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Limit Distribution of a One-dimensional Reflecting Process of Jump Type

Yasumasa SAISHO

Hiroshima University

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Abstract. We construct a one dimensional stochastic process with reflecting barrier of jump type and investigate a limit distribution and stationary distribution of the process.

1. Introduction

In applications of the theory of stochastic processes, it is in some instances important to get limit distributions or stationary (invariant) distributions of the processes. For example, Kawamura and Saisho [2] treated a process representing the accumulation of toxic substances in human bodies and it was necessary to consider the steady-state of the accumulation. Though limit distributions of one-dimensional Markov processes have been well investigated (e.g. [5]), the limit distributions or stationary distributions of stochastic processes with reflecting barriers described by Skorohod equations do not seem to be studied enough yet. Saisho and Tanaka [4] considered reflecting Brownian motions on multi-dimensional domains and showed symmetry of the processes with stationary distributions using the so-called penalty method. Kawamura and Saisho [2] also studied the limit distribution of one-dimensional reflecting process representing a human metabolic system by using the penalty method. The limit distribution of the associated penalized process and the stationary distribution of the reflecting process were given ([2] §2.2). However these previous studies were not successful in determining the limit distribution of the reflecting processes. Our goal in the the present study is to provide a simple but not trivial example of a stochastic process with reflecting barriers that has a limit distribution: we construct a one-dimensional stochastic process of the jump type with reflecting barriers and find the limit distribution of it. Our process can be regarded as a mathematical model of some queuing system or a simple model of a neuronal excitation system (see §2).

In §2 we construct our process and discuss the limit distribution and stationary distribution of the process. Examples of some applications of our process are also given in this

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section. We find the limit distribution of the jump distances of the process in §3. The stationary distribution is also studied in this section.

2. Distribution of X_{τ_n}

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{Y_t\}_{t\geq 0}$ be a Poisson process with parameter $\lambda\,(>0)$, $Y_0 = 0$ on $(\Omega, \mathcal{F}, P)$, i.e., Y_t is Poisson-distributed with parameter λt and set $Z_t := Y_t - \gamma t$, $t \geq 0$ for some constant $\gamma(>0)$. Let $\tau_1 < \tau_2 < \cdots$ be jumping times of $\{Y_t\}_{t\geq 0}$ and set $\tau_0 \equiv 0$.

REMARK 1. $\tau_k - \tau_{k-1}$, $k = 1, 2, \ldots$ are independently and identically distributed $(i.i.d.)$ exponential distributed random variables of parameter λ.

Now we define $\{X_t\}_{t\geq 0}$ as follows:

$$\begin{aligned}
&X_0 = 0\,,\\
&X_{\tau_k} = \min\{X_{\tau_k -} + 1,\ K\},\ k = 1, 2, \ldots\,,\\
&X_t = \max\left\{X_{\tau_k} - \gamma(t - \tau_k),\ 0\right\},\ \tau_k \leq t < \tau_{k+1},\ k = 0, 1, 2, \ldots\,,
\end{aligned}$$

where K is a positive constant. We denote

$$\triangle X_t := X_t - X_{t-},\ t \geq 0\,, \quad \triangle\tau_k := \tau_k - \tau_{k-1},\ k = 1, 2, \ldots\,.$$

REMARK 2. $\{X_t\}_{t\geq 0}$ can be considered to be a solution of the following one-dimensional Skorohod equation of the interval $I := (0,\ K)$ for $\{Z_t\}_{t\geq 0}$:

$$X_t = Z_t + \Phi_t\,, \quad t \geq 0\,,$$

where X and Φ satisfy the following conditions:

$$\begin{cases}
X_t \in \overline{I} = [0,\ K]\,,\\
\Phi_t \text{ is of bounded variation with } \Phi_0 = 0\,,\\
\Phi_t = \displaystyle\int_0^t \boldsymbol{n}(X_s)d|\Phi|_s\,, \quad |\Phi|_t = \int_0^t \mathbf{1}_{\{s|X_s\in\partial I\}}(s)d|\Phi|_s\,,\\
\boldsymbol{n}(x) = \begin{cases} 1\,, & x = 0\,,\\ -1\,, & x = K\,. \end{cases}
\end{cases}$$

Here, $\mathbf{1}_A(s) = 1\ (s \in A), = 0\ (s \notin A)$. The uniqueness of the solution of the Skorohod equation follows from Lemma 2.2 and Lemma 2.3 (i) of [3]. For more details of the Skorohod equations, see [3] and [6].

REMARK 3. Since we can write

$$P(X_t \leq x) = \sum_{k=0}^{\infty} P(X_{\tau_k} \leq x + \gamma y,\ 0 < y < \tau_{k+1} - \tau_k)\,,$$

we see that the distribution of X_t is determined by the distributions of $\{X_{\tau_n}\}_{n=1,2,\dots}$ and $\{\tau_{n+1} - \tau_n\}_{n=0,1,2,\dots}$.

We investigate herein the distribution of X_t in the case that $K = 2$. However, we now give two examples of our model for general $K > 0$. For $K = 1$ (Example 1), the process is very simple and we have the limit distribution of the process.

EXAMPLE 1. It is known that the intervals of applications of stimuli to cells can be naturally regarded as independent exponentially distributed random variables and that for two successive stimuli separated by a brief interval (refractory period), the response (action potential) to the second stimulus is very small or almost negligible. Furthermore, it is known that the strength of the action potential for the second stimulus depends only on the length of the interval between the two stimuli and is independent of the strength of the given stimulus (for more details see [1]). In case $K = 1$, $\{\triangle X_{\tau_n}\}_{n=1,2,\dots}$ is regarded as a simple mathematical model describing the action potentials of the cell. In this context we are also interested in the distribution of $\{X_{\tau_n}\}_{n=1,2,\dots}$ and that of jump distances $\{\triangle X_{\tau_n}\}_{n=1,2,\dots}$ for the process. Indeed, if $K = 1$ we have $X_{\tau_0} = 0,\ X_{\tau_n} = 1,\ n = 1, 2, 3, \dots$ and

$$\triangle X_{\tau_1} = 1\,,$$

$$\triangle X_{\tau_n} = \begin{cases} 1\,, & \gamma\triangle\tau_n \geq 1 \\ \gamma\triangle\tau_n\,, & \text{otherwise}\,, \quad n = 2, 3, 4, \dots . \end{cases}$$

Thus, we have

$$\begin{aligned} P(\triangle X_{\tau_n} = 1) &= P\left(\triangle\tau_n \geq \frac{1}{\gamma}\right) \\ &= \int_{1/\gamma}^{\infty} \lambda e^{-\lambda x} dx \\ &= e^{-\lambda/\gamma}\,, \quad n = 2, 3, \dots\,, \end{aligned}$$

$$P(\triangle X_{\tau_n} = \gamma\triangle\tau_n) = 1 - e^{-\lambda/\gamma}\,, \quad n = 2, 3, \dots\,,$$

which shows that this process has a limit distribution.

