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A Simple Proof of Nowicki's Conjecture on the Kernel of an Elementary Derivation

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Abstract. Khoury solved Nowicki's conjecture on the kernel of an elementary derivation of a polynomial ring using Gröbner basis theory. In this paper, we give a simple new proof of the conjecture.

1. Introduction

Let $A[\mathbf{x}] = A[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an integral domain A for $n \in \mathbf{N}$, and D an A-derivation of $A[\mathbf{x}]$, i.e., an A-linear map $D : A[\mathbf{x}] \to A[\mathbf{x}]$ satisfying $D(fg) = D(f)g + fD(g)$ for each $f, g \in A[\mathbf{x}]$. We say that D is *elementary* if $D(x_i)$ belongs to A for each i. Then, the kernel $\ker D$ of D is an A-subalgebra of $A[\mathbf{x}]$ containing

$$L_{i,j}^D := D(x_j)x_i - D(x_i)x_j \quad \text{for each } i, j \in \{1, \ldots, n\}.$$

In general, it is difficult to determine the structure of $\ker D$. The problem of finite generation of $\ker D$ is a special case of the Fourteenth Problem of Hilbert when A is a polynomial ring over a field. This problem was settled in the negative by Nagata [11], while Roberts [13] gave a new type of counterexample obtained as the kernel of an elementary derivation (see [7] and [9] for generalizations of Roberts' counterexample). For a certain elementary derivation D, Kurano [8, Proposition 3.1] found a finite set of generators of $\ker D$, which cannot be generated by $L_{i,j}^D$'s (see also [3] and [5] for affirmative results).

Recently, Khoury [6] solved the following conjecture of Nowicki in the affirmative by calculating a Gröbner basis for some ideal.

CONJECTURE (Nowicki [12, Conjecture 6.9.10]). *Assume that $k[\mathbf{y}] = k[y_1, \ldots, y_n]$ is the polynomial ring in n variables over a field k of characteristic zero. If Δ_n is the $k[\mathbf{y}]$-derivation of $k[\mathbf{y}][\mathbf{x}]$ defined by $\Delta_n(x_i) = y_i$ for $i = 1, \ldots, n$, then $\ker \Delta_n$ is generated by $L_{i,j}^{\Delta_n}$ for $1 \leq i < j \leq n$ over $k[\mathbf{y}]$.*

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Khoury's Gröbner basis consists of several families of polynomials, and he checked many cases to show that all the S-polynomials are reduced to zero. The aim of this paper is to give a simple new proof of Nowicki's conjecture by a method similar to that used in the proof of Kurano [8, Proposition 3.1].

For each A-domain B and an elementary A-derivation D of $A[\mathbf{x}]$, the B-derivation $D_B := \mathrm{id}_B \otimes D$ of $B \otimes_A A[\mathbf{x}] = B[x_1, \ldots, x_n]$ is elementary. Moreover, if B is flat over A, then $\ker D_B = B \otimes_A \ker D$. Therefore, the result on $\ker \Delta_n$ implies the following theorem.

THEOREM. *Let A be an integral domain containing a field k of characteristic zero, and let D be an elementary A-derivation of $A[\mathbf{x}]$ such that A is flat over $k[D(x_1), \ldots, D(x_n)]$ and $D(x_1), \ldots, D(x_n)$ are algebraically independent over k. Then, $\ker D$ is generated by $L_{i,j}^D$ for $1 \leq i < j \leq n$ over A.*

Actually, D induces an elementary R-derivation D' of $R[x_1, \ldots, x_n]$, for which $\ker D = A \otimes_R \ker D'$, where $R = k[D(x_1), \ldots, D(x_n)] \simeq k[\mathbf{y}]$. We note that Khoury [6, Theorem 1.1] showed that $\ker D$ is generated by $L_{i,j}^D$ for $1 \leq i < j \leq n$ over $k[\mathbf{y}]$ for the $k[\mathbf{y}]$-derivation D of $k[\mathbf{y}][\mathbf{x}]$ defined by $D(x_i) = y_i^{t_i}$ with $t_i \in \mathbf{N}$ for $i = 1, \ldots, n$. In this case, $y_1^{t_1}, \ldots, y_n^{t_n}$ are algebraically independent over k, and $k[\mathbf{y}]$ is free over $k[y_1^{t_1}, \ldots, y_n^{t_n}]$.

