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Trace Formula for Partial Isometry Case

Dedicated to Professors Shôichi Ôta and Mitsuru Uchiyama on their sixtieth birthdays

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Abstract. Let $T = U|T|$ be the polar decomposition of T. For a semi-hyponormal operator $T = U|T|$ with partial isometry U, if $|T| - |T^*| \in \mathcal{C}_1$, then we give the trace formula for the polar decomposition of T.

1. Introduction

Let $T = U|T|$ be an operator with partial isometry U and put $Q = |T| - |T^*|$. Then $U|T| = (|T| - Q)U$. If Q is a trace class operator, Carey-Pincus' Theorem [2] gives a trace formula associated with the decomposition $T = U|T|$. In this paper, using a result [4], we give a simple proof of the trace formula of semi-hyponormal operator.

An operator below means a bounded linear operator on a separable infinite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{C}_1$ be the set of all trace class operators. An operator T is said to be semi-hyponormal if $(T^*T)^{1/2} \geq (TT^*)^{1/2}$, that is, $|T| \geq |T^*|$. For a polynomials $p(r) = \sum_{k=0}^N a_k r^k$, put $p(|T|) = a_0 + \sum_{k=1}^N a_k |T|^k$. $\phi(r, z)$ is said to be *Laurent polynomial* if there exist a non-negative integer N and polynomials $p_k(r)$ such that $\phi(r, z) = \sum_{k=-N}^N p_k(r) z^k$. Put $\phi(|T|, U) = \sum_{k=-N}^{-1} p_k(|T|) U^{*|k|} + p_0(|T|) + \sum_{k=1}^N p_k(|T|) U^k$.

Let $\mathcal{A}$ be the linear space of all Laurent polynomials. For differentiable functions ϕ, ψ of two variables (r, z), let $J(\phi, \psi)(r, z) = \phi_r(r, z) \cdot \psi_z(r, z) - \phi_z(r, z) \cdot \psi_r(r, z)$ be the Jacobian of ϕ and ψ. Then we have the following.

THEOREM A ([5, Theorem 7]). *Let $T = U|T|$ be a semi-hyponormal operator with unitary U and $[|T|, U] \in \mathcal{C}_1$. Then there exists a summable function g and it holds*

$$\mathrm{Tr}([\phi(|T|, U), \psi(|T|, U)]) = \frac{1}{2\pi} \iint J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g(r\cos\theta, r\sin\theta) dr d\theta$$

for any $\phi, \psi \in \mathcal{A}$.

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If $A \in \mathcal{C}_1$, then $\mathrm{Tr}([A, B]) = 0$ for any operator B. This implies that if $[X, Y], [X, Z], [Y, Z] \in \mathcal{C}_1$, then $[XY, Z] \in \mathcal{C}_1$, $[YX, Z] \in \mathcal{C}_1$ and $\mathrm{Tr}([XY, Z]) = \mathrm{Tr}([YX, Z])$.

If $|T| - |T^*| \in \mathcal{C}_1$, then

$$\begin{aligned}
[|T|, U] &= (|T| - |T^*|)U \in \mathcal{C}_1 ,\\
|T|(I - F) &= (|T| - U^*|T|U) + (U^*|T|U - |T|UU^*)\\
&= (|T|U^* - U^*|T|)U + [U^*, |T|U]\\
&= [|T|, U^*]U + [U^*, (|T|U - U|T|)] + [U^*, U|T|]\\
&= [U, |T|]^*U + [U^*, -[U, |T|]] + (|T| - |T^*|) \in \mathcal{C}_1 .
\end{aligned}$$

Hence,

$$\mathrm{Tr}([XU|T|U^*Y, Z]) = \mathrm{Tr}([X|T|UU^*Y, Z]) = \mathrm{Tr}([X|T|FY, Z]) = \mathrm{Tr}([X|T|Y, Z]) .$$

We also have that $\mathrm{Tr}([XU^*|T|UY, Z]) = \mathrm{Tr}([X|T|Y, Z])$. In this case, we consider Laurent polynomials such that $\sum_{k=-N}^{-1} p_k(|T|)U^{*|k|} + p_0(|T|) + \sum_{k=1}^{N} p_k(|T|)U^k$ with polynomial $p_k(0) = 0$ for $k = -N, -N + 1, \dots, N$. In addition, if $[U, U^*] \in \mathcal{C}_1$, we consider Laurent polynomials such that $\sum_{k=-N}^{-1} p_k(|T|)U^{*|k|} + p_0(|T|) + \sum_{k=1}^{N} p_k(|T|)U^k$ with polynomial p_k.

THEOREM 1. *Let $T = U|T|$ be a semi-hyponormal operator with $|T| - |T^*| \in \mathcal{C}_1$. Then there exists a summable function g and it holds*

$$\mathrm{Tr}([\phi(|T|, U), \psi(|T|, U)]) = \frac{1}{2\pi} \iint J(\phi, \psi)(r, e^{i\theta})e^{i\theta} g(r\cos\theta, r\sin\theta)drd\theta ,$$

where $\phi(r, z)$, $\psi(r, z)$ are Laurent polynomials such that $\phi(0, z) = \psi(0, z) = 0$

2. Proof

PROOF OF THEOREM 1. Let T act on a Hilbert space $\mathcal{H}$. Let $T = U|T|$ be the polar decomposition of T and let $E = U^*U$, $F = UU^*$. Put $V = U + (I - E)$. Then $V = U_{|E(\mathcal{H})} \oplus (I - E)$, $V|T| = U|T|$ and $|T|V = |T|U$.

