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Projectively Invariant Cocycles of Holomorphic Vector Fields on an Open Riemann Surface

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Abstract. Let Σ be an open Riemann surface and $\mathrm{Hol}(\Sigma)$ be the Lie algebra of holomorphic vector fields on Σ. We fix a projective structure (i.e. a local $\mathrm{SL}_2(\mathbf{C})$-structure) on Σ. We calculate the first group of cohomology of $\mathrm{Hol}(\Sigma)$ with coefficients in the space of linear holomorphic operators acting on tensor densities, vanishing on the Lie algebra $\mathrm{sl}_2(\mathbf{C})$. The result is independent on the choice of the projective structure. We give explicit formulas of 1-cocycles generating this cohomology group.

1. Introduction.

The first group of cohomology of the Lie algebra of (formal) vector fields on the circle S^1 with coefficients in the space $\mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$, where $\mathcal{F}_\lambda$ is the space of tensor densities of degree λ on S^1, was first calculated in [6]. This group of cohomology measures all extensions of exact sequences $0 \to \mathcal{F}_\mu \to \cdot \to \mathcal{F}_\lambda \to 0$ of modules. The first group of cohomology of the Lie algebra $\mathrm{Vect}(S^1)$ of (smooth) vectors fields on the circle with coefficients in the space of linear differential operators acting from λ-densities to μ-densities, vanishing on the subalgebra $\mathrm{sl}_2(\mathbf{R}) \subset \mathrm{Vect}(S^1)$, was calculated in [2]. This group of cohomology appears as an obstruction to the equivariant quantization (see [2], [8]). The computation is based on the following observation: any 1-cocycle vanishing on the subalgebra $\mathrm{sl}_2(\mathbf{R})$ is an $\mathrm{sl}_2(\mathbf{R})$-invariant operator. The $\mathrm{sl}_2(\mathbf{R})$-invariant differential operators acting on tensor densities, which are called “Transvectants”, were classified by Gordan (see [7], [11]). To find the 1-cocycles generating the group of cohomology means, therefore, to determine which from the Transvectants are 1-cocycles.

In this paper, we study the complex analog of the above group of cohomology on an open Riemann surface Σ endowed with a flat projective structure.

The aim of this paper is to describe the first group of cohomology

$$\mathrm{H}^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbf{C}); \mathcal{D}_{\lambda,\mu}(\Sigma)), \tag{1.1}$$

of holomorphic vector fields on Σ with coefficients in the space of linear holomorphic operators acting on tensor densities, vanishing on the Lie algebra $\mathrm{sl}_2(\mathbf{C})$. We give explicit formulas

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of 1-cocycles generating the group (1.1). These 1-cocycles are the complex analog of the 1-cocycles given in [2].

The main tool of this paper is the existence of affine and projective connection on any open Riemann surface (see [10], [13]). These notions has been recently used in [16] to compute the second group of cohomology of the Lie algebra $\mathrm{Hol}(\Sigma)$ with coefficients in the space of λ-densities.

2. Affine and projective structure.

Let Σ be a Riemann surface, and let $\{U_\alpha, z_\alpha\}$ be an atlas of Σ.

A holomorphic affine connection is a family of holomorphic functions $\Gamma(z_\alpha)$ on U_α such that for non-empty $U_\alpha \cap U_\beta$, we have

$$\Gamma(z_\beta)\frac{dz_\beta}{dz_\alpha} = \Gamma(z_\alpha) + \frac{d^2 z_\beta}{dz_\alpha^2}\frac{dz_\alpha}{dz_\beta}\,.$$

Affine connection exists in any open Riemann surface in contrast with the compact case where affine connection exists only if the genus of Σ is one (see [10]).

A holomorphic projective connection is a family of holomorphic functions $R(z_\alpha)$ on U_α such that for non-empty $U_\alpha \cap U_\beta$, we have

$$R(z_\beta)\left(\frac{dz_b}{dz_\alpha}\right)^2 = R(z_\alpha) + S(z_\beta, z_\alpha)\,,$$

where $S(z_\beta, z_\alpha) = \left(\frac{d^2 z_\beta}{dz_\alpha^2}\right)' - \frac{1}{2}\left(\frac{d^2 z_\beta}{dz_\alpha^2}\right)^2$ is the Schwarzian derivative.

Recall that any holomorphic affine connection Γ defines naturally a holomorphic projective connection given in a local coordiates z by

$$R(z) = \frac{d\Gamma(z)}{dz} - \frac{1}{2}\Gamma^2(z)\,. \tag{2.1}$$

For any projective connection R there exits locally an affine connection Γ satisfying (2.1).

We say that two affine connections are projectively equivalent if they define the same projective connection.

Let us define the notion of projective structure.

A Riemann surface admits a projective structure if there exists an atlas of charts $\{U_\alpha, z_\alpha\}$ such that the coordinate change $z_\alpha \circ z_\beta^{-1}$ are projective transformations.

In this case, the Lie algebra $\mathrm{sl}_2(\mathbf{C})$, in each chart of the projective structure, is generated by the following vector fields

$$\frac{d}{dz},\quad z\frac{d}{dz},\quad z^2\frac{d}{dz}\,. \tag{2.2}$$

There exists a 1-1 correspondence between projective structures and projective connections on an open Riemann surface (cf. [10]). We use implicitly this correspondence along this paper.

3. Hol(Σ)-module structures on the space of holomorphic operators.

The modules of linear differential operators on the space of tensor densities on a (real) smooth manifold has been studied in series of recent papers (see [1], [2], [3], [4], [8], [9], [14], [15]). Note that this space viewed as a module over the Lie algebra of vector fields has already been studied in the classical monograph [17].