2. Proof of the conjecture

We prove the conjecture by induction on n. The assertion is clear when $n = 1$. Assume that $n \geq 2$, and let S_l be the set of $L_{i,j} := L_{i,j}^{\Delta_n}$ for $1 \leq i < j \leq l$ for each $l \leq n$. By the assumption on induction, $\ker \Delta_{n-1}$ is generated by S_{n-1} over $k[\mathbf{y}'] := k[y_1, \ldots, y_{n-1}]$, since $L_{i,j}^{\Delta_{n-1}} = L_{i,j}^{\Delta_n}$ for each i, j. As discussed in Section 1, the $k[\mathbf{y}']$-derivation Δ_{n-1} naturally extends to a $k[\mathbf{y}]$-derivation $(\Delta_{n-1})_{k[\mathbf{y}]}$ of $k[\mathbf{y}][\mathbf{x}'] := k[\mathbf{y}][x_1, \ldots, x_{n-1}]$. Then, $(\Delta_{n-1})_{k[\mathbf{y}]} = \Delta_n|_{k[\mathbf{y}][\mathbf{x}']}$, so we have $\ker(\Delta_{n-1})_{k[\mathbf{y}]} = k[\mathbf{y}][\mathbf{x}'] \cap \ker \Delta_n$. Moreover, $\ker(\Delta_{n-1})_{k[\mathbf{y}]} = k[\mathbf{y}] \otimes_{k[\mathbf{y}']} \ker \Delta_{n-1}$, since $k[\mathbf{y}]$ is flat over $k[\mathbf{y}']$. Thus, we get

$$k[\mathbf{y}][\mathbf{x}'] \cap \ker \Delta_n = k[\mathbf{y}][S_{n-1}]\,. \tag{1}$$

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the coordinate unit vectors of $\mathbf{R}^n$, M the $\mathbf{Z}$-submodule of $(\mathbf{Z}^n)^2$ generated by $(\mathbf{e}_j - \mathbf{e}_i, \mathbf{e}_i - \mathbf{e}_j)$ for $1 \leq i < j \leq n$, and $\Gamma = (\mathbf{Z}^n)^2/M$. Then, Γ-gradings are defined on $k[\mathbf{y}][\mathbf{x}]$ and $k[\mathbf{y}^{\pm 1}][\mathbf{x}] := k[\mathbf{y}][\mathbf{x}][(y_1 \cdots y_n)^{-1}]$ as follows. Here, a k-algebra R is said to be Γ-graded if there exists a k-vector subspace R_γ of R for each $\gamma \in \Gamma$ such that $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ and $R_\gamma R_\mu \subset R_{\gamma+\mu}$ for $\gamma, \mu \in \Gamma$. Let $\mathbf{Z}_{\geq 0}$ denote the set of nonnegative integers, and $\mathbf{y}^a = y_1^{a_1} \cdots y_n^{a_n}$ and $\mathbf{x}^b = x_1^{b_1} \cdots x_n^{b_n}$ for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. For each $\gamma \in \Gamma$, we define $k[\mathbf{y}][\mathbf{x}]_\gamma$ (resp. $k[\mathbf{y}^{\pm 1}][\mathbf{x}]_\gamma$) to be the k-vector space generated by $\mathbf{y}^a \mathbf{x}^b$ for $a, b \in (\mathbf{Z}_{\geq 0})^n$ (resp. $a \in \mathbf{Z}^n$, $b \in (\mathbf{Z}_{\geq 0})^n$) such that the image of (a, b) in Γ is equal to γ. Then, Γ-gradings are defined on $k[\mathbf{y}][\mathbf{x}]$ and $k[\mathbf{y}^{\pm 1}][\mathbf{x}]$. Note that $\Delta_n(k[\mathbf{y}][\mathbf{x}]_\gamma)$ is contained in $k[\mathbf{y}][\mathbf{x}]_{\gamma-\delta}$ for each $\gamma \in \Gamma$, where δ is the image of $(-\mathbf{e}_n, \mathbf{e}_n)$ in Γ. From this, we know that

$$\ker \Delta_n = \bigoplus_{\gamma \in \Gamma} \left(k[\mathbf{y}][\mathbf{x}]_\gamma \cap \ker \Delta_n \right) .$$

