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Linear Topologies on a Field and Completions of Valuation Rings

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Introduction.

For an integral local ring A, we consider the linear topology on QA with fundamental system of neighborhoods of 0:

$$\Sigma_A = \{a\mathfrak{m}(A) \mid a \in A, a \neq 0\}.$$

This topology is said to be the A-topology on QA. Here QA is the quotient field of A and $\mathfrak{m}(A)$ is the unique maximal ideal of A. In general, the A-topology is stronger than the $\mathfrak{m}(A)$-adic topology.

For an integral local ring A, we consider the completion

$$\hat{A} = \text{proj.lim}\, A/\mathfrak{a} \quad (\mathfrak{a} \in \Sigma_A)$$

with respect to the A-topology.

In this paper we shall study the fundamental properties of the completion $\hat{A}$ of an integral local ring A with respect to the A-topology and show some related examples. The A-topology and the completion $\hat{A}$ are very important conceptions for a valuation ring A, especially in the case that A is not noetherian. The main results are as follows:

THEOREM 1. *Let A be an integral local ring. Then*

$$A \text{ is a valuation ring} \Leftrightarrow \hat{A} \text{ is a valuation ring}.$$

Moreover, if A is a valuation ring, then the residue field of $\hat{A}$ is isomorphic to the residue field of A and the value group of $\hat{A}$ is isomorphic to the value group of A.

For a field K and a subring A of K, let $Zar(K|A)$ denote the set of valuation rings of K which contain A. Then the set $Zar(K|A)$ has a structure of local ringed spaces (see [4, §1]).

THEOREM 2. *Suppose that A is a valuation ring.*

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(i) *The morphism of local ringed spaces defined by*

$$\begin{array}{ccc} Zar(Q\hat{A}|\hat{A}) & \rightarrow & Zar(QA|A) \\ \Downarrow & & \Downarrow \\ R & \mapsto & QA\cap R \end{array}$$

is a homeomorphism. Moreover the inverse mapping is given by

$$\hat{B} \leftarrow\!\shortmid B$$

for $B \neq QA$.

(ii) $A = k \oplus \mathfrak{m}(A) \Leftrightarrow k \subset QA,\ \hat{A} = k \oplus \mathfrak{m}(\hat{A})$ *for any subfield* k *of* $Q\hat{A}$.

(iii) *If the exact sequence* $1 \to A^{\times} \to (QA)^{\times} \to (QA)^{\times}/A^{\times} \to 1$ *splits, then* $1 \to \hat{A}^{\times} \to (Q\hat{A})^{\times} \to (Q\hat{A})^{\times}/\hat{A}^{\times} \to 1$ *also splits.*

(iv) *If the* A*-topology on* QA *is metrizable, then* $\hat{A}$ *is the completion of* A *and* $Q\hat{A}$ *is the completion of* QA *as metric spaces.*

REMARK 0. There exists an integral local ring A such that $\hat{A}$ is not integral. See Example 1, (iii).

REMARK 1. Theorem 1 can be proved by the use of the theory of completion of uniform spaces. See [2, Chapter 6, §5.3, Proposition 5]. Here we prove Theorem 1 without using the theory of uniform spaces.

REMARK 2. There exists an equal characteristic complete valuation ring which does not have the coefficient field. See Example 2.

REMARK 3. The completion $\hat{A}$ of a valuation ring A is not determined uniquely from the residue field and the value group, if A is not noetherian. See Example 3 and Proposition 4.

REMARK 4. The converse of Theorem 2, (iii) does not hold. See Example 4.

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1. Here we consider the topologies and the completions of integral local rings.
The following results are well-known.

LEMMA 1. *For an integral local ring* A, *we consider the* A*-topology on* QA.
(i) *All the sub* A*-modules of* QA *are closed.*
(ii) QA *is a separable topological field.*

For a sub A-module M of QA, we consider the completion

$$\hat{M} = \text{proj.lim}\, M/M \cap \mathfrak{a} \quad (\mathfrak{a} \in \Sigma_A)$$

with respect to the A-topology. Then the family $\hat{\Sigma}_A = \{\hat{\mathfrak{a}} \mid \mathfrak{a} \in \Sigma_A\}$ defines a separable linear topology on $\widehat{QA}$ as a fundamental system of neighborhoods of 0.

LEMMA 2. *Let A be an integral local ring.*

(i) $\hat{A} \hookrightarrow \widehat{QA}$, $QA \hookrightarrow \widehat{QA}$ *and* $A = QA \cap \hat{A}$. *Moreover* $\widehat{QA}$ *gives rise to a* QA-*module by the natural way.*

(ii) $\widehat{S^{-1}\mathfrak{a}} = S^{-1}\hat{\mathfrak{a}} = S^{-1}(\mathfrak{a}\hat{A}) = \mathfrak{a}(S^{-1}\hat{A})$ *for any ideal* $\mathfrak{a}$ *and multiplicative system* S *of* A. *Therefore* $\widehat{S^{-1}A}$ *is a ring and* $\widehat{S^{-1}\mathfrak{a}}$ *is an ideal of* $\widehat{S^{-1}A}$.

