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On the Stability of Conditional Bases in $L^2[-\pi, \pi]$

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Abstract. We consider the conditions for the multiplication by a weight $w(t)$ that make the system $\{w(t)e^{i\lambda_n t}\}$ a conditional basis for $L^2[-\pi, \pi]$. Furthermore, some stability result is investigated on such bases.

1. Introduction

A sequence $\{x_n\}$ of vectors in a Banach space X is said to be a *bounded basis* for X if it satisfies the following conditions,

(i) to each $x \in X$ there corresponds a unique sequence $\{\alpha_n\}$ of scalars such that

$$x = \sum_n \alpha_n x_n$$

with respect to the norm, and

(ii) $0 < \inf_n \|x_n\| \leq \sup_n \|x_n\| < \infty$.

Moreover a basis $\{x_n\}$ is said to be an *unconditional basis* if every convergent series of the form $\sum_n \alpha_n x_n$ is unconditionally convergent. Especially it is said to be a *Riesz basis* if X is a Hilbert space. The basis $\{x_n\}$ is said to be a *conditional basis* if it is non-unconditional. About the theory of bases in a Banach space, we refer to Lindenstrauss and Tzafriri [LT] or Singer [S]. In this note, we deal with the space $L^2[-\pi, \pi]$ of square summable functions on $[-\pi, \pi]$ as X and the system $\{w(t)e^{i\lambda_n t}\}$ multiplied by a nonnegative measurable function $w(t)$ as $\{x_n\}$, where $w(t)$ is called *weight*. As usual, we identify two functions f and g on $[-\pi, \pi]$, if $f(x) = g(x)$ for almost every x in $[-\pi, \pi]$.

EXAMPLE A (see [S, Ch. II, Example 11.1 and Example 14.3]). *The system* $\{e^{int}\}_{n=-\infty}^{\infty}$ *is a Riesz basis for* $L^2[-\pi, \pi]$ *and a conditional basis for* $L^p[-\pi, \pi]$ *with* $1 < p < \infty$, $p \neq 2$.

The class of Riesz bases is so large that the existence of conditional bases is an important problem. Babenko gave the following examples in [B].

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THEOREM A ([B, p.160]; see [S, p. 428, Example 14.4]). *The systems*

$$\{|t|^{-\beta}e^{int}\}_{n=-\infty}^{\infty} \quad \textit{and} \quad \{|t|^{\beta}e^{int}\}_{n=-\infty}^{\infty}$$

with $0 < \beta < 1/2$, *are bounded conditional bases for* $L^2[-\pi, \pi]$.

We say that a nonnegative measurable function $w(t)$ on the real line **R** satisfies the *Muckenhoupt condition* (A_2) if

$$\left(\frac{1}{|I|}\int_I w(t)dt\right)\cdot\left(\frac{1}{|I|}\int_I w(t)^{-1}dt\right) \le K\,,$$

where I is any finite interval of **R** and K is a positive constant independent of I.

Now, we consider the system $\{w(t)e^{int}\}_{n=-\infty}^{\infty}$ for a nonnegative measurable function $w(t)$. By applying Theorem 8 in [HMW] with $p = 2$, a necessary and sufficient condition for this system to be a basis of $L^2[-\pi, \pi]$ is that $w^2(t)$ satisfies the Muckenhoupt condition (A_2). Moreover, when $w(t)$ is unbounded, we see that $\{w(t)e^{int}\}_{n=-\infty}^{\infty}$ becomes a conditional basis by the same way as in the proof of [B, p. 160] or [S, p. 353] in the case of $w(t) = |t|^{-\beta}$. Therefore, by biorthogonality, we also find that $\{w(t)^{-1}e^{int}\}_{n=-\infty}^{\infty}$ becomes a conditional basis. We obtain a similar result when $w(t)^{-1}$ is unbounded. In addition, we remark that the operator $T_w(f) = wf$ is an isomorphism on $L^2[-\pi, \pi]$ if $w(t)$ and $w(t)^{-1}$ are bounded at the same time. Then, $\{w(t)e^{int}\}_{n=-\infty}^{\infty}$ becomes a Riesz basis of $L^2[-\pi, \pi]$. By the above obserbation, Theorem A may be extended as follows:

THEOREM B. *Let* $w(t) \ge 0$ *be a function with period* 2π. *Then the system* $\{w(t)e^{int}\}_{n=-\infty}^{\infty}$ *(or* $\{w(t)^{-1}e^{int}\}_{n=-\infty}^{\infty}$*) is a conditional basis for* $L^2[-\pi, \pi]$ *if and only if*

(1) *the function* $w^2(t)$ *satisfies the Muckenhoupt condition* (A_2), *and*
(2) *one of the functions* $w(t)$, $w(t)^{-1}$ *is unbounded.*

Tha same result is stated in Kazarian [K, p. 241]. Olevskii [O, Corollary 1] gave conditions under which multiplication by a bounded measurable function in $L^2[a, b]$ transforms a complete orthonormal system into a conditional basis. By the way, it is supposed that $w(t)$ and $f \in L^2[-\pi, \pi]$ have the period of 2π in Theorems A and B. This is due to the fact that every exponential function e^{int} has the period of 2π. On the other hand, generally, the nonharmonic functions $e^{i\lambda_n t}$ have no period 2π. In this note, we obtain some results for conditional bases about the nonharmonic case which does not assume the periodicity. In what follows we always assume that $w(t)$ and $f(t)$ are 0 almost everywhere outside the interval $[-\pi, \pi]$. Then we find some conditions on $w(t)$ under which $\{w(t)e^{i\lambda_n t}\}$ becomes a conditional basis for $L^2[-\pi, \pi]$. First, we give some sufficient conditions (Proposition 2.1). Using the conditions and an arguement in [Y1, Theorem 1], we investigate the stability of conditional bases.

2. Main Results

Following [G], we define the *conjugate function*,

$$\tilde{f}(t) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_{\varepsilon < |t-s| < \pi} \frac{f(s)}{2\tan\left(\frac{t-s}{2}\right)} ds$$

and the *Hilbert transform*,

$$Hf(t) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_{\varepsilon < |t-s| < \pi} \frac{f(s)}{x - s} ds$$

for $f \in L^2[-\pi, \pi]$. We will simply express $\tilde{f}(t)$ as

$$\tilde{f}(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(s)}{2\tan\left(\frac{t-s}{2}\right)} ds \, .$$

It is well known that $Hf(t)$ exists almost everywhere for $f \in L^2[-\pi, \pi]$. Consequently, $\tilde{f}(t)$ also exists almost everywhere for $f \in L^2[-\pi, \pi]$ (see [G, p. 105]). From now on, we denote by δ, C, C' and C'' positive constants such that C, C' and C'' depend only on δ.

LEMMA 2.1. *Assume that a function $w(t)$ on $\mathbf{R}$ satisfies the following conditions,*
(i) $w(t) \geq \delta > 0$, $-\pi \leq t \leq \pi$, *and*
(ii) $w^2(t)$ *satisfies the Muckenhoupt condition* (A_2).
Then

$$\left(\int_{-\pi}^{\pi} \left| \tilde{f}(t) \right|^2 w^2(t) dt \right)^{\frac{1}{2}} \leq C \left(\int_{-\pi}^{\pi} |f(t)|^2 w^2(t) dt \right)^{\frac{1}{2}} \tag{2.1}$$

for $f \in L^2[-\pi, \pi]$.

PROOF. When $w(t)$ and $f(t)$ are functions with period 2π, it is known that the condition (ii) is equivalent to the inequality (2.1) by [HMW, Theorem 1].