EXAMPLE 2. We consider the following queuing system:

1. there is only one server,
2. time intervals of arrival of customers are mutually independent and exponentially distributed (*i.i.d.*),
3. each customer brings in a constant quantity of work unless the amount of task that the server is performing at that instant is under constant amount K,
4. the server performs a constant task γ per hour.

This system can therefore be expressed by $\{X_t\}_{t\geq 0}$. That is, X_t describes the size of the task that the server is performing at time t. We are interested in the stationary state of the system, that is, the distribution of $\{X_t\}_{t\geq 0}$ for sufficiently large t.

In what follows we assume $K = 2$ and put $\nu := \frac{\lambda}{\gamma}$. In this case, we have $X_{\tau_0} = 0$, $X_{\tau_1} = 1$,

$$X_{\tau_2} = \begin{cases} 1\,, & \text{if } \gamma\triangle\tau_2 \geq 1\,, \\ 2-\gamma\triangle\tau_2\,, & \text{if } \gamma\triangle\tau_2 < 1\,, \end{cases}$$

$$X_{\tau_n} = \begin{cases} 2\,, & \text{if } X_{\tau_{n-1}} - \gamma\triangle\tau_n \geq 1\,, \\ X_{\tau_{n-1}} - \gamma\triangle\tau_n + 1\,, & \text{if } 0 \leq X_{\tau_{n-1}} - \gamma\triangle\tau_n < 1\,, \\ 1\,, & \text{if } X_{\tau_{n-1}} - \gamma\triangle\tau_n < 0\,, \quad n = 3, 4, \dots\,, \end{cases}$$

$\triangle X_{\tau_1} \equiv 1,\ \triangle X_{\tau_2} \equiv 1$ and

$$\triangle X_{\tau_n} = \begin{cases} 2 - X_{\tau_{n-1}} + \gamma\triangle\tau_n\,, & \text{if } X_{\tau_{n-1}} - \gamma\triangle\tau_n \geq 1\,, \\ 1\,, & \text{otherwise}\,, \quad n = 3, 4, \dots\,. \end{cases}$$

Next, we write

$$p(x, y) := P(X_{\tau_{k+1}} = y | X_{\tau_k} = x)\,,$$
$$p(x, \Lambda) := P(X_{\tau_{k+1}} \in \Lambda | X_{\tau_k} = x)\,, \quad x, y \in \mathbf{R},\ \Lambda \subset (1, 2)\,.$$

Then we have the following lemma.

LEMMA 1.

$$p(2, 2) = 1 - e^{-\nu}\,,$$
$$p(2, (a, b]) = e^{-(3-b)\nu} - e^{-(3-a)\nu}\,, \quad 1 < a < b < 2\,,$$
$$p(2, 1) = e^{-2\nu}\,,$$
$$p(x, 2) = 1 - e^{-(x-1)\nu}\,, \quad 1 < x < 2\,,$$
$$p(x, 1) = e^{-x\nu}, \quad 1 < x < 2\,,$$
$$p(1, 2) = 0\,,$$
$$p(1, (a, b]) = e^{-(2-b)\nu} - e^{-(2-a)\nu}\,, \quad 1 < a < b < 2\,,$$
$$p(1, 1) = e^{-\nu}\,.$$

PROOF. For $p(2, 2)$ we have

$$p(2, 2) = P(\gamma\triangle\tau_{k+1} \leq 1)$$

$$= \int_0^{1/\gamma} \lambda e^{-\lambda x} dx = 1 - e^{-\nu} .$$

For $p(2, (a, b])$ we have

$$\begin{aligned} p(2, (a, b]) &= P(a < 2 - \gamma \triangle\tau_{k+1} + 1 \le b) \\ &= P(3 - b \le \gamma \triangle\tau_{k+1} < 3 - a) = e^{-(3-b)\nu} - e^{-(3-a)\nu} . \end{aligned}$$

Similarly we have

$$\begin{aligned} p(2, 1) &= P(\gamma \triangle\tau_{k+1} \ge 2) = e^{-2\nu} , \\ p(x, 2) &= P(x - 1 \ge \gamma \triangle\tau_{k+1}) = 1 - e^{-(x-1)\nu} , \\ p(x, 1) &= P(\gamma \triangle\tau_{k+1} \ge x) = e^{-x\nu} , \\ p(1, (a, b]) &= P(2 - b \le \gamma \triangle\tau_{k+1} < 2 - a) = e^{-(2-b)\nu} - e^{-(2-a)\nu} , \\ p(1, 1) &= P(\gamma \triangle\tau_{k+1} \ge 1) = e^{-\nu} . \end{aligned}$$

The fact that $p(1, 2) = 0$ follows immediately from the definition of X_t. The proof is complete. □

From Lemma 1, we have the following theorem.

THEOREM 1.

$$\begin{aligned} p_{\tau_k}^{(1)} &:= P(X_{\tau_k} = 1) = \frac{e^{-2\nu} - e^{-2\nu}(\nu e^{-\nu})^{k-2}}{1 - \nu e^{-\nu}} + \nu^{k-2} e^{-(k-1)\nu} , \\ p_{\tau_k}^{(2)} &:= P(X_{\tau_k} = 2) = 1 - \frac{e^{-\nu} - e^{-\nu}(\nu e^{-\nu})^{k-2}}{1 - \nu e^{-\nu}} - \nu^{k-2} e^{-(k-2)\nu} , \\ P(a < X_{\tau_k} \le b) &= \int_a^b p_{\tau_k}(x) dx \\ &= \nu^{k-2}\{e^{-(k-b)\nu} - e^{-(k-a)\nu}\} \\ &\quad + \frac{1}{1 - \nu e^{-\nu}}\{e^{-(3-b)\nu} - e^{-(3-a)\nu}\}\{1 - (\nu e^{-\nu})^{k-2}\} , \\ &\qquad 1 < a < b < 2 , \quad k \ge 2 , \end{aligned}$$

where

$$p_{\tau_k}(x) := \frac{\nu e^{-(3-x)\nu} - \nu e^{-(3-x)\nu}(\nu e^{-\nu})^{k-2}}{1 - \nu e^{-\nu}} + \nu^{k-1} e^{-(k-x)\nu} .$$