(iii) $\hat{A}$ *is a local ring and* $\mathfrak{m}(\hat{A}) = \widehat{\mathfrak{m}(A)}$. *Thus* $A/\mathfrak{m}(A) \cong \hat{A}/\mathfrak{m}(\hat{A})$.

(iv) *If* $\hat{A}$ *is integral, then* $\hat{\Sigma}_A \subset \Sigma_{\hat{A}}$.

(v) *If* $\hat{A}$ *is integral, then the following conditions are equivalent*:

(a) $\hat{\Sigma}_A = \Sigma_{\hat{A}}$.

(a′) $\hat{\Sigma}_A$ *and* $\Sigma_{\hat{A}}$ *define the same topology on* $\hat{A}$.

(b) $A \hookrightarrow \hat{A}$ *is continuous with respect to the* $\hat{A}$-*topology.*

(c) *For any* $\alpha \in \hat{A}$, *there exists* $a \in A$ *such that* $\alpha\hat{A} = a\hat{A}$.

(d) $A \cap \mathfrak{A} = 0 \Rightarrow \mathfrak{A} = 0$ *for any closed ideal* $\mathfrak{A}$ *of* $\hat{A}$ *with respect to the topology defined by* $\hat{\Sigma}_A$.

(e) $\widehat{QA} = Q\hat{A}$.

(e′) $(QA)^{\times}\hat{A}^{\times} = (Q\hat{A})^{\times}$.

(vi) *If* $\hat{A}$ *is integral and* $\hat{\Sigma}_A = \Sigma_{\hat{A}}$, *then* $\hat{A} \cong \hat{\hat{A}}$ *and* $(QA)^{\times}/A^{\times} \cong (Q\hat{A})^{\times}/\hat{A}^{\times}$.

PROOF. (i) Since $\widehat{QA}$ is a torsion-free and divisible A-module, the field QA acts naturally on $\widehat{QA}$.

(ii), (iii) and (iv) are easy.

(v) (c) ⇒ (a) ⇒ (a′) ⇒ (b): Obvious.

(b) ⇒ (c): Take any $\alpha \in \hat{A}$. We can assume $\alpha \neq 0$. Since $\alpha \in \hat{A} = A + \alpha\mathfrak{m}(\hat{A})$ by (b), there exist $a \in A$ and $\beta \in \mathfrak{m}(\hat{A})$ such that $\alpha = a + \alpha\beta$. Then $a = \alpha(1-\beta)$ and $1-\beta \in \hat{A}^{\times}$ imply (c).

(c) ⇒ (d): Easy.

(d) ⇒ (b): For any $\alpha \in \hat{A}$, $\alpha \neq 0$, we put $\mathfrak{A} = \overline{\alpha\mathfrak{m}(\hat{A})}$. Then $\mathfrak{A}$ is a non 0 closed ideal of $\hat{A}$. Therefore $A \cap \mathfrak{A} \neq 0$ by (d). Thus there exists $a \in A \cap \mathfrak{A}$ such that $a \neq 0$. Since $a \in \mathfrak{A} \subset \alpha\mathfrak{m}(\hat{A}) + a\mathfrak{m}(\hat{A})$, there exist $\beta, \gamma \in \mathfrak{m}(\hat{A})$ such that $a = \alpha\beta + a\gamma$. Then $a(1-\gamma) = \alpha\beta$ and $1-\gamma \in \hat{A}^{\times}$ imply (b).

(c) ⇒ (e): Put $\mathfrak{a} = A$ and $S = A - \{0\}$ in (ii). Then $\widehat{QA} = (A - \{0\})^{-1}\hat{A} \subset Q\hat{A}$. Conversely, for any $\alpha \in \hat{A}$ and $\beta \in \hat{A}$, $\beta \neq 0$, there exists $b \in A$ such that $b \neq 0$ and $b\hat{A} = \beta\hat{A}$ by (c). Since $b\alpha \in b\hat{A} = \beta\hat{A}$, there exists $\gamma \in \hat{A}$ such that $b\alpha = \beta\gamma$. Therefore $\frac{\alpha}{\beta} = \frac{\gamma}{b} \in (A - \{0\})^{-1}\hat{A} = \widehat{QA}$.

(e) ⇒ (b): Take any $\alpha \in \hat{A}$, $\alpha \neq 0$. Since $\frac{1}{\alpha} \in Q\hat{A} = \widehat{QA} = (A - \{0\})^{-1}\hat{A}$, there exist $a \in A - \{0\}$ and $\beta \in \hat{A}$ such that $\frac{1}{\alpha} = \frac{\beta}{a}$. Since $a\mathfrak{m}(\hat{A}) = \alpha\beta\mathfrak{m}(\hat{A}) \subset \alpha\mathfrak{m}(\hat{A})$, we obtain (b).