Let $f \in L^2[-\pi, \pi]$. Then we have

$$\|f\|_2 \leq \delta^{-1} \left(\int_{-\pi}^{\pi} |f(t)|^2 w^2(t) dt \right)^{\frac{1}{2}}$$

by the condition (i) for $w(t)$. And we see from (1.6) in Garnett [G, p. 105] that

$$\left| \tilde{f}(t) \right| \leq |Hf(t)| + \frac{2}{\pi} \|f\|_1$$

for $f \in L^2[-\pi, \pi]$. Together with [HMW, Theorem 9], we have

$$
\begin{aligned}
&\left(\int_{-\pi}^{\pi}\left|\tilde{f}(t)\right|^2 w^2(t)dt\right)^{\frac{1}{2}}\\
&\le \left(\int_{-\pi}^{\pi}|Hf(t)|^2 w^2(t)dt\right)^{\frac{1}{2}} + \frac{2}{\pi}\|f\|_1\left(\int_{-\pi}^{\pi} w^2(t)dt\right)^{\frac{1}{2}}\\
&\le \left(\int_{-\infty}^{\infty}|Hf(t)|^2 w^2(t)dt\right)^{\frac{1}{2}} + C\|f\|_2\\
&\le C'\left(\int_{-\infty}^{\infty}|f(t)|^2 w^2(t)dt\right)^{\frac{1}{2}} + C\|f\|_2\\
&= C'\left(\int_{-\pi}^{\pi}|f(t)|^2 w^2(t)dt\right)^{\frac{1}{2}} + C\|f\|_2\\
&\le C''\left(\int_{-\pi}^{\pi}|f(t)|^2 w^2(t)dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

□

Now, we give a sufficient condition on $w(t)$ where $\{w(t)e^{i\lambda_n t}\}$ is a conditional basis for $L^2[-\pi,\pi]$ without assuming the periodicity of the functions $w(t)$ and $f \in L^2[-\pi,\pi]$.

PROPOSITION 2.1. *Assume that a function $w(t)$ on $\mathbf{R}$ satisfies the following conditions,*

(i) $w(t) \ge \delta > 0,\ -\pi \le t \le \pi$,

(ii) $w(t)$ *is unbounded on* $-\pi \le t \le \pi$, *and*

(iii) $w^2(t)$ *satisfies the Muckenhoupt condition* (A_2).

Then the systems

$$\{w(t)e^{int}\}_{n=-\infty}^{\infty} \quad \text{and} \quad \{w(t)^{-1}e^{int}\}_{n=-\infty}^{\infty}$$

are bounded conditional bases for $L^2[-\pi,\pi]$.

PROOF. We use the argument in [S, Ch. II, §11, Example 11.2], which rests on the periodicity of $f \in L^2[-\pi,\pi]$ to evaluate the convolution operator. As we deal with the non-periodic case, it is necessary to evaluate it without using the periodicity of $f \in L^2[-\pi,\pi]$.

Now, recall that the Dirichlet kernel $D_n(s)$ is represented as

$$D_n(s) = \sum_{k=-n}^{n} e^{iks} = \frac{\sin\left(n+\frac{1}{2}\right)s}{\sin\dfrac{s}{2}}.$$

Denote by $S_n(f)$ the partial sum of $f \in L^2[-\pi,\pi]$ associated with the system $\{w(t)e^{int}\}_{n=-\infty}^{\infty}$. Since

$$\{w(t)e^{int}\}_{n=-\infty}^{\infty} \quad \text{and} \quad \{w(t)^{-1}e^{int}\}_{n=-\infty}^{\infty}$$

are biorthogonal, we have

$$
\begin{aligned}
S_n(f)(t) &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)w(s)^{-1} e^{-iks} ds \right) w(t)e^{ikt} \\
&= w(t)\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)w(s)^{-1} \sum_{k=-n}^{n} e^{ik(t-s)} ds \\
&= \frac{1}{\pi} w(t) \int_{-\pi}^{\pi} f(s)w(s)^{-1} \frac{\sin\left(n+\frac{1}{2}\right)(t-s)}{2\sin\left(\frac{t-s}{2}\right)} ds \\
&= \frac{1}{\pi} w(t) \left\{ \int_{-\pi}^{\pi} f(s)w(s)^{-1} \frac{\sin n(t-s)}{2\tan\left(\frac{t-s}{2}\right)} ds + \frac{1}{2} \int_{-\pi}^{\pi} f(s)w(s)^{-1} \cos n(t-s) ds \right\}.
\end{aligned}
$$

