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Inverse Scattering for the Nonlinear Schrödinger Equation with Cubic Convolution Nonlinearity

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Dedicated to Professor Kiyoshi Mochizuki on his sixtieth birthday

Abstract. In this paper it will be shown that a potential $V(x)$ and a constant λ are uniquely determined from the scattering operator S associated with the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + (-\Delta + V)u + \lambda(|x|^{-\sigma} * |u|^2)u = 0,$$

and the corresponding unperturbed equation

$$i\frac{\partial u}{\partial t} - \Delta u = 0.$$

1. Introduction.

Let $H = H_0 + V(x)$, where $H_0 = -\Delta = -\sum_{j=1}^{n} \partial^2/\partial x_j^2$, and $V(x)$ is a real valued function defined on $\mathbf{R}^n$ which satisfies conditions described below. Let $F(u)$ be the cubic convolution nonlinearity:

$$F(u) = \lambda(|x|^{-\sigma} * |u|^2)u = \lambda\left(\int_{\mathbf{R}^n} |x-y|^{-\sigma}|u(y)|^2 dy\right)u(x),$$

where $\lambda > 0$ is a constant.

In this paper we shall consider the inverse scattering problem for the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + Hu + F(u) = 0 \tag{1.1}$$

for $(x, t) \in \mathbf{R}^n \times \mathbf{R}$, $n \geq 3$.

The nonlinear inverse scattering problem has been studied in Weder [10] for the equation (1.1) in the case of power nonlinearity

$$F(u) = f(|u|)\frac{u}{|u|}$$

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with $f \in C^1(\mathbf{R})$ satisfies $f(0) = 0$ and

$$\left|\frac{df}{d\mu}\right| \le c|\mu|^{p-1}$$

for some $1 + 4/n \le p < 1 + 4/(n-2)$, $\mu \in \mathbf{R}$. In this case, if V satisfies the following conditions

1. $(1 + |x|^2)^{\alpha/2}V(x)$ is a bounded operator from $H^{\beta,2}$ into $H^{\beta,2}$($H^{r,p}$ is a usual Sobolev space of order r in L^p) for some $\alpha > n + 4$ and $\beta > 0$,
2. $\hat{V} \in L^1$, ($\hat{}$ denotes the Fourier transfrom),
3. $H = H_0 + V$ with the domain $H^{2,2}$ has no eigenvalues and that zero is not a resonance for H, (0 is said to be a resonance of H if there exists a solution of $Hu = 0$ such that $(1 + |x|^2)^{-\gamma}u(x) \in L^2(\mathbf{R}^n)$ for any $\gamma > 1/2$ but not for $\gamma = 0$.),

then there is a $\delta > 0$ such that for every $\phi_- \in \mathcal{H}_\delta := \{\phi \in H^{1,2} : \|\phi\|_{H^{1,2}} < \delta\}$, there exist a unique solution u of (1.1) such that

$$\|u(t) - e^{itH_0}\phi_-\|_{H^{1,2}} \to 0 \quad \text{as } t \to -\infty\,.$$

Furthermore, there exists a unique $\phi_+ \in H^{1,2}$ which satisfies

$$\|u(t) - e^{itH_0}\phi_+\|_{H^{1,2}} \to 0 \quad \text{as } t \to \infty\,.$$

And in the above conditions of $V(x)$, as is well known (see for example [1], [2]), the wave operators for H and H_0,

$$W_\pm = s - \lim_{t\to\pm\infty} e^{-itH}e^{itH_0}$$

exist and are complete, where $s-$ indicates the strong limit in $L^2(\mathbf{R}^n)$. The mapping $S : \phi_- \to \phi_+$ defined on a neighborhood of 0 in $H^{1,2}$ is called the nonlinear scattering operator. Put

$$S_F\phi = \phi(x) + \int_{-\infty}^{\infty} e^{-i\tau H}F(u(\tau))d\tau\,, \tag{1.2}$$