(c) ⇒ (e′): Easy.

(e′) ⇒ (b): Take any $\alpha \in \hat{A}$, $\alpha \neq 0$. Since $\alpha \in (Q\hat{A})^{\times} = (QA)^{\times}\hat{A}^{\times}$, there exist $a \in (QA)^{\times}$ and $\beta \in \hat{A}^{\times}$ such that $\alpha = a\beta$. Since $a \in A$ and $a\mathfrak{m}(\hat{A}) = \alpha\mathfrak{m}(\hat{A})$, we obtain (b).

(vi) Since $A/\mathfrak{a} \cong \hat{A}/\hat{\mathfrak{a}}$ holds for any $\mathfrak{a} \in \Sigma_A$, we get $\hat{A} \cong \hat{\hat{A}}$. Moreover we obtain $A^\times = (QA)^\times \cap \hat{A}^\times$ by (i). Therefore $(QA)^\times/A^\times \cong (Q\hat{A})^\times/\hat{A}^\times$ by (v).

COROLLARY. *$\hat{A}$ is integral and $\hat{\Sigma}_A = \Sigma_{\hat{A}} \Leftrightarrow \widehat{QA}$ is a field.*

REMARK. For an integral local ring A, the family $\Sigma_A^\times = \{1 + \mathfrak{a} \mid \mathfrak{a} \in \Sigma_A\}$ is an fundamental system of neighborhoods of 1 with respect to the A-topology on the multiplicative group $(QA)^\times$. Moreover we have

(i) $\hat{A}^\times = \widehat{A^\times} = \text{proj.lim}\, A^\times/U \ (U \in \Sigma_A^\times)$.

(ii) $(\widehat{QA})^\times = \widehat{(QA)^\times} = \text{proj.lim}(QA)^\times/U \ (U \in \Sigma_A^\times)$.

EXAMPLE 1. Let k be a field and t an indeterminate over k. For $a, b \in k$, we put

$$A = k[t^2 + at + b, t^3 + at^2 + bt]_{(t^2+at+b, t^3+at^2+bt)}\,.$$

Then A is an integral local ring.

(i) If $t^2 + at + b \in k[t]$ is irreducible, then $\hat{A} = k \oplus (t^2 + at + b)k'[[t^2 + at + b]]$ and $\widehat{QA} = k'((t^2 + at + b))$. Here $k' = k[t]/(t^2 + at + b)$.

(ii) If $t^2 + at + b = 0$ has a double root $\alpha \in k$, then $\hat{A} = k \oplus (t - \alpha)^2 k[[t - \alpha]]$ and $\widehat{QA} = k((t - \alpha))$.

(iii) If $t^2 + at + b = 0$ has distinct two roots $\alpha, \beta \in k$, then

$$\hat{A} = \{(f, g) \in k[[t - \alpha]] \times k[[t - \beta]] \mid f(\alpha) = g(\beta)\}$$

and $\widehat{QA} = k((t - \alpha)) \times k((t - \beta))$.

The proof is obvious from the fact that the A-topology coincides with the $\mathfrak{m}(A)$-adic topology.

2. Here we consider the topologies and the completions of valuation rings.

Suppose that A is a valuation ring. Then any separable linear topology on A defined by ideals is either the discrete topology or the A-topology. Moreover

A is noetherian $\Leftrightarrow$ the A-topology coincides with the $\mathfrak{m}(A)$-adic topology.

Let K be a field. For various valuation rings A with quotient field K, we consider the A-topology on K.

LEMMA 3. *Let K be a field and A_0 a subring of K. For $A, B \in Zar(K|A_0)$, we define*

$$A \sim B \Leftrightarrow \text{the } A\text{-topology coincides with the } B\text{-topology on } K\,.$$

Then $\sim$ is an equivalence relation on $Zar(K|A_0)$.

(i) *For $A, B \in Zar(K|A_0)$, we put $A \vee B = A[B] = B[A] \in Zar(K|A_0)$. Then $Zar(K|A \vee B) = Zar(K|A) \cap Zar(K|B)$ and*

$$A \sim B \Leftrightarrow A \vee B \neq K \ \text{ or } \ A = B = K\,.$$

(ii) *If* $\dim Zar(K|A_0) < \infty$, *then the corresponding disjoint union is given by*

$$Zar(K|A_0) = \{K\} \cup \bigcup_{\substack{A \in Zar(K|A_0) \\ \dim A = 1}} \overline{\{A\}}\,.$$

(ii′) *If* $A_0 \in Zar K$, *then the corresponding disjoint union is given by*

$$Zar(K|A_0) = \{K\} \cup (Zar(K|A_0) - \{K\})\,.$$

PROOF. (i) $\Leftarrow$: Since $A \subset B \subsetneq K \Rightarrow A \sim B$ holds for any $A, B \in Zar(K|A_0)$, we obtain $A \vee B \neq K \Rightarrow A \sim B$.