It is easy to see that

$$
\left| \int_{-\pi}^{\pi} f(s)w(s)^{-1} \cos n(t-s) ds \right| \le C\|f\|_2 .
$$

Next, if we set

$$
g_1(s) = f(s)w(s)^{-1} \sin ns, \quad g_2(s) = f(s)w(s)^{-1} \cos ns ,
$$

we have

$$
\begin{aligned}
&\left| \frac{1}{\pi} w(t) \int_{-\pi}^{\pi} f(s)w(s)^{-1} \frac{\sin n(t-s)}{2\tan\left(\frac{t-s}{2}\right)} ds \right| \\
&= \left| \frac{1}{\pi} w(t) \int_{-\pi}^{\pi} f(s)w(s)^{-1} \frac{\sin nt \cos ns - \cos nt \sin ns}{2\tan\left(\frac{t-s}{2}\right)} ds \right| \\
&= \left| \frac{1}{\pi} w(t) \sin nt \int_{-\pi}^{\pi} \frac{f(s)w(s)^{-1}\cos ns}{2\tan\left(\frac{t-s}{2}\right)} ds - \frac{1}{\pi} w(t) \cos nt \int_{-\pi}^{\pi} \frac{f(s)w(s)^{-1}\sin ns}{2\tan\left(\frac{t-s}{2}\right)} ds \right| \\
&= \left| w(t) \left(\tilde{g}_2(t) \sin nt - \tilde{g}_1(t) \cos nt \right) \right| \\
&\le w(t) \left(|\tilde{g}_1(t)| + |\tilde{g}_2(t)| \right) .
\end{aligned}
$$

Since $w(t)\,|g_i(t)| \le |f(t)|$ for $i = 1, 2$, it follows from Lemma 2.1 that

$$
\begin{aligned}
&\left\{ \int_{-\pi}^{\pi} w^2(t) \left(|\tilde{g}_1(t)| + |\tilde{g}_2(t)| \right)^2 dt \right\}^{\frac{1}{2}} \\
&\le \left(\int_{-\pi}^{\pi} w^2(t) \, |\tilde{g}_1(t)|^2 \, dt \right)^{\frac{1}{2}} + \left(\int_{-\pi}^{\pi} w^2(t) \, |\tilde{g}_2(t)|^2 \, dt \right)^{\frac{1}{2}} \\
&\le C \left\{ \left(\int_{-\pi}^{\pi} w^2(t) \, |g_1(t)|^2 \, dt \right)^{\frac{1}{2}} + \left(\int_{-\pi}^{\pi} w^2(t) \, |g_2(t)|^2 \, dt \right)^{\frac{1}{2}} \right\} \\
&\le 2C\|f\|_2 .
\end{aligned}
$$

Thus we get

$$\begin{aligned}
&\left(\int_{-\pi}^{\pi} |S_n(f)(t)|^2\,dt\right)^{\frac{1}{2}} \\
&\le C\left[\int_{-\pi}^{\pi} w^2(t)\,\{(|\tilde{g_1}(t)| + |\tilde{g_2}(t)|) + C\|f\|_2\}^2\,dt\right]^{\frac{1}{2}} \\
&\le C\left[\left\{\int_{-\pi}^{\pi} w^2(t)\,(|\tilde{g_1}(t)| + |\tilde{g_2}(t)|)^2\,dt\right\}^{\frac{1}{2}} + C\|f\|_2\left(\int_{-\pi}^{\pi} w^2(t)dt\right)^{\frac{1}{2}}\right] \\
&\le C\left(2C\|f\|_2 + C'\|f\|_2\right) \\
&= C''\|f\|_2\,.
\end{aligned}$$