PROOF. We prove the theorem by induction. For $k = 2$, we have $P(X_{\tau_2} = 1) = P(\gamma \triangle\tau_2 \ge 1) = e^{-\nu}$ and $P(X_{\tau_2} = 2) = 0$. For $1 < a < b < 2$, we have

$$P(a < X_{\tau_2} \le b) = P(a < 2 - \gamma \triangle\tau_2 \le b,\ \gamma \triangle\tau_2 < 1)$$

$$= \int_{(2-b)/\gamma}^{(2-a)/\gamma} \lambda e^{-\lambda x} dx = e^{-(2-b)\nu} - e^{-(2-a)\nu}.$$

Thus, the statement holds for $k = 2$. Now we suppose that the statement holds for k (≥ 2). Then we have

$$\begin{aligned}
P(X_{\tau_{k+1}} = 1) &= p(1,1) \cdot P(X_{\tau_k} = 1) + p(2,1) \cdot P(X_{\tau_k} = 2) \\
&\quad + \int_1^2 p(x,1) P(x < X_{\tau_k} \leq x + dx) \\
&= e^{-\nu} \left\{ \frac{e^{-2\nu} - e^{-2\nu}(\nu e^{-\nu})^{k-2}}{1 - \nu e^{-\nu}} + \nu^{k-2} e^{-(k-1)\nu} \right\} \\
&\quad + e^{-2\nu} \left\{ 1 - \frac{e^{-\nu} - e^{-\nu}(\nu e^{-\nu})^{k-2}}{1 - \nu e^{-\nu}} - \nu^{k-2} e^{-(k-2)\nu} \right\} \\
&\quad + \int_1^2 e^{-x\nu} \left\{ \frac{\nu e^{-(3-x)\nu} - \nu e^{-(3-x)\nu}(\nu e^{-\nu})^{k-2}}{1 - \nu e^{-\nu}} + \nu^{k-1} e^{-(k-x)\nu} \right\} dx \\
&= e^{-2\nu} + \frac{\nu e^{-3\nu} - \nu e^{-3\nu}(\nu e^{-\nu})^{k-2}}{1 - \nu e^{-\nu}} + \nu^{k-1} e^{-k\nu} \\
&= \frac{e^{-2\nu} - e^{-2\nu}(\nu e^{-\nu})^{k-1}}{1 - \nu e^{-\nu}} + \nu^{k-1} e^{-k\nu}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
P(X_{\tau_{k+1}} = 2) &= p(1,2) \cdot P(X_{\tau_k} = 1) + p(2,2) \cdot P(X_{\tau_k} = 2) \\
&\quad + \int_1^2 p(x,2) P(x < X_{\tau_k} \leq x + dx) \\
&= 1 - \frac{e^{-\nu} - e^{-\nu}(\nu e^{-\nu})^{k-1}}{1 - \nu e^{-\nu}} - \nu^{k-1} e^{-(k-1)\nu}.
\end{aligned}$$

Furthermore, for any $1 < a < b < 2$, we have

$$\begin{aligned}
P(a < X_{\tau_{k+1}} \leq b) &= p(1,(a,b]) \cdot P(X_{\tau_k} = 1) + p(2,(a,b]) \cdot P(X_{\tau_k} = 2) \\
&\quad + \int_1^2 P(a < x - \gamma \triangle \tau_{k+1} + 1 \leq b) P(x < X_{\tau_k} \leq x + dx)
\end{aligned}$$

$$
\begin{aligned}
&= \{e^{-(2-b)\nu} - e^{-(2-a)\nu}\}\left\{\frac{e^{-2\nu} - e^{-2\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}} + \nu^{k-2}e^{-(k-1)\nu}\right\} \\
&\quad + \{e^{-(3-b)\nu} - e^{-(3-a)\nu}\}\left\{1 - \frac{e^{-\nu} - e^{-\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}} - \nu^{k-2}e^{-(k-2)\nu}\right\} \\
&\quad + \int_1^2 \{e^{-(x+1-b)\nu} - e^{-(x+1-a)\nu}\} \\
&\qquad \cdot \left\{\frac{\nu e^{-(3-x)\nu} - \nu e^{-(3-x)\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}} + \nu^{k-1}e^{-(k-x)\nu}\right\}dx \\
&= \nu^{k-1}\{e^{-(k+1-b)\nu} - e^{-(k+1-a)\nu}\} \\
&\quad + \frac{1}{1-\nu e^{-\nu}}[(1-\nu e^{\nu})\{e^{-(3-b)\nu} - e^{-(3-a)\nu}\} \\
&\quad + \nu\{e^{-(3-b)\nu} - e^{-(3-a)\nu}\}\{e^{-\nu} - e^{-\nu}(\nu e^{-\nu})^{k-2}\}] \\
&= \nu^{k-1}\{e^{(k+1-b)\nu} - e^{(k+1-a)\nu}\} + \frac{1}{1-\nu e^{-\nu}}\{e^{-(3-b)\nu} - e^{-(3-a)\nu}\}\{1 - (\nu e^{-\nu})^{k-1}\},
\end{aligned}
$$

which yields that the statement of the theorem holds for $k+1$. The proof is finished. □

DEFINITION 1. We call the triple $(p^{(1)}, p^{(2)}, p(x))$, $1 < x < 2$ a stationary distribution of $\{X_{\tau_n}\}_{n=1,2,\ldots}$ if $P(X_{\tau_n} = 1) = p^{(1)}$, $P(X_{\tau_n} = 2) = p^{(2)}$ and $\int_a^b p(x)dx = P(a < X_{\tau_n} \le b)$ imply $P(X_{\tau_{n+1}} = 1) = p^{(1)}$, $P(X_{\tau_{n+1}} = 2) = p^{(2)}$ and $\int_a^b p(x)dx = P(a < X_{\tau_{n+1}} \le b)$ for any $n(\ge 3)$ and $1 < a < b < 2$.

