

PROOF. Note that

$$\begin{aligned}\sum_{u\in\mathbf{Z}} N_{r^{-\alpha n}t}(u)E[\xi_n^\varepsilon(u)] &= E[\xi_n^\varepsilon(0)]\sum_{u\in\mathbf{Z}} N_{r^{-\alpha n}t}(u)\\ &= E[\xi_n^\varepsilon(0)](r^{-\alpha n}t+1)\,.\end{aligned}$$

Thus in order to get Lemma 4.7, it is enough to prove that for $0<\beta<2$,

$$|E[\xi_n^\varepsilon(0)]| \le C_6(\varepsilon)r^{n-\frac{n}{\beta}} \quad \text{for some } C_6(\varepsilon)>0\,. \tag{4.9}$$

When $0<\beta<1$, we can show (4.9) by the same argument as for (4.8). When $\beta=1$, we can obtain $|E[\xi_n^\varepsilon(0)]| \le C_6(\varepsilon)$ by (1.5). When $1<\beta<2$, we first show $E[\xi(0)]=0$. To this end, observe, by (1.2),

$$q\varphi^{q-1}(\theta)\varphi'(\theta) = \varphi'(q^{\frac{1}{\beta}}\theta)q^{\frac{1}{\beta}}\,, \quad \theta\in\mathbf{R}^d\,,$$

where $q=1/r$. Thus, we have

$$(q^{\frac{1}{\beta}}-q)\varphi'(0)=0\,,$$

and hence $E[\xi(0)]=0$. This together with Lemma 4.3 yields

$$\begin{aligned}|E[\xi_n^\varepsilon(0)]| &= |E[\xi(0);\, |\xi(0)|<\rho(\varepsilon)r^{-n/\beta}]|\\ &= |E[\xi(0);\, |\xi(0)|\ge\rho(\varepsilon)r^{-n/\beta}]|\\ &\le \int_{\rho(\varepsilon)r^{-n}/\beta}^{\infty} P(|\xi(0)|>y)dy + \rho(\varepsilon)r^{-n/\beta}P(|\xi(0)|>\rho(\varepsilon)r^{-n/\beta})\\ &\le C_2\left\{\int_{\rho(\varepsilon)r^{-n/\beta}}^{\infty} H(y)y^{-\beta}dy + \rho(\varepsilon)r^{-n/\beta}H(\rho(\varepsilon)r^{-n/\beta})(\rho(\varepsilon)r^{-n/\beta})^{-\beta}\right\}\\ &\le C_2\left\{\sum_{k=0}^{\infty}\int_{\rho(\varepsilon)r^{-(n+k)/\beta}}^{\rho(\varepsilon)r^{-(n+k+1)/\beta}} H(\rho(\varepsilon)r^{-(n+k)/\beta})(\rho(\varepsilon)r^{-(n+k)/\beta})^{-\beta}dy\right.\\ &\qquad\qquad \left. + \rho(\varepsilon)^{1-\beta}r^{r(1-1/\beta)}H(\rho(\varepsilon))\right\}\\ &\le C_6(\varepsilon)r^{n(1-1/\beta)}\,.\end{aligned}$$

The proof of Lemma 4.7 is thus completed. □

By Lemma 4.7, we have

$$\lim_{\delta\to 0}\sup_{n\ge 1} P\left(\sup_{\substack{0\le t_1,t_2\le T\\ |t_2-t_1|\le\delta}} |E_n^\varepsilon(t_2)-E_n^\varepsilon(t_1)| \ge \frac{\eta}{4}\right) = 0\,. \tag{4.10}$$

LEMMA 4.8. *For any $\varepsilon>0$ and $\eta>0$, we have*

$$P\left(\sup_{t\le T}|D_n(t)-D_n^\varepsilon(t)-E_n^\varepsilon(t)| \ge \frac{\eta}{4}\right) \le \frac{\varepsilon}{2}$$

for all $n\ge 1$.

PROOF. By the definition of $D_n(t)$, $D_n^\varepsilon(t)$ and $E_n^\varepsilon(t)$,

$$D_n(t) - D_n^\varepsilon(t) - E_n^\varepsilon(t) = r^{-\alpha n H} \sum_{u\in\mathbf{Z}} N_{r^{-\alpha n}t}(u)(\xi(u) - \xi_n^\varepsilon(u)) .$$

Thus, by Lemmas 4.1 and 4.2, for all $n \geq 1$, we have

$$\begin{aligned} P&\left(\left|\sum_{u\in\mathbf{Z}} N_{r^{-\alpha n}t}(u)(\xi(u) - \xi_n^\varepsilon(u))\right| > 0 \text{ for some } t \leq T\right) \\ &\leq P\left(\left|\sum_{|u|\leq A(\varepsilon)r^{-n}} N_{r^{-\alpha n}t}(u)(\xi(u) - \xi_n^\varepsilon(u))\right| > 0 \text{ for some } t \leq T\right) \\ &\quad + P\left(\left|\sum_{|u|> A(\varepsilon)r^{-n}} N_{r^{-\alpha n}t}(u)(\xi(u) - \xi_n^\varepsilon(u))\right| > 0 \text{ for some } t \leq T\right) \\ &\leq P(\xi(u) \neq \xi_n^\varepsilon(u) \text{ for some } |u| \leq A(\varepsilon)r^{-n}) + \frac{\varepsilon}{4} \\ &\leq (2A(\varepsilon)+1)r^{-n} P(\xi(0) \neq \xi_n^\varepsilon(0)) + \frac{\varepsilon}{4} \\ &\leq \frac{\varepsilon}{2} . \end{aligned}$$

This completes the proof of Lemma 4.8. □

Finally, we have

$$\begin{aligned} (4.11) \quad |D_n(t_2) - D_n(t_1)| &\leq |D_n(t_2) - D_n^\varepsilon(t_2) - E_n^\varepsilon(t_2)| \\ &+ |D_n(t_1) - D_n^\varepsilon(t_1) - E_n^\varepsilon(t_1)| + |D_n^\varepsilon(t_2) - D_n^\varepsilon(t_1)| + |E_n^\varepsilon(t_2) - E_n^\varepsilon(t_1)| . \end{aligned}$$

Consequently, by (4.10), (4.11), Lemmas 4.5 and 4.8, we obtain

$$\lim_{\delta\to 0} \sup_{n\geq 1} P\left(\sup_{\substack{0\leq t_1,t_2\leq T \\ |t_2-t_1|\leq\delta}} |D_n(t_2) - D_n(t_1)| \geq \eta\right) = 0 .$$

Thus the family $\{D_n(t), 0 \leq t \leq T\}$, $n = 1, 2, \cdots$, is tight in $C([0,T]; \mathbf{R}^d)$ and hence in $C([0,\infty); \mathbf{R}^d)$. The proof of Proposition 1.3 is finished.

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Present address:
DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY,
HIYOSHI, KOHOKU-KU, YOKOHAMA, 223–8522 JAPAN.