where u is the solution of (1.1) with the initial data ϕ at $t = -\infty$ (see (3.1)). Then the nonlinear scattering operator S is represented as

$$S = W_+^* S_F W_-\,. \tag{1.3}$$

Weder [10] has proved that the potential V is uniquely determined from the nonlinear scattering operator S. To be more percise, the following equality has been proved. For any $\phi, \psi \in H^{1,2}$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(S(\varepsilon\phi), \psi) = (S_L\phi, \psi)\,, \tag{1.4}$$

where S_L is the linear scattering operator associated with perturbed Schrödinger equation

$$i\frac{\partial u}{\partial t} + Hu = 0\,, \tag{1.5}$$

and the corresponding unperturbed Schrödinger equation

$$i\frac{\partial u}{\partial t} + H_0u = 0\,. \tag{1.6}$$

As is well known, the sattering operator S_L can be defined as

$$S_L = W_+^* W_- .$$

From (1.4) and the well known uniqueness results for the inverse scattering problem for the linear Schrödinger equation (for example [3], [4]), the nonlinear scattering operator S determines uniquely the potential $V(x)$.

From (1.3) and (1.2),

$$\frac{1}{\varepsilon}S(\varepsilon\phi) - S_L\phi = \frac{1}{\varepsilon}W_+^* S_F W_-(\varepsilon\phi) - W_+^* W_-\phi \tag{1.7}$$

$$= \frac{1}{\varepsilon}W_+^*(S_F - I)W_-(\varepsilon\phi) . \tag{1.8}$$

It is found from (1.8) and (1.2) that one of the key point to prove (1.4) is the estimate on nonlinearity like (2.8) in Section 2. In Weder [10], to obtain it, the $L^p - L^{p'}$ estimate and the Strichartz type estimate on e^{-itH}, which have been shown by Journé, Soffer, Sogge [5], and estimates on power nonlinearity $F(u)$ like (2.6) and (2.7) in Section 2, which have been shown by Strauss ([8], [9]), are used.

Estimates on cubic convolution nonlinearity $F(u)$ like (2.6) and (2.7) in Section 2 also have been proved in Strauss [9] when σ satisfies $2 \le \sigma < 4$ and $\sigma < n$. Hence, as for our case of cubic convolition nonlineaity, it can also be shown that (1.4) follows when σ satisfies $2 \le \sigma < 4$ and $\sigma < n$. But in this way the case $\sigma = 4$ is not included.

On the other hand, Yajima ([11], [12]) has shown the boundedness of the wave operators $W_\pm$ in $H^{1,p}$ and the $H^{1,p} - H^{1,p'}$ estimate on e^{-itH} under the Assumption 1.1 which is described below. Mochizuki [7] has shown the Strichartz type estimate on e^{-itH} using the $H^{1,p} - H^{1,p'}$ estimate. Moreover, using these estimates Mochizuki [7] has given estimates on the nonlinearity as in Lemma 2.3 including the case $\sigma = 4$.

In this paper it will be proved by Weder's method that the nonlinear scattering operator S uniquely determines the potential V, not only the case $2 \le \sigma < 4$ and $\sigma < n$ but also the case $2 \le \sigma \le 4$ and $\sigma < n$. Moreover, the topology of the convergence in (1.4) will be improved. Namely, we can prove that $(1/\varepsilon)S(\varepsilon\phi)$ converges to $S_L\phi$ strongly as $\varepsilon \to 0$.

Let $\mathfrak{F}$ be the Fourier transfrom. We use the notation: $n_* = (n-1)/(n-2)$ $(n \ge 3)$, $\langle x\rangle = (1+|x|^2)^{1/2}$. We assume that $V(x)$ and σ satisfy the following conditions.