PROPOSITION 1. *$\{X_{\tau_n}\}_{n=1,2,\ldots}$ has at most one stationary distribution.*

PROOF. Suppose that $(p^{(1)}, p^{(2)}, p(x))$, $(q^{(1)}, q^{(2)}, q(x))$, $1 < x < 2$ are two pairs of stationary distributions of $\{X_{\tau_n}\}_{n=1,2,\ldots}$ and set

$$
\begin{aligned}
r^{(1)} &:= p^{(1)} - q^{(1)}, \\
r^{(2)} &:= p^{(2)} - q^{(2)}, \\
r(x) &:= p(x) - q(x).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
p^{(1)} &= p^{(1)} \cdot p(1,1) + p^{(2)} \cdot p(2,1) + \int_1^2 p(x,1)p(x)dx \\
&= p^{(1)}e^{-\nu} + p^{(2)}e^{-2\nu} + \int_1^2 e^{-x\nu}p(x)dx,
\end{aligned}
$$

$$
p^{(2)} = p^{(1)} \cdot p(1,2) + p^{(2)} \cdot p(2,2) + \int_1^2 p(x,2)p(x)dx
$$

$$= p^{(2)} \cdot \left(1 - e^{-\nu}\right) + \int_1^2 \{1 - e^{-(x-1)\nu}\} p(x)dx \,,$$

$$\begin{aligned}
\int_a^b p(x)dx &= p^{(1)} \cdot p(1, (a, b]) + p^{(2)} \cdot p(2, (a, b]) \\
&\quad + \int_{y=a}^b \int_{x=1}^2 P(x - \gamma \triangle \tau_{n+1} + 1 \in (y, y + dy]) p(x)dx \\
&= p^{(1)} \left\{ e^{-(2-b)\nu} - e^{-(2-a)\nu} \right\} + p^{(2)} \left\{ e^{-(3-b)\nu} - e^{-(3-a)\nu} \right\} \\
&\quad + \int_{y=a}^b \int_{x=1}^2 \nu e^{-(x+1-y)\nu} p(x)dxdy \,, \quad 1 < a < b < 2 \,,
\end{aligned}$$

and

$$p^{(1)} + p^{(2)} + \int_1^2 p(x)dx = 1 \,.$$

Since $(q^{(1)}, q^{(2)}, q(x))$ also satisfies the corresponding equalities, we have

$$\begin{cases}
r^{(1)} = r^{(1)} e^{-\nu} + r^{(2)} e^{-2\nu} + \int_1^2 e^{-x\nu} r(x)dx \\
r^{(2)} = r^{(2)} \left(1 - e^{-\nu}\right) + \int_1^2 \{1 - e^{-(x-1)\nu}\} r(x)dx \\
\int_a^b r(x)dx = r^{(1)} \{e^{-(2-b)\nu} - e^{-(2-a)\nu}\} + r^{(2)} \{e^{-(3-b)\nu} - e^{-(3-a)\nu}\} \\
\qquad\qquad + \int_{y=a}^b \int_{x=1}^2 \nu e^{-(x+1-y)\nu} r(x)dxdy \\
r^{(1)} + r^{(2)} + \int_1^2 r(x)dx = 0 \,.
\end{cases}$$

Thus we have

$$r^{(2)} e^{-\nu} = \int_1^2 \{1 - e^{-(x-1)\nu}\} r(x)dx \,. \tag{1}$$

Therefore,

$$\begin{aligned}
r^{(1)} &= r^{(1)} e^{-\nu} + e^{-\nu} \int_1^2 \{1 - e^{-(x-1)\nu}\} r(x)dx + \int_1^2 e^{-x\nu} r(x)dx \\
&= r^{(1)} e^{-\nu} + e^{-\nu} \int_1^2 r(x)dx \\
&= r^{(1)} e^{-\nu} + e^{-\nu} (-r^{(1)} - r^{(2)}) = -e^{-\nu} r^{(2)} \,,
\end{aligned}$$

$$\int_a^b r(x)dx = -e^{-\nu} r^{(2)} \{e^{-(2-b)\nu} - e^{-(2-a)\nu}\} + r^{(2)} \{e^{-(3-b)\nu} - e^{-(3-a)\nu}\}$$

$$+\int_{y=a}^{b}\int_{x=1}^{2} \nu e^{-(x+1-y)\nu} r(x)dxdy$$
$$= \int_{a}^{b} \nu e^{y\nu} dy \int_{1}^{2} e^{-(x+1)\nu} r(x)dx\,.$$

Setting $J := \int_1^2 e^{-(x+1)\nu} r(x)dx$, we have

$$\int_a^b r(x)dx = (e^{b\nu} - e^{a\nu})J$$

for any $1 < a < b < 2$. This implies

$$r(x) = \nu e^{\nu x} J\,. \tag{2}$$

Combining (2) with (1), we have

$$r^{(2)} e^{-\nu} = -r^{(1)} - r^{(2)} - \int_1^2 e^{-(x-1)\nu} r(x)dx$$
$$= (e^{-\nu} - 1)r^{(2)} - e^{2\nu} J\,.$$

Therefore, we have $r^{(2)} = -e^{2\nu} J$ and $r^{(1)} = e^{\nu} J$.

On the other hand, by (2) we have

$$J = \int_1^2 e^{-(x+1)\nu} \cdot \nu e^{\nu x} J dx\,.$$

Thus, we have $(1 - \nu e^{-\nu})J = 0$, which implies $J = 0$. Consequently, we have $r^{(1)} = r^{(2)} = 0$ and $r(x) \equiv 0$. The proof is complete. □

The limit distribution of $\{X_{\tau_n}\}_{n=1,2,\dots}$ is given by the following theorem.

THEOREM 2. *As n tends to ∞, $(p_{\tau_n}^{(1)},\ p_{\tau_n}^{(2)},\ p_{\tau_n}(x))$ converges weakly to $(p_{\tau_\infty}^{(1)},\ p_{\tau_\infty}^{(2)},\ p_{\tau_\infty}(x))$ given by*

$$p_{\tau_\infty}^{(1)} = \frac{e^{-2\nu}}{1 - \nu e^{-\nu}}\,,$$
$$p_{\tau_\infty}^{(2)} = 1 - \frac{e^{-\nu}}{1 - \nu e^{-\nu}}$$
$$p_{\tau_\infty}(x) = \frac{\nu e^{-(3-x)\nu}}{1 - \nu e^{-\nu}}\,, \quad 1 < x < 2\,,$$

and $(p_{\tau_\infty}^{(1)},\ p_{\tau_\infty}^{(2)},\ p_{\tau_\infty}(x))$ is the unique stationary distribution of $\{X_{\tau_n}\}_{n=1,2,\dots}$.