Hence, we are reduced to showing that each $0 \neq \Phi \in k[\mathbf{y}][\mathbf{x}]_\gamma \cap \ker \Delta_n$ belongs to $k[\mathbf{y}][S_n]$ for $\gamma \in \Gamma$. We may find $a = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ and $l \in \mathbf{Z}_{\geq 0}$ such that the image of $(a, l\mathbf{e}_n)$ in Γ is equal to γ. Let m be the x_n-degree of Φ, where $0 \leq m \leq l$, and $\phi \in k[\mathbf{y}][\mathbf{x}']$ the coefficient of x_n^m in Φ. Then, ϕ belongs to $k[\mathbf{y}][\mathbf{x}]_\mu$, where μ is the image of $(a, (l-m)\mathbf{e}_n)$ in Γ. Furthermore, $0 = \Delta_n(\Phi) = \Delta_n(\phi)x_n^m + m\phi y_n x_n^{m-1} + \Delta_n(\Phi - \phi x_n^m)$, and the x_n-degrees of $m\phi y_n x_n^{m-1}$ and $\Delta_n(\Phi - \phi x_n^m)$ are at most $m-1$. Hence, $\Delta_n(\phi) = 0$. Thus, ϕ belongs to $k[\mathbf{y}][S_{n-1}]$ by (1). Write $\phi = \sum_{b,\mathbf{u}} r'_{b,\mathbf{u}} \mathbf{y}^b \hat{\mathbf{y}}^{-\mathbf{u}} L^{\mathbf{u}}$, where the sum is taken over $b \in (\mathbf{Z}_{\geq 0})^n$ and $\mathbf{u} = (u_{i,j})_{i,j}$ with $u_{i,j} \in \mathbf{Z}_{\geq 0}$ for $1 \leq i < j \leq n-1$, $r'_{b,\mathbf{u}} \in k$ for each b and $\mathbf{u}$, and

$$\hat{\mathbf{y}}^{-\mathbf{u}} = \prod_{1 \leq i < j \leq n-1} (y_i y_j)^{-u_{i,j}} , \quad L^{\mathbf{u}} = \prod_{1 \leq i < j \leq n-1} L_{i,j}^{u_{i,j}} \quad \text{for each } \mathbf{u} .$$

We may assume that $r'_{b,\mathbf{u}} = 0$ if $\mathbf{y}^b \hat{\mathbf{y}}^{-\mathbf{u}}$ is not in $k[\mathbf{y}]$. Let $\eta(b, \mathbf{u})$ be the image of $(b - |\mathbf{u}|\mathbf{e}_n, |\mathbf{u}|\mathbf{e}_n)$ in Γ, where $|\mathbf{u}| = \sum_{i,j} u_{i,j}$. Then, $\mathbf{y}^b \hat{\mathbf{y}}^{-\mathbf{u}} L^{\mathbf{u}}$ belongs to $k[\mathbf{y}^{\pm 1}][\mathbf{x}]_{\eta(b,\mathbf{u})}$ for each b and $\mathbf{u}$, since $(y_i y_j)^{-1} L_{i,j}$ belongs to $k[\mathbf{y}^{\pm 1}][\mathbf{x}]_\delta$ for each i, j. Since ϕ is in $k[\mathbf{y}][\mathbf{x}]_\mu$, and μ is the image of $(a, (l-m)\mathbf{e}_n)$, we may assume that $r'_{b,\mathbf{u}} = 0$ unless $|\mathbf{u}| = l - m$ and $b = a + (l-m)\mathbf{e}_n$. For each $\mathbf{u}$ with $r_{\mathbf{u}} := r'_{a+(l-m)\mathbf{e}_n,\mathbf{u}} \neq 0$, write $\mathbf{y}^a y_n^{l-m} \hat{\mathbf{y}}^{-\mathbf{u}} = y_1^{\rho_1(\mathbf{u})} \cdots y_{n-1}^{\rho_{n-1}(\mathbf{u})} y_n^s$, where $\rho_i(\mathbf{u}) \in \mathbf{Z}_{\geq 0}$ for $i = 1, \ldots, n-1$, and $s = a_n + l - m$. Then, we have $\phi = y_n^s \sum_{\mathbf{u}} r_{\mathbf{u}} y_1^{\rho_1(\mathbf{u})} \cdots y_{n-1}^{\rho_{n-1}(\mathbf{u})} L^{\mathbf{u}}$. Since $|\mathbf{u}| = l - m$, it follows that

$$\sum_{i=1}^{n-1} \rho_i(\mathbf{u}) = \sum_{i=1}^{n-1} a_i - 2(l-m) \quad \text{for each } \mathbf{u} . \tag{2}$$