Put $\mathbf{H} = \mathcal{H} \oplus \mathcal{H}$. Define operators $\tilde{U}$, $|\tilde{T}|$ and $\tilde{T}$ on $\mathbf{H}$ by

$$\tilde{U} = \begin{pmatrix} V & I - VV^* \\ 0 & -V^* \end{pmatrix}, \quad |\tilde{T}| = \begin{pmatrix} |T| & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{T} = \tilde{U}|\tilde{T}| .$$

Then $\tilde{U}$ is a unitary operator. We obtain

$$[V, |T|] = [U, |T|] = U|T|U^*U - |T|U = (|T^*| - |T|)U \in \mathcal{C}_1$$

Hence

$$[\tilde{U}, |\tilde{T}|] = \begin{pmatrix} [V, |T|] & -|T|(I - VV^*) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [V, |T|] & -|T|(I - F) \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_1.$$

We have for $n \geq 1$,

$$\tilde{U}^n|\tilde{T}| = \begin{pmatrix} U^n|T| & 0 \\ 0 & 0 \end{pmatrix}, \quad |\tilde{T}|\tilde{U}^n = \begin{pmatrix} |T|U^n & * \\ 0 & 0 \end{pmatrix}.$$

It also holds that

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} XA & XB \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} AX & 0 \\ CX & 0 \end{pmatrix}$$

Hence, for a positive integer n, we have

$$\mathrm{Tr}_{\mathbf{H}}[\tilde{U}^{*n}|\tilde{T}|, \tilde{U}^*|\tilde{T}|] = \mathrm{Tr}_{\mathbf{H}}[|\tilde{T}|\tilde{U}^{*n}, \tilde{U}^*|\tilde{T}|]$$

$$= \mathrm{Tr}[|T|U^{*n}, U^*|T|] = \mathrm{Tr}[U^{*n}|T|, U^*|T|]\,.$$

Define

$$U^{[n]} = \begin{cases} U^{*|n|} & (n < 0) \\ I & (n = 0) \\ U^n & (n > 0)\,. \end{cases}$$

It is easy to check that, for integers $m,\ p$, positive integers $n,\ q$,

$$\mathrm{Tr}_{\mathbf{H}}([\tilde{U}^m|\tilde{T}|^n, \tilde{U}^p|\tilde{T}|^q]) = \mathrm{Tr}([U^{[m]}|T|^n, U^{[p]}|T|^q])\,. \tag{1}$$

Since $n,\ q > 0$ and $\phi(r,z)$, $\psi(r,z)$ are Laurent polynomials with $\phi(0,z) = \psi(0,z) = 0$, by (1) we have

$$\mathrm{Tr}_{\mathbf{H}}([\phi(|\tilde{T}|, \tilde{U}), \psi(|\tilde{T}|, \tilde{U})]) = \mathrm{Tr}([\phi(|T|, U), \psi(|T|, U)])\,. \tag{2}$$

Since $\tilde{T} = \tilde{U}|\tilde{T}|$ is semi-hyponormal with unitary $\tilde{U}$ and $[|\tilde{T}|, \tilde{U}] \in \mathcal{C}_1$, by Theorem A there exists a summable function g and it holds

$$\mathrm{Tr}_{\mathbf{H}}([\phi(|\tilde{T}|, \tilde{U}), \psi(|\tilde{T}|, \tilde{U})]) = \frac{1}{2\pi}\iint J(\phi,\psi)(r, e^{i\theta})e^{i\theta}g(r\cos\theta, r\sin\theta)drd\theta\,.$$

Hence, by (2) we obtain

$$\mathrm{Tr}([\phi(|T|, U), \psi(|T|, U)]) = \frac{1}{2\pi}\iint J(\phi,\psi)(r, e^{i\theta})e^{i\theta}g(r\cos\theta, r\sin\theta)drd\theta\,.$$

□

COROLLARY 2. *Let $T = U|T|$ be a semi-hyponormal operator with $|T| - |T^*| \in \mathcal{C}_1$ and $[U^*, U] \in \mathcal{C}_1$. Then there exists a summable function g and it holds*

$$\mathrm{Tr}([\phi(|T|, U), \psi(|T|, U)]) = \frac{1}{2\pi}\iint J(\phi,\psi)(r, e^{i\theta})e^{i\theta}g(r\cos\theta, r\sin\theta)drd\theta\,,$$

where $\phi(r,z)$, $\psi(r,z)$ are Laurent polynomials with $\phi(0,z) = 0$.