$\Rightarrow$: It suffices to prove that $A \sim B$, $A \neq K$, $B \neq K \Rightarrow A \vee B \neq K$ for any A, $B \in Zar(K|A_0)$. Put $\mathfrak{p}_A = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \mathfrak{p} \neq 0}} \mathfrak{p}$. Then

$$\mathfrak{p}_A = \{a \in K \mid \lim_{i \to +\infty} a^i = 0 \text{ with respect to the } A\text{-topology}\} \in \operatorname{Spec} A\,.$$

Therefore $A \sim B$ implies $\mathfrak{p}_A = \mathfrak{p}_B$. We put $\mathfrak{p} = \mathfrak{p}_A = \mathfrak{p}_B$. Assume that $\mathfrak{p} \neq 0$. If we put $C = A_{\mathfrak{p}}$, then $C = B_{\mathfrak{p}}$. Thus $A \vee B \subset C \neq K$. Assume that $\mathfrak{p} = 0$. Then there exists $\mathfrak{q} \in \operatorname{Spec} B$ such that $0 \neq \mathfrak{q} \subset \mathfrak{m}(A)$. If we put $C = B_{\mathfrak{q}}$, then $\mathfrak{q} = \mathfrak{m}(C)$ and $A \subset C \neq K$. Thus $A \vee B \subset C \neq K$.

(ii) and (ii′) are easy from (i).

LEMMA 4. *Suppose that A is a valuation ring. Then*

(i) $\widehat{A \cap \mathfrak{A}} = \mathfrak{A}$ *for any closed ideal* $\mathfrak{A}$ *of* $\hat{A}$ *with respect to the topology defined by* $\hat{\Sigma}_A$.

(ii) $\hat{A}$ *is an integral local ring and* $\hat{\Sigma}_A = \Sigma_{\hat{A}}$.

PROOF. For $\mathfrak{a} \in \Sigma_A$, we denote by $p_{\mathfrak{a}} : \hat{A} \to A/\mathfrak{a}$ the natural projection.

(i) For any ideal $\mathfrak{A}$, take an open ideal $\mathfrak{a}_{\mathfrak{A}}$ of A such that $p_{\mathfrak{a}}(\mathfrak{A}) = \mathfrak{a}_{\mathfrak{A}}/\mathfrak{a}$. Then $\mathfrak{a}_{\mathfrak{A}} \supset (A \cap \mathfrak{A}) + \mathfrak{a}$ and $\mathfrak{A} \subset \hat{\mathfrak{a}} \Leftrightarrow \mathfrak{a}_{\mathfrak{A}} = \mathfrak{a}$. Since A is a valuation ring and $\mathfrak{A}$ is closed, we obtain $\mathfrak{A} \not\subset \hat{\mathfrak{a}} \Rightarrow \mathfrak{a}_{\mathfrak{A}} = A \cap \mathfrak{A}$. Therefore $\mathfrak{a}_{\mathfrak{A}} = (A \cap \mathfrak{A}) + \mathfrak{a}$ holds for any $\mathfrak{a} \in \Sigma_A$. Thus

$$\widehat{A \cap \mathfrak{A}} = \text{proj.lim}((A \cap \mathfrak{A}) + \mathfrak{a})/\mathfrak{a} = \text{proj.lim}\, p_{\mathfrak{a}}(\mathfrak{A}) = \mathfrak{A}\,.$$

(ii) $\hat{A}$ is integral: It suffices to prove

$$\alpha, \beta \in \hat{A}\,, \quad \beta \neq 0\,, \quad \alpha\beta = 0 \Rightarrow \alpha = 0\,.$$

Since $\beta \neq 0$, there exists $\mathfrak{a}_0 \in \Sigma_A$ such that $\beta \notin \hat{\mathfrak{a}}_0$. For any $\mathfrak{a} \in \Sigma_A$, there exists $\mathfrak{b} \in \Sigma_A$ such that $\mathfrak{b} \subset \mathfrak{a}\mathfrak{a}_0$. If we write

$$p_{\mathfrak{b}}(\alpha) = a \bmod \mathfrak{b}\,, \quad p_{\mathfrak{b}}(\beta) = b \bmod \mathfrak{b} \quad (a, b \in A)\,,$$

then $b \notin \mathfrak{a}_0$ and hence $Ab \not\subset \mathfrak{a}_0$. Since A is a valuation ring, we get $\mathfrak{a}_0 \subset Ab$. Moreover $\alpha\beta = 0$ implies that

$$ab \in \mathfrak{b} \subset \mathfrak{a}\mathfrak{a}_0 \subset \mathfrak{a} \cdot Ab = \mathfrak{a}b\,.$$

Therefore $a \in \mathfrak{a}$ and hence $\alpha \in \hat{\mathfrak{a}}$. Since $\mathfrak{a} \in \Sigma_A$ is arbitrary, we obtain $\alpha = 0$.

$\hat{\Sigma}_A = \Sigma_{\hat{A}}$: Obvious from (i) and Lemma 2, (v).