ASSUMPTION 1.1. $V(x)$ is a real valued function on $\mathbf{R}^n$, $n \ge 3$. For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| \le 1$ and some $\rho > 2/n_*$, $\mathfrak{F}(\langle x\rangle^\rho D^\alpha V) \in L^{n_*}(\mathbf{R}^n)$ holds, where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ $(D_j = -i\partial/\partial x_j)$. H has no eigenvalues and 0 is not a resonance for H. Moreover, $V(x)$ satisfies one of the following conditions:

1. $\|\mathfrak{F}(\langle x\rangle^\rho V)\|_{L^{n_*}}$ is sufficiently small.
2. The spatial dimension $n = 2n' - 1$ is odd. For any $|\alpha| \le \max\{1, 1 + n' - 4\}$, there exists a constant $C_\alpha > 0$ such that

$$\left|\frac{\partial^\alpha V}{\partial x^\alpha}\right| \le C_\alpha \langle x\rangle^{-\delta}$$

where $\delta > \max\{n+2, 3n/2-2\}$.

3. n is even, $V(x) > 0$. For any $|\alpha| \leq n+1$, there exists a constant $C_\alpha > 0$ such that

$$\left|\frac{\partial^\alpha V}{\partial x^\alpha}\right| \leq C_\alpha \langle x \rangle^{-\delta}$$

where $\delta > 3n/2+1$.

ASSUMPTION 1.2. $2 \leq \sigma \leq 4$ and $\sigma < n$.

REMARK 1.1. It follows from Yajima ([11], [12]) that $V(x)$ is short range in the sense of Agmon [1]. Hence, the wave operators $W_\pm$ exist and are complete. The scattering operator S_L can be defined as

$$S_L = W_+^* W_- .$$

It will be shown in Propositon 3.1 that for the equation (1.1), the nonlinear scattering operator S can be defined on a neighborhood of 0 in $H^{1,2}$ and represented as (1.3).

The main result is the following

THEOREM 1.1. *Under Assumptions* 1.1 *and* 1.2 *the following holds, for any* $\phi \in H^{1,2}$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} S(\varepsilon\phi) = S_L \phi \quad in \;\; H^{1,2} .$$

From the well known results (for example [3], [4]) that V is uniquely determined from the scattering operator S_L, we obtain the following corollary.

COROLLARY 1.1. *Assume that Assumptions* 1.1 *and* 1.2 *are satisfied, the scattering operator S determines uniquely the potential V.*

THEOREM 1.2. *Under Assumptions* 1.1 *and* 1.2, *the scattering operator S determines uniquely a constant* λ. *Furthermore, for any* $\phi \in H^{1,2}$

$$\lambda = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \frac{\|(S_F - I)(\varepsilon\phi)\|_{H^{1,2}}}{\| \int_{-\infty}^{\infty} e^{-itH}(|x|^{-\sigma} * |e^{itH}\phi|^2) e^{itH}\phi dt \|_{H^{1,2}}}, \tag{1.9}$$

where S_F *is the scattering operator associated with the nonlinear equation* (1.1) *and the purturbed equation* (1.5), *and* S_F *is defined as* (1.2).

In Section 2 we give notations and several results used throughout this paper. The scattering problem for the equation (1.1) is discussed in Section 3. Theorem 1.1 and Theorem 1.2 are proved in Section 4.

2. Notations and preliminary lemmas.

First we give notations which will be freely used in the sequel. $C_0^\infty = C_0^\infty(\mathbf{R}^n) = \bigcap_{k=1}^\infty C_0^k(\mathbf{R}^n)$, where $C_0^k(\mathbf{R}^n)$ is the space of all k-times continuously differentiable function with compact support in $\mathbf{R}^n$. For a Banach space Z, $L^p(Z) = L^p(\mathbf{R}; Z)$ is the space of all Z valued L^p functions in $\mathbf{R}^n$. $\hat{}$ denotes the Fourier transfrom and $\mathfrak{F}^{-1}$ is its inverse. For $r \in \mathbf{R}$

and $1 \le p \le \infty$, let $H^{r,p} = H^{r,p}(\mathbf{R}^n)$ be the completions of C_0^∞ with respect to the norm,

$$\|f\|_{H^{r,p}} = \|\mathfrak{F}^{-1}\{\langle\xi\rangle^r \hat{f}(\xi)\}\|_{L^p} . \tag{2.1}$$

Conjugate exponents are denoted by q, q'; s, s' etc. c will denote a constant that may differ in different equation or inequalities.