PROOF. By Theorem 1 and Proposition 1, it is enough to prove that $(p_{\tau_\infty}^{(1)},\ p_{\tau_\infty}^{(2)},\ p_{\tau_\infty}(x))$ is the stationary distribution of $\{X_{\tau_n}\}_{n=1,2,3,\dots}$. That is, we will prove

$$p^{(1)}_{\tau_\infty} = p^{(1)}_{\tau_\infty} \cdot p(1,1) + p^{(2)}_{\tau_\infty} \cdot p(2,1) + \int_1^2 p(x,1) p_{\tau_\infty}(x) dx \,,$$

$$p^{(2)}_{\tau_\infty} = p^{(1)}_{\tau_\infty} \cdot p(1,2) + p^{(2)}_{\tau_\infty} \cdot p(2,2) + \int_1^2 p(x,2) p_{\tau_\infty}(x) dx \,,$$

$$\int_a^b p_{\tau_\infty}(x) dx = p^{(1)}_{\tau_\infty} \cdot p(1,(a,b]) + p^{(2)}_{\tau_\infty} \cdot p(2,(a,b]) + \int_{y=a}^b \int_{x=1}^2 p(x,(y,y+dy]) p_{\tau_\infty}(x) dx \,.$$

We have

$$\begin{aligned}
p^{(1)}_{\tau_\infty} &\cdot p(1,1) + p^{(2)}_{\tau_\infty} \cdot p(2,1) + \int_1^2 p(x,1) p_{\tau_\infty}(x) dx \\
&= \frac{e^{-2\nu}}{1-\nu e^{-\nu}} \cdot e^{-\nu} + \left(1 - \frac{e^{-\nu}}{1-\nu e^{-\nu}}\right) e^{-2\nu} + \int_1^2 e^{-\nu x} \frac{\nu e^{-(3-x)\nu}}{1-\nu e^{-\nu}} dx \\
&= e^{-2\nu} + \frac{e^{-3\nu} - e^{-3\nu} + \nu e^{-3\nu}}{1-\nu e^{-\nu}} \\
&= \frac{e^{-2\nu}}{1-\nu e^{-\nu}} = p^{(1)}_{\tau_\infty} \,,
\end{aligned}$$

$$\begin{aligned}
p^{(1)}_{\tau_\infty} &\cdot p(1,2) + p^{(2)}_{\tau_\infty} \cdot p(2,2) + \int_1^2 p(x,2) p_{\tau_\infty}(x) dx \\
&= \left(1 - \frac{e^{-\nu}}{1-\nu e^{-\nu}}\right)(1 - e^{-\nu}) + \int_1^2 \left(1 - e^{-\nu(x-1)}\right) \frac{\nu e^{-(3-x)\nu}}{1-\nu e^{-\nu}} dx \\
&= 1 - \frac{e^{-\nu}}{1-\nu e^{-\nu}} - e^{-\nu} + \frac{e^{-2\nu}}{1-\nu e^{-\nu}} + \int_1^2 \frac{\nu e^{-(3-x)\nu}}{1-\nu e^{-\nu}} dx - \frac{\nu e^{-2\nu}}{1-\nu e^{-\nu}} \\
&= 1 - \frac{e^{-\nu}}{1-\nu e^{-\nu}} = p^{(2)}_{\tau_\infty}
\end{aligned}$$

and

$$\begin{aligned}
p^{(1)}_{\tau_\infty} &\cdot p(1,(a,b]) + p^{(2)}_{\tau_\infty} \cdot p(2,(a,b]) \\
&\qquad + \int_{y=a}^b \int_{x=1}^2 p(x,(y,y+dy]) p_{\tau_\infty}(x) dx \\
&= \frac{e^{-2\nu}}{1-\nu e^{-\nu}} \{e^{-(2-b)\nu} - e^{-(2-a)\nu}\} + \left(1 - \frac{e^{-\nu}}{1-\nu e^{-\nu}}\right) \{e^{-(3-b)\nu} - e^{-(3-a)\nu}\} \\
&\qquad + \int_{y=a}^b \int_{x=1}^2 \nu e^{-(x+1-y)\nu} \frac{\nu e^{-(3-x)\nu}}{1-\nu e^{-\nu}} dx dy
\end{aligned}$$

$$= e^{-(3-b)\nu} - e^{-(3-a)\nu} + \frac{\nu e^{-4\nu}}{1-\nu e^{-\nu}}(e^{b\nu} - e^{a\nu})$$
$$= \frac{e^{-(3-b)\nu} - e^{-(3-a)\nu}}{1-\nu e^{-\nu}} = \int_a^b p_{\tau_\infty}(x)dx\,,$$

which prove the theorem. □

3. Distribution of $\triangle X_{\tau_n}$

In this section, we investigate the distribution of $\triangle X_{\tau_n}$, $n = 1, 2, \ldots$. Put

$$q(x,y) := P(\triangle X_{\tau_{k+1}} = y \mid X_{\tau_k} = x)\,,$$
$$q(x,\Lambda) := P(\triangle X_{\tau_{k+1}} \in \Lambda \mid X_{\tau_k} = x)\,,$$
$$q(A,\Lambda) := P(\triangle X_{\tau_{k+1}} \in \Lambda \mid X_{\tau_k} \in A)\,,\quad x, y \in \mathbf{R}\,,\ \ \Lambda \subset (0,1)\,,\ \ A \subset (1,2)\,.$$

Then we have the following lemma.

LEMMA 2. *For* $1 < a < b < 2$ *and* $0 < x < y < 1$ *we have*

$$q(2,1) = e^{-\nu}\,,$$
$$q(2,(a,b]) = e^{-a\nu} - e^{-b\nu}\,,$$
$$q((a,b],(x,y]) = \begin{cases} 0\,, & \text{if} \quad b < 2-y\,,\\ \left\{\dfrac{\nu e^{-3\nu} - \nu e^{-3\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}} + \nu^{k-1}e^{-k\nu}\right\}\{e^{-(x-2)\nu} - e^{-(y-2)\nu}\} & \\ \qquad\cdot(b-a)/P(X_{\tau_k} \in (a,b]), & \text{if} \quad 2-x < a\,,\end{cases}$$
$$q((a,b],1) = \left\{\frac{\nu e^{-2\nu} - \nu e^{-2\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}} + \nu^{k-1}e^{-(k-1)\nu}\right\}(b-a)/P(X_{\tau_k} \in (a,b])\,,$$
$$q(1,(x,y]) = 0\,,$$
$$q(1,1) = 1\,.$$

PROOF. For $q(2,1)$, by the independence of X_{τ_k} and τ_{k+1}, we have

$$q(2,1) = P(\gamma\triangle\tau_{k+1} \geq 1 | X_{\tau_k} = 2)$$
$$= P(\gamma\triangle\tau_{k+1} \geq 1) = e^{-\nu}\,.$$