Then the proof of Theorem 1 is complete from Lemma 2 and Lemma 4.

LEMMA 5. *Let A be a valuation ring. If $\mathfrak{p} \in \operatorname{Spec} A$ and $\mathfrak{p} \neq 0$, then $\widehat{A_\mathfrak{p}} = \hat{A}_{\hat{\mathfrak{p}}}$.*

PROOF. Since $A_\mathfrak{p} \neq QA$, the $A_\mathfrak{p}$-topology coincides with the A-topology on QA by Lemma 3, (i). Let $\bar{S}$ denote the saturation of the multiplicative system $S = A - \mathfrak{p}$ in $\hat{A}$. Then, by Lemma 2, (v), we get $\bar{S} = \hat{A} - \hat{\mathfrak{p}}$. By Lemma 2, (ii), we obtain

$$\widehat{A_\mathfrak{p}} = \widehat{S^{-1}A} = S^{-1}\hat{A} = \bar{S}^{-1}\hat{A} = \hat{A}_{\hat{\mathfrak{p}}} .$$

COROLLARY. *If A is complete and $B \in Zar(QA|A)$, $thenBisalso$ complete.*

PROOF OF THEOREM 2. (i) Since $(QA \cap R)^\times \hat{A}^\times = R^\times$ holds for any $R \in Zar(Q\hat{A}\,|\,\hat{A})$, the square in topological spaces:

$$\begin{array}{ccc} Zar(Q\hat{A}|\hat{A}) & \rightarrow & Zar(QA|A) \\ \downarrow & & \downarrow \\ i.Sub((Q\hat{A})^\times/\hat{A}^\times) & \rightarrow & i.Sub((QA)^\times/A^\times) \end{array}$$

commutes. Here $i.Sub(\Gamma)$ denotes the set of isolated subgroups of a totally ordered abelian group Γ. Therefore the mapping: $Zar(Q\hat{A}|\hat{A}) \rightarrow Zar(QA|A)$ is a homeomorphism. Moreover, for any $B \in Zar(QA|A)$, $B \neq QA$, we get $\hat{B} \in Zar(Q\hat{A}|\hat{A})$ by Lemma 5 and $QA \cap \hat{B} = B$ by Lemma 2, (i). Therefore the completion gives the inverse mapping.

(ii) and (iii) are easy to prove.

(iv) For $B \in Zar(QA|A)$, let $C_A(B)$ denote the set of Cauchy sequences of B and $C_A^0(B)$ the set of zero sequences of B with respect to the A-topology. Then we obtain $\hat{B} \cong C_A(B)/C_A^0(B)$. Therefore $\hat{A}$ and $Q\hat{A} = \widehat{QA}$ are the completions as metric spaces.

REMARK. The morphism: $Zar(Q\hat{A}|\hat{A}) \rightarrow Zar(QA|A)$ of local ringed spaces defined in Theorem 2, (i) is an isomorphism if and only if $A \cong \hat{A}$.

The following result is induced from Lemma 3, (i) and Theorem 2, (i).

COROLLARY. *Let K be a field and A_0 a subring of K. Then*

$$A \sim B \Rightarrow Q\hat{A} = Q\hat{B}$$

for any A, $B \in Zar(K|A_0)$. Here $\sim$ is the equivalence relation on $Zar(K|A_0)$ defined in Lemma 3.

PROOF. By Theorem 2, (i), we have $A \subset B \subsetneq K \Rightarrow Q\hat{A} = Q\hat{B}$ for any A, $B \in Zar(K|A_0)$. Therefore $A \vee B \neq K \Rightarrow Q\hat{A} = Q\hat{B}$. By Lemma 3, (i), we obtain $A \sim B \Rightarrow Q\hat{A} = Q\hat{B}$.

3. Here we show some examples of completions of valuation rings.

PROPOSITION 1. *Let K be a field and t an indeterminate over K. If $A \in Zar\,K$ is noetherian, then*

(i) *$B_1 = A[[t]]_{\mathfrak{m}(A)[[t]]}$ is a valuation ring with quotient field $(A - \{0\})^{-1}B_1$ and satisfies $A = K \cap B_1$, $B_1/\mathfrak{m}(B_1) \cong A/\mathfrak{m}(A)((t \bmod \mathfrak{m}(B_1)))$ and $K^\times/A^\times \cong (QB_1)^\times/B_1^\times$.*

(ii) $\hat{A}\{\{t\}\} = \{\sum_{i=-\infty}^{\infty} a_i t^i \mid a_i \in \hat{A}, \lim_{i\to-\infty} a_i = 0\}$ *is a complete valuation ring with quotient field* $\hat{K}_A\{\{t\}\} = \{\sum_{i=-\infty}^{\infty} \frac{a_i}{b} t^i \mid b \in A, b \neq 0, a_i \in \hat{A}, \lim_{i\to-\infty} a_i = 0\}$ *and* $\widehat{B_1} = \hat{A}\{\{t\}\}$.