Next we summarize several lemmas which will be used throughout this paper.

LEMMA 2.1 ([11], [12]). *Under Assumption* 1.1, *there exists a constant* $c > 0$ *such that, for all* $t \neq 0$,

$$\|e^{itH} f\|_{H^{1,q}} \le c|t|^{-d}\|f\|_{H^{1,q'}} , \tag{2.2}$$

for $2 \le q \le \infty$ *and* $f \in L^2 \cap H^{1,q'}$, *where*

$$d = n\left(\frac{1}{q'} - \frac{1}{2}\right) . \tag{2.3}$$

Put

$$X = H^{1,2} , \quad Y = H^{1,q} , \quad Y' = H^{1,q'}$$

and

$$W = L^3(\mathbf{R};\, H^{1,q}) \cap L^\infty(\mathbf{R};\, H^{1,2}) ,$$

where q satisfies

$$\frac{1}{q} = \frac{1}{2} - \frac{2}{3n} . \tag{2.4}$$

LEMMA 2.2 ([7]). *Let* $\phi \in X$. *Then* $e^{itH}\phi \in L^3(Y)$ *and there exists* $c \ge 1$ *such that*

$$\|e^{itH}\phi\|_{L^3(Y)} \le c\|\phi\|_X . \tag{2.5}$$

The cubic convolution nonlinearity $F(u) = \lambda(|x|^{-\sigma} * |u|^2)u$ is estimated as follows.

LEMMA 2.3 ([7]). *Assume that Assumption* 1.2. *Then*

$$\begin{aligned} \|F(u) - F(v)\|_{Y'} \le\ & c\|u - v\|_X\{\|u\|_Y^2 + \|v\|_Y^2\} \\ & + c\{\|u\|_X + \|v\|_X\}\|u - v\|_Y\{\|u\|_Y + \|v\|_Y\} , \end{aligned} \tag{2.6}$$

$$\|F(u) - F(v)\|_X \le c\|u - v\|_Y(\|u\|_Y^2 + \|v\|_Y^2) . \tag{2.7}$$

It follows from Lemmas 2.1 and 2.3 that

$$\left\| \int_{-\infty}^t e^{i(t-\tau)H}\{F(u(\tau)) - F(v(\tau))\}d\tau \right\|_W \le c_1\|u - v\|_W(\|u\|_W^2 + \|v\|_W^2) . \tag{2.8}$$

3. Nonlinear scattering with small data.

The theory of small data scattering with cubic convolution nonlinearity for the free Schrödinger operator H_0 has been studied in Mochizuki [7]. In this section we shall extend it to the purturbed Schrödinger operator $H = H_0 + V$.

First we consider the equation

$$\begin{cases} i\dfrac{\partial u}{\partial t} + Hu + F(u) = 0\,, & t \in \mathbf{R} \\ \|u(t) - e^{itH}\phi_-\|_X \to 0\,, & \text{as} \quad t \to -\infty\,, \end{cases} \tag{3.1}$$

where $F(u) = \lambda(|x|^{-\sigma} * |u|^2)u$ and $\phi_- \in X$. It is convenient to rewrite (3.1) into the integral form;

$$u(t) = e^{itH}\phi_- + \int_{-\infty}^{t} e^{i(t-\tau)H}F(u(\tau))d\tau\,. \tag{3.2}$$