This result implies that

$$\sup_n \|S_n\| < \infty\,.$$

By the similar way as in the case where $w(t) = |t|^{-\beta}$ $(0 < \beta < 1/2)$ in [S, p. 353], we find that $\{w(t)e^{int}\}_{n=-\infty}^{\infty}$ is complete in $L^2[-\pi,\pi]$ and becomes a bounded conditional basis.

Finally, by biorthogonality, $\{w(t)^{-1}e^{int}\}_{n=-\infty}^{\infty}$ is also a bounded conditional basis. □

THEOREM 2.1. *Let M, L and δ be positive constants. Assume that a function $w(t)$ on* **R** *satisfies the following conditions,*

(i) $w(t) \ge \delta > 0,\ -\pi \le t \le \pi$,

(ii) *$w(t)$ is unbounded and $|t|w(t) \le M,\ -\pi \le t \le \pi$, and*

(iii) *$w^2(t)$ satisfies the Muckenhoupt condition (A_2).*

If a sequence $\{\lambda_n\}$ of complex numbers satisfies the inequality $|\lambda_n - n| \le L$ with

$$0 < L < \frac{1}{\pi}\log\left(\frac{\pi\delta}{M} + 1\right),$$

then $\{w(t)e^{i\lambda_n t}\}_{n=-\infty}^{\infty}$ forms a bounded conditional basis for $L^2[-\pi,\pi]$.

PROOF. We see by Proposition 2.1 that the system $\{w(t)e^{int}\}_{n=-\infty}^{\infty}$ is a bounded conditional basis for $L^2[-\pi,\pi]$. By regarding [S, p. 84, Theorem 9.1], it suffices to find a constant θ with $0 < \theta < 1$ such that

$$\left\|\sum_n c_n w(t)\left(e^{int} - e^{i\lambda_n t}\right)\right\|_2 \le \theta\left\|\sum_n c_n w(t)e^{int}\right\|_2 \tag{2.2}$$

for any finite sequence $\{c_n\}$ of complex numbers. By virtue of (i), we observe

$$\left\|\sum_n c_n e^{int}\right\|_2 \le \frac{1}{\delta}\left\|\sum_n c_n w(t)e^{int}\right\|_2 . \tag{2.3}$$

By the same way as in the proof of [Y1, Theorem 1], we have

$$\begin{aligned}\left\|\sum_n c_n w(t)\left(e^{int}-e^{i\lambda_n t}\right)\right\|_2 &= \left\|\sum_n c_n w(t)\left(e^{i(\lambda_n-n)t}-1\right)e^{int}\right\|_2\\ &= \left\|\sum_n c_n w(t)\sum_{k=1}^{\infty}\frac{\{i(\lambda_n-n)\}^k}{k!}t^k e^{int}\right\|_2\\ &= \left\|\sum_{k=1}^{\infty}\sum_n c_n w(t)\frac{\{i(\lambda_n-n)\}^k}{k!}t^k e^{int}\right\|_2\\ &\le \sum_{k=1}^{\infty}\left\|\frac{t^k w(t)}{k!}\sum_n c_n\{i(\lambda_n-n)\}^k e^{int}\right\|_2 .\end{aligned} \tag{2.4}$$