(iii) $A = k \oplus \mathfrak{m}(A) \Leftrightarrow B_1 = k((t)) \oplus \mathfrak{m}(B_1)$ *for any subfield* k *of* K_{alg}.

The proof is easy.

Note that the valuation ring $A[t]_{\mathfrak{m}(A)[t]} = K(t) \cap B_1 = K(t) \cap \hat{A}\{\{t\}\}$ is said to be the trivial extension of A.

PROPOSITION 2. *Let* K *be a field and* t *an indeterminate over* K. *If* $A \in Zar\,K$, *then*

(i) $B_0 = (A \oplus tK[t])_{\mathfrak{m}(A)\oplus tK[t]} = A \oplus tK[t]_{tK[t]}$ *is a valuation ring with quotient field* $K(t)$ *and satisfies* $A = K \cap B_0$, $A/\mathfrak{m}(A) \cong B_0/\mathfrak{m}(B_0)$ *and* $K(t)^\times/B_0^\times \cong (t \bmod B_0^\times)^{\mathbf{Z}} \times K^\times/A^\times$ (*lexicographical order*).

(ii) $B = A \oplus tK[[t]]$ *is a complete valuation ring with quotient field* $K((t))$ *and* $\widehat{B_0} = B$.

(iii) $A = k \oplus \mathfrak{m}(A) \Leftrightarrow B_0 = k \oplus \mathfrak{m}(B_0)$ *for any subfield* k *of* K.

(iv) *The following conditions are equivalent*:

(a) *The exact sequence* $1 \to A^\times \to K^\times \to K^\times/A^\times \to 1$ *splits.*

(b) *The exact sequence* $1 \to B_0^\times \to K(t)^\times \to K(t)^\times/B_0^\times \to 1$ *splits.*

(c) *The exact sequence* $1 \to B^\times \to K((t))^\times \to K((t))^\times/B^\times \to 1$ *splits.*

The proof is easy.

COROLLARY. *Both the mappings*

$$f: \begin{array}{ccc} Zar\,K & \to & Zar\,K(t) \\ \Psi & & \Psi \\ A & \mapsto & B_0 \end{array} \quad \text{and} \quad g: \begin{array}{ccc} Zar\,K & \to & Zar\,K((t)) \\ \Psi & & \Psi \\ A & \mapsto & B \end{array}$$

are closed immersions.

PROOF. If we put $R_0 = K[t]_{tK[t]} \in Zar\,K(t)$ and $R = K[[t]] \in Zar\,K((t))$, then there exist isomorphisms: $Zar\,K \cong Zar(R_0/\mathfrak{m}(R_0)) \cong Zar(R/\mathfrak{m}(R))$ of local ringed spaces. Therefore $f : Zar\,K \to \overline{\{R_0\}}$ and $g : Zar\,K \to \overline{\{R\}}$ are homeomorphisms, and hence f and g are closed immersions.

EXAMPLE 2. Let k be a field. Assume that k is not algebraically closed. Take two indeterminates t, u over k and an irreducible polynomial $p \in k[u]$ such that $deg\,p \geqq 2$, and put $B = k[u]_{(p)} \oplus tk(u)[[t]]$. Then B is an equal characteristic complete discrete valuation ring of dimension two, but does not have the coefficient field.

Let A be a ring and Γ a totally ordered abelian group. Then the set

$$A((\Gamma)) = \{x \in A^\Gamma \mid \{\gamma \in \Gamma \mid x(\gamma) \neq 0\} \text{ is a well-ordered subset of } \Gamma\}$$

is a sub A-module of the direct product A^{Γ}. For $x, y \in A(\!(\Gamma)\!)$, we define $xy \in A(\!(\Gamma)\!)$ by

$$xy : \begin{array}{ccc} \Gamma & \to & A \\ \cup\!\!\!| & & \cup\!\!\!| \\ \gamma & \mapsto & \displaystyle\sum_{\alpha \in \Gamma} x(\alpha) y(\gamma - \alpha) . \end{array}$$

Then $A(\!(\Gamma)\!)$ turns out to be a ring with this product (see [2, Chapter 6, §3, Exercise 2]), and the following two subsets

$$A[[\Gamma]] = \{x \in A(\!(\Gamma)\!) \mid x(\gamma) \neq 0 \Rightarrow \gamma \geqq 0\} ,$$
$$A[\Gamma] = \{x \in A[[\Gamma]] \mid \{\gamma \in \Gamma \mid x(\gamma) \neq 0\} \text{ is a finite subset of } \Gamma\}$$

are subrings of $A(\!(\Gamma)\!)$. Moreover $\mathfrak{n} = \{x \in A[\Gamma] \mid x(0) = 0\}$ is an ideal of $A[\Gamma]$ and satisfies $A[\Gamma] = A \oplus \mathfrak{n}$.