By Lemma 2.2 and the boundedness of e^{itH} in X, there exists a constant $c_2 \geq 1$ such that

$$\|e^{itH}\phi\|_W \leq c_2\|\phi\|_X\,. \tag{3.3}$$

Let $\phi_- \in \mathcal{H}_\delta := \{\phi \in X : \|\phi\|_X < \delta\}$ and put

$$(\Phi u)(t) = e^{itH}\phi_- + \int_{-\infty}^{t} e^{i(t-\tau)H}F(u(\tau))d\tau\,.$$

We consider the nonlinear operator Φ in the ball $\mathfrak{B}(\delta_1) = \{u \in W;\, \|u\|_W \leq \delta_1\}$, where the constants $\delta_1 > 0$ and $\delta > 0$ are chosen to satisfy

$$2c_1\delta_1^2 \leq \frac{1}{2}, \quad \text{and} \quad c_2\delta \leq \frac{3}{4}\delta_1\,. \tag{3.4}$$

(c_1 and c_2 are constants used in (2.8) and (3.3) respectively.)

Let $u \in \mathfrak{B}(\delta_1)$. Then, by (3.3) with $\phi = \phi_-$ and by (2.8) with $v = 0$,

$$\|\Phi u\|_W \leq c_2\|\phi_-\|_X + c_1\|u\|_W^3 \leq \delta_1\,. \tag{3.5}$$

On the other hand, it follows from (2.8) that

$$\|\Phi u - \Phi v\|_W \leq \frac{1}{2}\|u - v\|_W \tag{3.6}$$

for any $u, v \in \mathfrak{B}(\delta_1)$. From (3.5) and (3.6) Φ defines a contraction map of $\mathfrak{B}(\delta_1)$ into itself. Thus there exists a unique fixed point $u \in \mathfrak{B}(\delta_1)$ which solves (3.2). Moreover, by (3.2) and (2.7) with $v = 0$, we have

$$\|u(t) - e^{itH}\phi_-\|_X \leq c\int_{-\infty}^{t} \|u(\tau)\|_Y^3 d\tau \to 0 \quad \text{as } t \to -\infty\,.$$

Next put

$$\phi_+ = \phi_- + \int_{-\infty}^{\infty} e^{-i\tau H}F(u(\tau))d\tau\,. \tag{3.7}$$

Then $\phi_+ \in X$ and

$$e^{itH}\phi_+ = u(t) + \int_{t}^{\infty} e^{i(t-\tau)H}F(u(\tau))d\tau\,.$$

Thus

$$\|u(t) - e^{itH}\phi_+\|_X \to 0 \quad \text{as } t \to \infty\,.$$

Hence we can define the scattering operator $S_F : \phi_- \to \phi_+$ on $\mathcal{H}_\delta$.

Next as in [10], put

$$S := W_+^* S_F W_- .$$

Then it is easy to see that solution of (1.6) has the same asymptotics as the solution u of the (1.1) when $t \to \pm\infty$ (see Weder [10]). Thus if we take $\rho = \|W_-\|_{X\to X}\delta$ (the operator norm is denoted by $\|\cdot\|_{X\to X}$ and δ is a constant which satisfies (3.4)), we obtain

PROPOSITION 3.1. *Under Assumptions* 1.1 *and* 1.2, *there exists* $\rho > 0$ *with the following properties*: *If* $\psi_- \in \mathcal{H}_\rho$, *then there exists a unique solution* $u \in W$ *of* (1.1) *such that*

$$\|u(t) - e^{itH_0}\psi_-\|_{H^{1,2}} \to 0 \quad \text{as } t \to -\infty .$$

Furthermore, put $\psi_+ = S\psi_-$. *Then* ψ_+ *satisfies*

$$\|u(t) - e^{itH_0}\psi_+\|_{H^{1,2}} \to 0 \quad \text{as } t \to \infty .$$

4. Proof of Theorems.

First we prove Theorem 1.1. By (3.2) and (2.8) for ε small enough, we have

$$\|u\|_W \le c\|e^{itH}\varepsilon\phi\|_W + \tilde{c}\|u\|_W$$

for a solution $u \in W$ of (3.2) with $\phi_- = \varepsilon\phi$ and some $\tilde{c} < 1$. Then, by (3.3),

$$\|u\|_W \le c\varepsilon\|e^{itH}\phi\|_W \le c\varepsilon\|\phi\|_X . \tag{4.1}$$