For $q(2,(a,b])$ we have

$$q(2,(a,b]) = P(\triangle X_{\tau_{k+1}} \in (a,b] | X_{\tau_k} = 2)$$
$$= P(a < \gamma\triangle\tau_{k+1} \leq b | X_{\tau_k} = 2)$$
$$= P(a < \gamma\triangle\tau_{k+1} \leq b) = e^{-a\nu} - e^{-b\nu}\,.$$

Next we calculate $q((a,b],1)\cdot P(X_{\tau_k}\in(a,b]),\ 1<a<b<2$:

$$\begin{aligned} q((a,b],1)\cdot P(X_{\tau_k}\in(a,b]) &= P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1,\, X_{\tau_k}\in(a,b]) \\ &= \int_a^b p_{\tau_k}(x)\int_{(x-1)/\gamma}^{\infty}\lambda e^{-\lambda y}dydx \\ &= \left\{\frac{\nu e^{-2\nu}-\nu e^{-2\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}}+\nu^{k-1}e^{-(k-1)\nu}\right\}(b-a)\,. \end{aligned}$$

Since

$$\begin{aligned} q((a,b],(x,y])\cdot P(X_{\tau_k}\in(a,b]) &= P(\triangle X_{\tau_{k+1}}\in(x,y],\, X_{\tau_k}\in(a,b]) \\ &= P(2-y\le X_{\tau_k}-\gamma\triangle\tau_{k+1}<2-x,\ X_{\tau_k}\in(a,b])\,, \end{aligned}$$

it is easy to see that $q((a,b],(x,y])=0$ if $b<2-y$. On the other hand, if $2-x<a$, we have

$$\begin{aligned} &P(2-y\le X_{\tau_k}-\gamma\triangle\tau_{k+1}<2-x,\ X_{\tau_k}\in(a,b]) \\ &\quad= \int_a^b p_{\tau_k}(u)\int_{(u+x-2)/\gamma}^{(u+y-2)/\gamma}\lambda e^{-\lambda v}dvdu \\ &\quad= \left\{\frac{\nu e^{-3\nu}-\nu e^{-3\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}}+\nu^{k-1}e^{-k\nu}\right\}\cdot\{e^{-(x-2)\nu}-e^{-(y-2)\nu}\}\cdot(b-a)\,. \end{aligned}$$

Last of all, it is easy to see $q(1,1)=0$, which yields $q(1,(x,y])=0$. The proof is finished. □

Now we have the distribution $(q^{(1)}_{\triangle_k}, q_{\triangle_k}(x)),\ 0<x<1$ of $\triangle X_{\tau_k}$ for $k\ge 3$.

THEOREM 3. *For $k\ge 3$, we have*

$$q^{(1)}_{\triangle_k} := P(\triangle X_{\tau_k}=1)=e^{-\nu}+\nu^{k-2}e^{-(k-2)\nu}+\frac{\nu e^{-2\nu}\{1-(\nu e^{-\nu})^{k-3}\}}{1-\nu e^{-\nu}}$$

and

$$P(a<\triangle X_{\tau_k}\le b)=\int_a^b q_{\triangle_k}(x)dx\,,\quad 0<a<b<1\,,$$

where

$$\begin{aligned} q_{\triangle_k}(x) :=& \nu e^{-\nu x}-\frac{\nu e^{-(x+1)\nu}-\nu e^{-(x+1)\nu}(\nu e^{-\nu})^{k-3}}{1-\nu e^{-\nu}}-\nu^{k-2}e^{-(k-3+x)\nu} \\ &+\frac{\nu e^{-3\nu}\{1-(\nu e^{-\nu})^{k-3}\}}{1-\nu e^{-\nu}}\cdot\nu x e^{-(x-2)\nu}+\nu^{k-1}xe^{-(k-3+x)\nu}\,. \end{aligned}$$

PROOF. We have

$$P(\triangle X_{\tau_k}=1)=P(X_{\tau_{k-1}}=2)\cdot q(2,1)+\int_1^2 q((x,x+dx],1)P(X_{\tau_{k-1}}\in(x,x+dx])$$

$$
\begin{aligned}
&+ P(X_{\tau_{k-1}} = 1) \cdot q(1, 1) \\
&= \left\{ 1 - \frac{e^{-\nu} - e^{-\nu} \left(\nu e^{-\nu}\right)^{k-3}}{1 - \nu e^{-\nu}} - \nu^{k-3} e^{-(k-3)\nu} \right\} e^{-\nu} \\
&\quad + \int_1^2 \left\{ \frac{\nu e^{-2\nu} - \nu e^{-2\nu} \left(\nu e^{-\nu}\right)^{k-3}}{1 - \nu e^{-\nu}} + \nu^{k-2} e^{-(k-2)\nu} \right\} dx \\
&\quad + \left\{ \frac{e^{-2\nu} - e^{-2\nu} \left(\nu e^{-\nu}\right)^{k-3}}{1 - \nu e^{-\nu}} + \nu^{k-3} e^{-(k-2)\nu} \right\} \cdot 1 \\
&= e^{-\nu} + \frac{\nu e^{-2\nu} - \nu e^{-2\nu} \left(\nu e^{-\nu}\right)^{k-3}}{1 - \nu e^{-\nu}} + \nu^{k-2} e^{-(k-2)\nu} .
\end{aligned}
$$

On the other hand, for $0 < a < b < 1$ we have

$$
\begin{aligned}
P(a < \triangle X_{\tau_k} \le b) &= \int_a^b P(X_{\tau_{k-1}} = 1) q(1, (x, x + dx]) \\
&\quad + \int_a^b P(X_{\tau_{k-1}} = 2) q(2, (x, x + dx]) \\
&\quad + \int_a^b \int_{u=1}^2 q((u, u + du], (x, x + dx]) P(X_{\tau_{k-1}} \in (u, u + du]) \\
&= \int_a^b \left\{ 1 - \frac{e^{-\nu} - e^{-\nu} (\nu e^{-\nu})^{k-3}}{1 - \nu e^{-\nu}} - \nu^{k-3} e^{-(k-3)\nu} \right\} \nu e^{-\nu x} dx \\
&\quad + \int_a^b \int_{2-x}^2 \left\{ \frac{\nu e^{-3\nu} - \nu e^{-3\nu} (\nu e^{-\nu})^{k-3}}{1 - \nu e^{-\nu}} \right. \\
&\qquad\qquad \left. + \nu^{k-2} e^{-(k-1)\nu} \right\} \nu e^{-(x-2)\nu} du dx \\
&= \int_a^b q_{\triangle_k}(x) dx .
\end{aligned}
$$

The proof is complete. □

Now we put

$$
q_\triangle(x, y) := P(\triangle X_{\tau_{k+1}} = y | \triangle X_{\tau_k} = x) ,
$$

$$
q_\triangle(x, \Lambda) := P(\triangle X_{\tau_{k+1}} \in \Lambda | \triangle X_{\tau_k} = x) , \quad 0 < x , y \le 1 , \quad \Lambda \subset (0, 1) .
$$

Then we have the following lemma.