Now, we show that Φ belongs to $k[\mathbf{y}][S_n]$ by contradiction. By replacing Φ if necessary, we may assume that m is the minimum among the x_n-degrees of elements of $\ker \Delta_n \setminus k[\mathbf{y}][S_n]$. To obtain a contradiction, it suffices to deduce that

$$m \geq 2l - \sum_{i=1}^{n-1} a_i . \tag{3}$$

In fact, (3) implies that $\sum_{i=1}^{n-1} \rho_i(\mathbf{u}) \geq m$ by (2), so we have $\sum_{i=1}^{n-1} \rho'_i(\mathbf{u}) = m$ for some integers $0 \leq \rho'_i(\mathbf{u}) \leq \rho_i(\mathbf{u})$ for $i = 1, \ldots, n-1$ for each $\mathbf{u}$. Then,

$$\Phi' := y_n^s \sum_{\mathbf{u}} r_{\mathbf{u}} L^{\mathbf{u}} \prod_{i=1}^{n-1} y_i^{\rho_i(\mathbf{u}) - \rho'_i(\mathbf{u})} L_{n,i}^{\rho'_i(\mathbf{u})} = y_n^s \sum_{\mathbf{u}} r_{\mathbf{u}} L^{\mathbf{u}} \prod_{i=1}^{n-1} y_i^{\rho_i(\mathbf{u}) - \rho'_i(\mathbf{u})} (y_i x_n - y_n x_i)^{\rho'_i(\mathbf{u})}$$

is an element of $k[\mathbf{y}][S_n]$ having x_n-degree m, in which the coefficient of x_n^m is equal to ϕ. Hence, the x_n-degree of $\Phi - \Phi'$ is less than m. Since $\Phi - \Phi'$ is an element of $\ker \Delta_n \setminus k[\mathbf{y}][S_n]$, this contradicts the minimality of m.

We establish that (3) holds for any nonzero homogeneous element Φ of $\ker \Delta_n$ by contradiction. Take Φ which does not satisfy (3) so that m would be the minimum among the x_n-degrees of such polynomials. Then, $t := 2l - \sum_{i=1}^{n-1} a_i - m$ is positive, and $\sum_{i=1}^{n-1} \rho_i(\mathbf{u}) = m - t$ for each $\mathbf{u}$ by (2). Hence, the x_n-degree of

$$\Phi_1 := \sum_{\mathbf{u}} r_{\mathbf{u}} L^{\mathbf{u}} \prod_{i=1}^{n-1} L_{n,i}^{\rho_i(\mathbf{u})} = \sum_{u} r_{\mathbf{u}} L^{\mathbf{u}} \prod_{i=1}^{n-1} (y_i x_n - y_n x_i)^{\rho_i(\mathbf{u})}$$

is $m - t$. The coefficient of x_n^{m-t} in $y_n^s \Phi_1$ is equal to ϕ, so the coefficient of x_n^m in $y_n^s \Phi_1 L_{n,1}^t$ is equal to that in $y_1^t \Phi$. Consequently, the x_n-degree m' of $\Phi_2 := y_1^t \Phi - y_n^s \Phi_1 L_{n,1}^t$ is less than m. We claim that $\Phi_2 = 0$. In fact, if γ' is the image of $(a + t\mathbf{e}_1, l\mathbf{e}_n)$ in Γ, and $(a_1', \dots, a_n') := a + t\mathbf{e}_1$, then Φ_2 belongs to $k[\mathbf{y}][\mathbf{x}]_{\gamma'} \cap \ker \Delta_n$, and

$$2l - \sum_{i=1}^{n-1} a_i' = 2l - \sum_{i=1}^{n-1} a_i - t = m > m' .$$

This implies that $\Phi_2 = 0$ by the minimality of m. Hence, $y_1^t \Phi = y_n^s \Phi_1 L_{n,1}^t$. Thus, Φ_1 is divisible by y_1, since neither are y_n and $L_{n,1}$. Recall that the kernel of a locally nilpotent derivation D of an integral domain R containing $\mathbf{Q}$ is factorially closed in R, that is, $D(fg) = 0$ implies $D(f) = D(g) = 0$ for each $f, g \in R \setminus \{0\}$ (cf. [2, Proposition 1.3.32 (iii)]). Note that Δ_n is locally nilpotent, $\Delta_n(\Phi_1) = 0$, $\Phi_1 \neq 0$ and $\Delta_n(x_n) \neq 0$. Hence, Φ_1 is not divisible by x_n. By substituting zero for x_n, we obtain from Φ_1 a nonzero polynomial

$$\sum_{\mathbf{u}} r_{\mathbf{u}} L^{\mathbf{u}} \prod_{i=1}^{n-1} (-y_n x_i)^{\rho_i(\mathbf{u})} = (-y_n)^{m-t} \Psi , \quad \text{where } \Psi = \sum_{\mathbf{u}} r_{\mathbf{u}} L^{\mathbf{u}} \prod_{i=1}^{n-1} x_i^{\rho_i(\mathbf{u})} .$$