If A is integral, then $A(\!(\Gamma)\!)$, $A[[\Gamma]]$ and $A[\Gamma]$ are all integral. Let $A(\Gamma)$ denote the quotient field of $A[\Gamma]$. Since $\mathfrak{n}$ is a prime ideal of $A[\Gamma]$, the ring

$$R(A, \Gamma) = A[\Gamma]_{\mathfrak{n}}$$

is integral and local.

PROPOSITION 3. *Let k be a field and Γ a totally ordered abelian group.*

(i) *$R(k, \Gamma)$ is a valuation ring with quotient field $k(\Gamma)$, residue field k and value group Γ.*

(ii) *$R(k, \Gamma) = k \oplus \mathfrak{m}(R(k, \Gamma))$, $k(\Gamma)^{\times} \cong R(k, \Gamma)^{\times} \times \Gamma$ (isomorphism of groups) and $k(\Gamma)^{\times}/R(k, \Gamma)^{\times} \cong \Gamma$ (anti-isomorphism of ordered set).*

For a proof, see [2, Chapter 6, §3.4, Example 6]).

COROLLARY. *For a valuation ring A, the following conditions are equivalent:*

(a) *There exists an injective homomorphism $\varphi : R(k, \Gamma) \to A$ of local rings such that* $\mathrm{Im}(\varphi) + \mathfrak{m}(A) = \mathrm{Im}(\varphi) \cdot A^{\times} = A$.

(b) *$A = k \oplus \mathfrak{m}(A)$ and there exists an split exact sequence* $1 \to A^{\times} \to (QA)^{\times} \to \Gamma \to 0$.

EXAMPLE 3. Let k be a field and $\Gamma = \mathbf{Z}^n$ (lexicographical order). Then there exist algebraically independent indeterminates $t_1, \cdots, t_n$ over k such that

(1) $k(\!(\Gamma)\!) = k(\!(t_n)\!) \cdots (\!(t_1)\!)$,

(2) $k[[\Gamma]] = k \oplus \bigoplus_{i=1}^{n} t_i k(\!(t_n)\!) \cdots (\!(t_{i+1})\!)[[t_i]]$,

(3) $k[\Gamma] = t_1 k[t_n, t_n^{-1}, \cdots, t_2, t_2^{-1}][t_1] \oplus \cdots \oplus t_{n-1} k[t_n, t_n^{-1}][t_{n-1}] \oplus k[t_n]$,

(4) $k(\Gamma) = k(t_n, \cdots, t_1)$,

(5) $R(k, \Gamma) = k \oplus \bigoplus_{i=1}^{n} t_i k(t_n, \cdots, t_{i+1})[t_i]_{(t_i)}$,

(6) $\widehat{R(k, \Gamma)} = k \oplus t_1 k(t_n, \cdots, t_2)[[t_1]] \oplus \bigoplus_{i=2}^{n} t_i k(t_n, \cdots, t_{i+1})[t_i]_{(t_i)}$.

Therefore $\widehat{R(k, \Gamma)} \neq k[[\Gamma]]$, if $n \geqq 2$.

The proof is easy.

Let A be a ring and Γ a totally ordered abelian group. For $\alpha \in \Gamma$, we define $t_\alpha \in A(\!(\Gamma)\!)$ by $t_\alpha : \gamma \mapsto t_\alpha(\gamma) = \delta_{\alpha,\gamma}$. Then $(t_\alpha x)(\gamma) = x(\gamma - \alpha)$ for any $x \in A(\!(\Gamma)\!)$.

PROPOSITION 4. *Let k be a field and Γ a totally ordered abelian group. If rank $\Gamma = 1$, then*

$$\widehat{R(k,\Gamma)} = \left\{ \sum_{i=0}^{\infty} c_i t_{\gamma_i} \mid c_i \in k, \gamma_i \in \Gamma, 0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots, \lim_{i\to\infty} \gamma_i = +\infty \right\}.$$

Therefore $\widehat{R(k,\Gamma)} \neq k[[\Gamma]]$, if Γ is not discrete.

The proof is easy.

EXAMPLE 4. Let k be a field of characteristic p $(\neq 0)$ and $\Gamma = \mathbf{Z}[\frac{1}{p}] \subset \mathbf{R}$. Put

$$\begin{cases} x = t_1 \\ y = \displaystyle\sum_{i=1}^{\infty} t_{\gamma_i}, \quad \left(\gamma_i = \frac{ip^i + 1}{p^i}\right) \end{cases}$$

and $K = k(x, y)$, $A = K \cap \widehat{R(k,\Gamma)} \in Zar(K|k)$. Then

(i) K/k is the rational function field of two variables and A satisfies $A = k \oplus \mathfrak{m}(A)$, $K^\times/A^\times \cong \Gamma$. But the exact sequence $1 \to A^\times \to K^\times \to \Gamma \to 0$ does not split.

(ii) $\hat{A} = \widehat{R(k,\Gamma)}$. Thus the exact sequence $1 \to \hat{A}^\times \to (Q\hat{A})^\times \to \Gamma \to 0$ splits.