We also obtain

$$|t^k w(t)| = |t^{k-1}|\cdot|t|w(t) \le \pi^{k-1}\cdot M \tag{2.5}$$

by the condition (ii). Combining (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned}\left\|\sum_n c_n w(t)\left(e^{int}-e^{i\lambda_n t}\right)\right\|_2 &\le \sum_{k=1}^{\infty}\frac{\pi^{k-1}\cdot M}{k!}\left\|\sum_n c_n\{i(\lambda_n-n)\}^k e^{int}\right\|_2\\ &= \frac{M}{\pi}\sum_{k=1}^{\infty}\frac{\pi^k}{k!}\left(\sum_n |c_n|^2\cdot|\lambda_n-n|^{2k}\right)^{\frac{1}{2}}\\ &\le \frac{M}{\pi}\sum_{k=1}^{\infty}\frac{\pi^k\cdot L^k}{k!}\left(\sum_n |c_n|^2\right)^{\frac{1}{2}}\\ &= \frac{M}{\pi}\left(e^{\pi L}-1\right)\left\|\sum_n c_n e^{int}\right\|_2\\ &\le \frac{M}{\pi\delta}\left(e^{\pi L}-1\right)\left\|\sum_n c_n w(t)e^{int}\right\|_2 .\end{aligned}$$

Observe that L satisfies

$$\frac{M}{\pi\delta}\left(e^{\pi L}-1\right) < 1 .$$

If we take

$$\theta = \frac{M}{\pi\delta}\left(e^{\pi L}-1\right) ,$$

then (2.2) holds. □

REMARK 2.1. We remark that $\{w(t)e^{i\lambda_n t}\}$ and $\{w(t)^{-1}e^{i\lambda_n t}\}$ are not biorthogonal in general. Therefore, $\{w(t)^{-1}e^{i\lambda_n t}\}$ is not necessary a conditional basis even if $\{w(t)e^{i\lambda_n t}\}$ is a conditional basis.

PROBLEM A (e.g., see [Y2, p. 165]). *Is there a basis of complex exponentials* $\{e^{i\lambda_n t}\}$ *that is a conditional basis for* $L^2[-\pi,\pi]$?

Recently, a partial answer to this problem is obtained by [N]. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be defined by

$$\lambda_n = \begin{cases} n-\alpha\,, & n>0\,,\\ n+\alpha\,, & n<0\,, \end{cases} \tag{2.6}$$

and

$$\mu_n = \begin{cases} n+\alpha\,, & n>0\,,\\ 0\,, & n=0\,,\\ n-\alpha\,, & n<0\,, \end{cases} \tag{2.7}$$

for $0<\alpha<1$. If the completeness is deleted from the definition of a Riesz basis, such a sequence is said to be a *Riesz sequence*. A *basic sequence* is similarly defined.

THEOREM C ([N, Corollary 2.1]). *Let* $\{\gamma_n\}$ *be a sequence given by* (2.6) *or* (2.7), *then* $\{e^{i\gamma_n t}\}$ *is either a Riesz sequence or not a basic sequence in* $L^2[-\pi,\pi]$.

In other words, if the systems $\{e^{i\lambda_n t}\}$ and $\{e^{i\mu_n t}\}$ for the sequences given by (2.6) and (2.7) are bases, they are Riesz bases. Consequently, Problem A has a negative solution for these sequences. It is a problem whether the same results hold or not about other sequences.

PROBLEM 2.1. *We suppose that* $w(t)$ *satsisfies the conditions* (i) *and* (iii) *in Theorem* 2.1. *If the system* $\{w(t)e^{i\lambda_n t}\}$ *is a conditional basis for* $L^2[-\pi,\pi]$, *then is the function* $w(t)$ *unbounded on* $-\pi\le t\le\pi$?

If it is affirmative for $\{\lambda_n\}$ such that $|\lambda_n - n|\le L$ with a positive constant L, we see that every basis $\{e^{i\lambda_n t}\}$ for $L^2[-\pi,\pi]$ is an unconditional basis, i.e., a Riesz basis. If it is negative, in other words, there exists such a bounded function $w(t)$, then $\{e^{i\lambda_n t}\}$ is a conditional basis for $L^2[-\pi,\pi]$.

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