LEMMA 3. *For* $0 < a < b < 1$ *and* $0 < x < y < 1$ *we have*

$$
q_\triangle(1, 1) = e^{-\nu} + \nu e^{-\nu} ,
$$

$$
q_\triangle(1, (a, b]) = (1 + a\nu) e^{-a\nu} - (1 + b\nu) e^{-b\nu} ,
$$

$$q_\triangle((x,y],1)=q_\triangle(x,1)=e^{-\nu}\,,$$

$$q_\triangle((x,y],(a,b])=e^{-a\nu}-e^{-b\nu}\,.$$

PROOF. For $q_\triangle(1,1)$ we have

$$\begin{aligned}
q_\triangle(1,1) &= P(\triangle X_{\tau_{k+1}}=1|\triangle X_{\tau_k}=1)\\
&= P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1|\triangle X_{\tau_k}=1)\\
&= \frac{P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1,\ \triangle X_{\tau_k}=1)}{P(\triangle X_{\tau_k}=1)}\\
&= \frac{P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1,\ \triangle X_{\tau_k}=1,\ X_{\tau_k}=1)}{P(\triangle X_{\tau_k}=1)}\\
&\quad+\frac{P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1,\ \triangle X_{\tau_k}=1,\ 1<X_{\tau_k}<2)}{P(\triangle X_{\tau_k}=1)}\\
&\quad+\frac{P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1,\ \triangle X_{\tau_k}=1,\ X_{\tau_k}=2)}{P(\triangle X_{\tau_k}=1)}\\
&= \frac{P(X_{\tau_k}=1)+P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1,\ 1<X_{\tau_k}<2)}{P(\triangle X_{\tau_k}=1)}\\
&\quad+\frac{P(\triangle\tau_{k+1}\ge\frac{1}{\gamma})P(\triangle X_{\tau_k}=1,\ X_{\tau_k}=2)}{P(\triangle X_{\tau_k}=1)}\,.
\end{aligned}$$

Since

$$P(X_{\tau_k}=1)=p^{(1)}_{\tau_k}=\frac{e^{-2\nu}-e^{-2\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}}+\nu^{k-2}e^{-(k-1)\nu}\,,$$

$$\begin{aligned}
P(X_{\tau_k}-\gamma\triangle\tau_{k-1}\le 1,\ 1<X_{\tau_k}<2) &= \int_1^2 p_{\tau_k}(x)\int_{(x-1)/\gamma}^\infty \lambda e^{-\lambda y}dydx\\
&= \int_1^2\left\{\frac{\nu e^{-(3-x)\nu}-\nu e^{-(3-x)\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}}+\nu^{k-1}e^{-(k-x)\nu}\right\}e^{-\nu(x-1)}dx\\
&= \frac{\nu e^{-2\nu}-\nu e^{-2\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}}+\nu^{k-1}e^{-(k-1)\nu}\,,
\end{aligned}$$

$P\big(\triangle\tau_{k+1}\ge\frac{1}{\gamma}\big)=e^{-\nu}$ and $P(\triangle X_{\tau_k}=1,\ X_{\tau_k}=2)=0$, we have $q_\triangle(1,1)=(1+\nu)e^{-\nu}$.

For $q_\triangle((x,y],1)$ we have

$$\begin{aligned}
q_\triangle((x,y],1) &= P(X_{\tau_k}-\gamma\triangle\tau_{k+1}\le 1|\triangle X_{\tau_k}\in(x,y])\\
&= \frac{P(2-\gamma\triangle\tau_{k+1}\le 1)P(\triangle X_{\tau_k}\in(x,y])}{P(\triangle X_{\tau_k}\in(x,y])}\\
&= P(\triangle\tau_{k+1}\ge 1/\gamma)=e^{-\nu}\,.
\end{aligned}$$

Similarly, we also get $q_\triangle(x,1) = e^{-\nu}$.

For $q_\triangle(1,(a,b])$ we have

$$\begin{aligned}
q_\triangle(1,(a,b]) &= P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b \mid \triangle X_{\tau_k} = 1) \\
&= \frac{P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b,\ \triangle X_{\tau_k} = 1,\ 1 < X_{\tau_k} < 2)}{P(\triangle X_{\tau_k} = 1)} \\
&\quad + \frac{P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b,\ \triangle X_{\tau_k} = 1,\ X_{\tau_k} = 2)}{P(\triangle X_{\tau_k} = 1)} .
\end{aligned}$$

Since

$$P(\triangle X_{\tau_k} = 1) = q^{(1)}_{\triangle_k} = e^{-\nu} + \nu^{k-2}e^{-(k-2)\nu} + \frac{\nu e^{-2\nu}\{1-(\nu e^{-\nu})^{k-3}\}}{1-\nu e^{-\nu}} ,$$

$$\begin{aligned}
&P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b,\ \triangle X_{\tau_k} = 1,\ X_{\tau_k} = 2) \\
&\quad = P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b) P(\triangle X_{\tau_k} = 1,\ X_{\tau_k} = 2) = 0
\end{aligned}$$

and

$$\begin{aligned}
&P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b,\ \triangle X_{\tau_k} = 1,\ 1 < X_{\tau_k} < 2) \\
&\quad = P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b,\ 1 < X_{\tau_k} < 2) \\
&\quad = P\left(0 < \triangle\tau_{k+1} \le \frac{X_{\tau_k} - 2 + b}{\gamma},\ 2-b \le X_{\tau_k} < 2-a\right) \\
&\qquad + P\left(\frac{X_{\tau_k} - 2 + a}{\gamma} < \triangle\tau_{k+1} < \frac{X_{\tau_k} - 2 + b}{\gamma},\ 2-a \le X_{\tau_k} < 2\right) \\
&\quad = \int_{2-b}^{2-a} p_{\tau_k}(x) \int_0^{(x-2+b)/\gamma} \lambda e^{-\lambda y}\,dy dx + \int_{2-a}^{2} p_{\tau_k}(x) \int_{(x-2+a)/\gamma}^{(x-2+b)/\gamma} \lambda e^{-\lambda y}\,dy dx \\
&\quad = a\nu^{k-1}e^{-(k-2+a)\nu} + \nu^{k-2}e^{-(k-2+a)\nu} - b\nu^{k-1}e^{-(k-2+b)\nu} - \nu^{k-2}e^{-(k-2+b)\nu} \\
&\qquad + (a\nu+1)\frac{e^{-(1+a)\nu} + e^{-(1+a)\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}} \\
&\qquad - (b\nu+1)\frac{e^{-(1+b)\nu} + e^{-(1+b)\nu}(\nu e^{-\nu})^{k-2}}{1-\nu e^{-\nu}} ,
\end{aligned}$$

we can get

$$q_\triangle(1,(a,b]) = e^{-a\nu} - e^{-b\nu} + \nu(ae^{-a\nu} - be^{-b\nu}) .$$