Moreover, by (3.7) and (2.7),

$$\begin{aligned}\frac{1}{\varepsilon}\|(S_F - I)(\varepsilon\phi_-)\|_X &\le \frac{c}{\varepsilon}\int_{-\infty}^{\infty}\|u(\tau)\|_Y^3 d\tau \\ &= \frac{c}{\varepsilon}\|u\|^3_{L^3(Y)} .\end{aligned}$$

Hence for any $\phi \in X$, using (4.1), we have

$$\begin{aligned}\left\|\frac{1}{\varepsilon}S(\varepsilon\phi) - S_L\phi\right\|_X &= \frac{1}{\varepsilon}\|S(\varepsilon\phi) - S_L(\varepsilon\phi)\|_X \\ &\le \frac{c}{\varepsilon}\|(S_F - I)W_-(\varepsilon\phi)\|_X \\ &\le \frac{c}{\varepsilon}\|W_-(\varepsilon\phi)\|_X^3 \\ &\le c\varepsilon^2\|\phi\|_X^3 \to 0 \quad \text{as} \quad \varepsilon \to 0 .\end{aligned}$$

Thus Theorem 1.1 follows.

Next we prove Theorem 1.2. Note that for any $\phi \in X$,

$$\frac{1}{\varepsilon^3}\int_{-\infty}^{\infty} e^{-itH}F(e^{itH}\varepsilon\phi))dt = \lambda\int_{-\infty}^{\infty} e^{-itH}(|x|^{-\sigma} * |e^{itH}\phi|^2)e^{itH}\phi\, dt \in X$$

by (2.7) and (2.5).

Compute that

$$\begin{aligned}
&\left\| \frac{1}{\varepsilon^3} \int_{-\infty}^{\infty} e^{-itH} (F(u(t)) - F(e^{itH}\varepsilon\phi))dt \right\|_X \\
&\quad \le \frac{c}{\varepsilon^3} \int_{-\infty}^{\infty} \|u(t) - e^{itH}\varepsilon\phi\|_Y (\|u(t)\|_Y^2 + \|e^{itH}\varepsilon\phi\|_Y^2)dt \\
&\quad \le \frac{c}{\varepsilon^3} \|u - e^{itH}\varepsilon\phi\|_W (\|u\|_W^2 + \|e^{itH}\varepsilon\phi\|_W^2) \\
&\quad \le \frac{c}{\varepsilon^3} \left\| \int_{-\infty}^{t} e^{i(t-\tau)H} F(u(\tau))d\tau \right\|_W (\varepsilon^2 \|\phi\|_X^2) \\
&\quad \le \frac{c}{\varepsilon} \|u\|_W^3 \|\phi\|_X^2 \\
&\quad \le c\varepsilon^2 \|\phi\|_X^5 .
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{\varepsilon^3}(S_F - I)W_-(\varepsilon\phi) &= \frac{1}{\varepsilon^3} \int_{-\infty}^{\infty} e^{-itH} F(u(t))dt \\
&= \int_{-\infty}^{\infty} e^{-itH} F(e^{itH}\phi)dt + \frac{1}{\varepsilon^3} \int_{-\infty}^{\infty} e^{-itH}(F(u) - F(e^{itH}\varepsilon\phi))dt \\
&\to \int_{-\infty}^{\infty} e^{-itH} F(e^{itH}\phi)dt \text{ in } X, \quad \text{as } \varepsilon \to 0 .
\end{aligned}$$

Thus we obtain (1.9). From Corollary 1.1, V and, consequently, the wave operators $W_\pm$ are determined uniquely by S. Hence $S_F = W_+ S W_-^*$ is uniquely determined by S.

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