4. Here we show some examples of valuation rings of infinite dimension and completions of such valuation rings.

EXAMPLE 5. For a field k and algebraically independent countable indeterminates $t_1, t_2, t_3, \cdots$ over k, we put

$$K = k(\cdots, t_3, t_2, t_1), \quad A = k[\cdots, t_3, t_2, t_1].$$

For $n \geqq 0$, we put

$$A_n = k(\cdots, t_{n+1}) \oplus \bigoplus_{i=1}^{n} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}.$$

Moreover we put

$$A_\infty = k \oplus \bigoplus_{i=1}^{\infty} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}.$$

Then

(i) $A_n = R(k(\cdots, t_{n+1}), \mathbf{Z}^n)$, $A_\infty = R(k, \mathbf{Z}^{\oplus \mathbf{N}}) \in Zar(K|A)$.

(ii) $\widehat{A_n} = k(\cdots, t_{n+1}) \oplus t_1 k(\cdots, t_3, t_2)[[t_1]] \oplus \bigoplus_{i=2}^{n} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}$ for $n \geqq 1$ and $\widehat{A_\infty} = k \oplus t_1 k(\cdots, t_3, t_2)[[t_1]] \oplus \bigoplus_{i=2}^{\infty} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}$. Therefore $Q\widehat{A_\infty} = k(\cdots, t_3, t_2)(\!(t_1)\!)$.

(iii) $A_\infty \in Zar(K|A)_{cl}$ and $Zar(K|A_\infty) = \{A_\infty, \cdots, A_2, A_1, K\}$, $A_\infty \subset \cdots \subset A_2 \subset A_1 \subset A_0 = K$, $A_\infty = \bigcap_{n=0}^{\infty} A_n$.

EXAMPLE 6. For a field k and algebraically independent countable indeterminates $t_{-1}, t_{-2}, t_{-3}, \cdots$ over k, we put

$$K = k(t_{-1}, t_{-2}, t_{-3}, \cdots), \quad A = k[t_{-1}, t_{-2}, t_{-3}, \cdots].$$

For $n \geqq 0$, we put

$$B_n = k(t_{-1}, \cdots, t_{-n}) \oplus \bigoplus_{i=n+1}^{\infty} t_{-i} k(t_{-1}, \cdots, t_{-i+1})[t_{-i}]_{(t_{-i})}.$$

Then

(i) $B_n = R(k(t_{-1}, \cdots, t_{-n}), \Gamma_n) \in Zar(K|A)$. Here

$$\Gamma_n = \{(\cdots, 0, e_{-m}, \cdots, e_{-n-1}) \mid m \geqq n+1, e_{-m}, \cdots, e_{-n-1} \in \mathbf{Z}\}$$

is a totally ordered abelian group with the lexicographical order.

(ii) $\widehat{B_n} = k(t_{-1}, \cdots, t_{-n}) \times \prod_{i=n+1}^{\infty} t_{-i} k(t_{-1}, \cdots, t_{-i+1})[t_{-i}]_{(t_{-i})}$.

(iii) $B_0 = R(k, \Gamma_0) \in Zar(K|A)_{cl}$ and $Zar(K|B_0) = \{B_0, B_1, B_2, \cdots, K\}$, $B_0 \subset B_1 \subset B_2 \subset \cdots \subset K$, $\bigcup_{n=0}^{\infty} B_n = K$.

EXAMPLE 7. For a field k and algebraically independent countable indeterminates $(t_n)_{n \in \mathbf{Z}}$ over k, we put

$$K = k(\cdots, t_1, t_0, t_{-1}, \cdots), \quad A = k[\cdots, t_1, t_0, t_{-1}, \cdots].$$

For $n \in \mathbf{Z}$, we put

$$C_n = k(\cdots, t_{n+1}) \oplus \bigoplus_{i=-\infty}^{n} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}.$$

Moreover we put

$$C_\infty = k \oplus \bigoplus_{i=-\infty}^{\infty} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}.$$

Then

(i) $C_n, C_\infty = R(k, \mathbf{Z}^{\oplus \mathbf{Z}}) \in Zar(K|A)$.

(ii) $\widehat{C_n} = k(\cdots, t_{n+1}) \times \prod_{i=-\infty}^{n} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}$ and $\widehat{C_\infty} = k \times \bigoplus_{i=1}^{\infty} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)} \times \prod_{i=-\infty}^{0} t_i k(\cdots, t_{i+1})[t_i]_{(t_i)}$.

(iii) $C_\infty \in Zar(K|A)_{cl}$ and $Zar(K|C_\infty) = \{C_\infty, \cdots, C_1, C_0, \cdots, K\}$, $C_\infty \subset \cdots \subset C_1 \subset C_0 \subset \cdots \subset K$, $C_\infty = \bigcap_{n=-\infty}^{\infty} C_n$, $\bigcup_{n=-\infty}^{\infty} C_n = K$.

REMARK. We can use the field $k((\Gamma))$ in Examples 5, 6 and 7 instead of $k(\Gamma)$, to obtain similar results.

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