PROOF. (1) holds for $n,\ q \geq 0$ and $n+q \geq 1$. Hence, (2) holds for Laurent polynomials with $\phi(0,z)=0$ and $\psi(0,z) \neq 0$. □

REMARK. Let T be the unilateral shift on ℓ^2 and $T=U|T|$ be the polar decomposition of T. Hence $T=U$ and $|T|=I$. Let $\phi(r,z)=z^{-1}$ and $\psi(r,z)=z$. Then $\phi(0,z)\neq 0$ and $\psi(0,z)\neq 0$. And we have $J(\phi,\psi)=0$. On the other hand,

$$\mathrm{Tr}([\phi(|T|,U),\psi(|T|,U)])=\mathrm{Tr}([U^*,U])=1\,.$$

Hence the trace formula does not hold for ϕ and ψ.

For the Cartesian decomposition $T=X+iY$, we show the following. Proof is similar to the proof of Theorem 1 of [6]. For the completeness, we give a proof.

THEOREM 3. *Let $T=X+iY$ be the Cartesian decomposition of a semi-hyponormal operator T. If $|T|-|T^*|\in\mathcal{C}_1$, then there exists a summable function g_T and it holds*

$$\mathrm{Tr}([P(X,Y),Q(X,Y)])=\frac{1}{2\pi i}\iint J(P,Q)(x,y)g_T(x,y)dxdy\,,$$

for polynomials P and Q.

PROOF. Let $T=U|T|$ be the polar decomposition of T. Let P and Q be polynomials of two variables (x,y). According to the commutator and the Jacobian, we may assume $P(0,0)=Q(0,0)=0$. We note that

$$\mathrm{Tr}([P(X,Y),Q(X,Y)])=\mathrm{Tr}\left(\left[P\left(\frac{T+T^*}{2},\frac{T-T^*}{2i}\right),Q\left(\frac{T+T^*}{2},\frac{T-T^*}{2i}\right)\right]\right).$$

Put

$$\tilde{P}(r,z)=P\left(\frac{zr+r/z}{2},\frac{rz-r/z}{2i}\right)\quad\text{and}\quad\tilde{Q}(r,z)=Q\left(\frac{zr+r/z}{2},\frac{rz-r/z}{2i}\right).$$

Then both $\tilde{P}$ and $\tilde{Q}$ are Laurent polynomials with $\tilde{P}(0,z)=\tilde{Q}(0,z)=0$ and also the following equations hold:

$$\begin{aligned}
\tilde{P}_r(r,z)&=P_x(r,z)\frac{z+1/z}{2}+P_y(r,z)\frac{z-1/z}{2i}\,,\\
\tilde{P}_z(r,z)&=\frac{r}{2}P_x(r,z)\left(1-\frac{1}{z^2}\right)+\frac{r}{2i}P_y(r,z)\left(1+\frac{1}{z^2}\right),\\
\tilde{Q}_r(r,z)&=Q_x(r,z)\frac{z+1/z}{2}+Q_y(r,z)\frac{z-1/z}{2i}\,,\\
\tilde{Q}_z(r,z)&=\frac{r}{2}Q_x(r,z)\left(1-\frac{1}{z^2}\right)+\frac{r}{2i}Q_y(r,z)\left(1+\frac{1}{z^2}\right).
\end{aligned}$$

Hence we obtain

$$J(\tilde{P},\tilde{Q})(r,z)=J(P,Q)(x,y)\,\frac{r}{zi}\,.$$

Therefore, it holds

$$J(\tilde{P}, \tilde{Q})(r, e^{i\theta}) = J(P, Q)(x, y)\, \frac{r}{ie^{i\theta}}\,. \tag{3}$$

Since $P(X, Y) = \tilde{P}(|T|, U)$ and $Q(X, Y) = \tilde{Q}(|T|, U)$, it holds

$$\mathrm{Tr}([P(X, Y), Q(X, Y)]) = \mathrm{Tr}([\tilde{P}(|T|, U), \tilde{Q}(|T|, U)])\,, \tag{4}$$

Since $\tilde{P}(0, z) = \tilde{Q}(0, z) = 0$, by (3), (4) and Theorem 1, we have

$$\begin{aligned}\mathrm{Tr}([P(X, Y), Q(X, Y)]) &= \mathrm{Tr}([\tilde{P}(|T|, U), \tilde{Q}(|T|, U)])\\ &= \frac{1}{2\pi}\iint J(\tilde{P}, \tilde{Q})(r, e^{i\theta})e^{i\theta} g(r\cos\theta, r\sin\theta)drd\theta\\ &= \frac{1}{2\pi i}\iint J(P, Q)(x, y)g(r\cos\theta, r\sin\theta)\, r\, drd\theta\,.\end{aligned}$$

Put $g_T(x, y) = g(r\cos\theta, r\sin\theta)$ for $x + iy = re^{i\theta}$. Using the transformation $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\iint J(P, Q)(x, y)g(r\cos\theta, r\sin\theta)rdrd\theta\\ &\quad = \frac{1}{2\pi i}\iint J(P, Q)(x, y)g_T(x, y)dxdy = \mathrm{Tr}([P(X, Y), Q(X, Y)])\,.\end{aligned}$$

Hence, it completes the proof. □

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