Then, $\Psi \neq 0$, and Ψ is divisible by y_1, since so is Φ_1. Define $\sigma \in \operatorname{Aut}_k k[\mathbf{y}][\mathbf{x}]$ by $\sigma(x_i) = y_i$ and $\sigma(y_i) = x_i$ for $i = 1, \dots, n$. Then, $\sigma(\Psi)$ is divisible by x_1. On the other hand, $\sigma(L_{i,j}) = L_{j,i}$ and $\sigma(x_i) = y_i$ are in $\ker \Delta_n$ for each i, j, so $\sigma(\Psi)$ belongs to $\ker \Delta_n$. Thus, we have $\sigma(\Psi) = 0$, because x_1 is not in $\ker \Delta_n$ and $\ker \Delta_n$ is factorially closed in $k[\mathbf{y}][\mathbf{x}]$. This contradicts that $\Psi \neq 0$. Therefore, (3) holds true. Thereby, we have proved that Φ belongs to $k[\mathbf{y}][S_n]$. This completes the proof of the conjecture.

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Note. Recently, Drensky–Makar-Limanov [1] independently gave a simple proof of Nowicki's conjecture. Very recently, Professor Mitsuyasu Hashimoto informed the author that Goto-Hayasaka-Kurano-Nakamura [4, Theorem 3.2] and Miyazaki [10, Theorem 3.7] also gave results which imply that Nowicki's conjecture is true. Actually, $\ker \Delta$ is equal to the invariant subring for the G_a-action on $k[\mathbf{y}][\mathbf{x}]$ defined by $y_i \mapsto y_i$ and $x_i \mapsto x_i + ty_i$ for $i = 1, \ldots, n$ for each $t \in G_a$. On the other hand, Goto-Hayasaka-Kurano-Nakamura and Miyazaki determined sets of generators for certain invariant rings where $\ker \Delta$ is included.

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References

[1] V. DRENSKY and L. MAKAR-LIMANOV, The conjecture of Nowicki on Weitzenboeck derivations of polynomial algebras, arXiv:AC/0804.2933.

[2] A. VAN DEN ESSEN, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, Vol. 190, Birkhäuser, Basel, Boston, Berlin, 2000.

[3] A. VAN DEN ESSEN and T. JANSSEN, The kernels of elementary derivations, University of Nijmegen, Report No. 9548, Nijmegen, The Netherlands, 1995.

[4] S. GOTO, F. HAYASAKA, K. KURANO and Y. NAKAMURA, Rees algebra of the second syzygy module of the residue field of a regular local ring, Contemp. Math. Vol. **390** (2005), 97–108.

[5] J. KHOURY, On some properties of elementary monomial derivations in dimension six, J. Pure Appl. Algebra **156** (2001), 69–79.

[6] J. KHOURY, On a Conjecture of Nowicki, preprint 2006.

[7] H. KOJIMA and M. MIYANISHI, On Roberts' counterexample to the fourteenth problem of Hilbert, J. Pure Appl. Algebra **122** (1997), 277–292.

[8] K. KURANO, Positive characteristic finite generation of symbolic Rees algebra and Roberts' counterexamples to the fourteenth problem of Hilbert, Tokyo J. Math. **16** (1993), 473–496.

[9] S. KURODA, A generalization of Roberts' counterexample to the fourteenth problem of Hilbert, Tohoku Math. J. **56** (2004), 501–522.

[10] M. MIYAZAKI, Invariants of the unipotent radical of a Borel subgroup, Proceedings of the 29th Symposium on Commutative Algebra in Japan, Nagoya, Japan November 19–22, 2007, 43–50.

[11] M. NAGATA, On the fourteenth problem of Hilbert, in Proceedings of the International Congress of Mathematicians, 1958, Cambridge Univ. Press, London, New York, 1960, 459–462.

[12] A. NOWICKI, *Polynomial derivations and their rings of constants*, Uniwersytet Mikolaja Kopernika, Torun, 1994.

[13] P. ROBERTS, An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem, J. Algebra **132** (1990), 461–473.

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