Last of all, we have

$$\begin{aligned}
q_\triangle((x,y],(a,b]) &= \frac{P(a < 2 - (X_{\tau_k} - \gamma\triangle\tau_{k+1}) \le b,\ \triangle X_{\tau_k} \in (x,y])}{P(\triangle X_{\tau_k} \in (x,y])} \\
&= P\left(\frac{a}{\gamma} < \triangle\tau_{k+1} \le \frac{b}{\gamma}\right) = e^{-a\nu} - e^{-b\nu} .
\end{aligned}$$

□

DEFINITION 2. We call a pair $(p^{(1)}, p(x))$, $0 < x < 1$ a stationary distribution of $\{\triangle X_{\tau_n}\}_{n=1,2,\dots}$ if $P(\triangle X_{\tau_n} = 1) = p^{(1)}$ and $\int_a^b p(x)dx = P(a < \triangle X_{\tau_n} \le b)$ imply $P(\triangle X_{\tau_{n+1}} = 1) = p^{(1)}$ and $\int_a^b p(x)dx = P(a < \triangle X_{\tau_{n+1}} \le b)$ for any $n(\ge 3)$ and $0 < a < b < 1$.

PROPOSITION 2. *$(p_\triangle^{(1)}, p_\triangle(x))$, $0 < x < 1$ is a stationary distribution of $\{\triangle X_{\tau_n}\}_{n=1,2,\dots}$ if and only if*

$$p_\triangle^{(1)} = \frac{e^{-\nu}}{1-\nu e^{-\nu}}\,,$$

$$p_\triangle(x) = \nu e^{-x\nu} - \frac{\nu - \nu^2 x}{1-\nu e^{-\nu}} e^{-(x+1)\nu}\,.$$

PROOF. Suppose that $(p_\triangle^{(1)}, p_\triangle(x))$ is a stationary distribution of $\{\triangle X_{\tau_n}\}_{n=1,2,\dots}$. Then $(p_\triangle^{(1)}, p_\triangle(x))$ must satisfy

$$\begin{cases} p_\triangle^{(1)} = p_\triangle^{(1)} \cdot q_\triangle(1,1) + \int_0^1 q_\triangle(x,1) p_\triangle(x)dx \\ \int_a^b p_\triangle(x)dx = p_\triangle^{(1)} \cdot q_\triangle(1,(a,b]) + \int_0^1 p_\triangle(x) q_\triangle((x,x+dx],(a,b])dx\,. \end{cases}$$

By Lemma 3 and the fact $\int_0^1 p_\triangle(x)dx = 1 - p_\triangle^{(1)}$, we have

$$p_\triangle^{(1)} = p_\triangle^{(1)} \cdot (e^{-\nu} + \nu e^{-\nu}) + e^{-\nu}(1 - p_\triangle^{(1)})\,,$$

which yields

$$p_\triangle^{(1)} = \frac{e^{-\nu}}{1-\nu e^{-\nu}}\,.$$

On the other hand, using Lemma 3 again, $p_\triangle(x)$ satisfies

$$\int_a^b p_\triangle(x)dx = p_\triangle^{(1)} \cdot q_\triangle(1,(a,b]) + \int_0^1 (e^{-a\nu} - e^{-b\nu}) p_\triangle(x)dx$$

for any $0 < a < b < 1$. Then we easily get

$$p_\triangle(x) = \nu e^{-x\nu} - \frac{\nu - \nu^2 x}{1-\nu e^{-\nu}} e^{-(x+1)\nu}\,.$$

The converse is easy and is omitted. The proof is finished. □

By Theorem 3 and Proposition 2, we consequently get the following theorem.

THEOREM 4. *As n ends to* ∞, *the distribution* $(q^{(1)}_{\triangle_n}, q_{\triangle_n}(x))$ *of* $\triangle X_{\tau_n}$, $n = 1, 2, \ldots$ *tends to the following distribution* $(q^{(1)}_{\triangle_\infty}, q_{\triangle_\infty}(x))$ *weakly which is the unique stationary distribution of* $\{\triangle X_{\tau_n}\}_{n=1,2,\ldots}$:

$$q^{(1)}_{\triangle_\infty} := \frac{e^{-\nu}}{1 - \nu e^{-\nu}},$$

$$q_{\triangle_\infty}(x) := \nu e^{-x\nu} - \frac{\nu - \nu^2 x}{1 - \nu e^{-\nu}} e^{-(x+1)\nu}, \quad 0 < x < 1.$$

References

[1] J. S. GRIFFITH, *Mathematical Neurobiology, An Introduction to the Mathematics of the Nervous System*, Academic Press, London, 1971.

[2] T. KAWAMURA and Y. SAISHO, Stochastic models describing human metabolism processes using stochastic differential equations, Stochastic Models **22** (2006), 273–287.

[3] Y. SAISHO, Stochastic differential equations for multi-dimensional domain with reflecting boundary, Probab. Th. Rel. Fields **74** (1987), 455–477.

[4] Y. SAISHO and H. TANAKA, On the symmetry of a reflecting Brownian motion defined by Skorohod's equation for a multi-dimensional domain, Tokyo J. Math. **10** (1987), 419–435.

[5] H. TANAKA, On limiting distributions for one-dimensional diffusion processes, Bulletin of Mathematical Statistics **7** (1957), 84–91.

[6] H. TANAKA, Stochastic differential equations with reflecting boundary condition in convex regions, Hiroshima Math. J. **9** (1979), 163–177.

Present Address:
DEPARTMENT OF APPLIED MATHEMATICS,
GRADUATE SCHOOL OF ENGINEERING,
HIROSHIMA UNIVERSITY,
1–4–1 KAGAMIYAMA, HIGASHI-HIROSHIMA, 739–8527 JAPAN.
e-mail: saisho@hiroshima-u.ac.jp