Let us give the definition of the natural two-parameter family of modules over the Lie algebra of holomorphic vector fields on the space of linear holomorphic operators.

3.1. Tensor Densities. Let Σ be an open Riemann surface. Fix an affine connection Γ on it.

Space of tensor densities on Σ, noted $\mathcal{F}_\lambda$, is the space of sections of the line bundle $(T^*\Sigma)^{\otimes\lambda}$, where $\lambda \in \mathbf{C}$. This bundle is of course trivial, since any (holomorphic) bundle on an open Riemann surface is holomorphically trivial.

Fix a global section dz^λ and $\mathcal{F}_\lambda$. Any λ-density can be written in the form $\phi\ dz^\lambda$. Let us recall the definition of a covariant derivative of tensor densities. Let ∇ be the covariant derivative associated to the affine connection Γ. If $\phi \in \mathcal{F}_\lambda$, then $\nabla\phi \in \Omega^1(\Sigma) \otimes \mathcal{F}_\lambda$ is given by the formula

$$\nabla\phi = \frac{d\phi}{dz} - \lambda\Gamma\phi\,.$$

The standard action of Hol(Σ) on $\mathcal{F}_\lambda$ reads as follows (cf. [16]):

$$L_X^\lambda(\phi) = X\nabla\phi + \lambda\phi\nabla X\,, \tag{3.1}$$

where $X = X(z)\frac{d}{dz} \in \mathrm{Hol}(\Sigma)\,.$

3.2. Hol(Σ)-module of holomorphic operators. Consider holomorphic operators acting on tensor densities:

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu\,. \tag{3.2}$$

In local coordinates z, any operator A can be written in the form

$$A = a_k(z)\frac{d^k}{dz^k} + \cdots + a_0(z)\,,$$

where a_i, for $i = 0, \ldots, k$, are holomorphic functions on z.

A two-parameter family of actions of Hol(Σ) on the space of holomorphic operators is defined by

$$L_X^{\lambda,\mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda\,. \tag{3.3}$$

Demote by $\mathcal{D}_{\lambda,\mu}(\Sigma)$ the space of operators (3.2) endowed with the defined Hol(Σ)-module structure (3.3).

4. Main result.

Assume Σ is endowed with a projective structure, this defines locally an action of the Lie group $\mathrm{SL}_2(\mathbf{C})$ on Σ.

Consider cochains on $\mathrm{Hol}(\Sigma)$ with values in $\mathcal{D}_{\lambda,\mu}$, vanishing on the Lie algebra $\mathrm{sl}_2(\mathbf{C})$. One, therefore, obtains the so-called relative cohomology of the Lie algebra $\mathrm{Hol}(\Sigma)$, namely

$$\mathrm{H}^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbf{C}); \mathcal{D}_{\lambda,\mu}(\Sigma)) ,$$

(see [5]).

The purpose of this paper is the following:

THEOREM 4.1. *The first group of cohomology* $\mathrm{H}^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbf{C}); \mathcal{D}_{\lambda,\mu}(\Sigma))$ *is one-dimensional in the following cases*:

(a) $\mu - \lambda = 2, \lambda \neq -1/2,$
(b) $\mu - \lambda = 3, \lambda \neq -1,$
(c) $\mu - \lambda = 4, \lambda \neq -1/2,$
(d) $(\lambda, \mu) = (-4, 1), (0, 5).$

Otherwise, this cohomology group is trivial.

This Theorem generalizes the result of [2] in the case of the circle S^1.

Note that this result does not depend on the choice of the projective structure.

5. Construction of the 1-cocycles.

In this section, we give explicit formulas for the 1-cocycles generating the nontrivial cohomology classes from Theorem 4.1. Given a projective structure on Σ, we prove that there is a canonical choice of the 1-cocycles vanishing on $\mathrm{sl}_2(\mathbf{C})$.

Fix (locally) an affine connection Γ related to the projective structure. Denote by R the projective connection associated to Γ (see section 2).

LEMMA 5.1. *The following linear differential operators*

$$\begin{aligned}
\mathcal{I}_3 &= R\,,\\
\mathcal{I}_4 &= R\nabla - \frac{\lambda}{2}\nabla R\,,\\
\mathcal{I}_5 &= R\nabla^2 - \frac{2\lambda+1}{2}\nabla R\nabla + \frac{\lambda(2\lambda+1)}{10}\nabla^2 R + \frac{\lambda(\lambda+3)}{5}R^2\,,\\
\mathcal{I}_6 &= R\nabla^3 - \frac{3}{2}\nabla R\nabla^2 + \Big(\frac{3}{10}\nabla^2 R + \frac{4}{5}R^2\Big)\nabla\,,\\
\mathcal{I}_6' &= R\nabla^3 + \frac{9}{2}\nabla R\nabla^2 + \Big(\frac{63}{10}\nabla^2 R + \frac{4}{5}R^2\Big)\nabla + \frac{14}{5}\nabla^3 R + \frac{8}{5}R\nabla R\,,
\end{aligned}$$

are globally defined in $\mathcal{D}_{\lambda,\lambda+2}(\Sigma), \mathcal{D}_{\lambda,\lambda+3}(\Sigma), \mathcal{D}_{\lambda,\lambda+4}(\Sigma), \mathcal{D}_{0,5}(\Sigma), \mathcal{D}_{-4,1}(\Sigma)$, *respectively and depend only on the projective class of the connection* Γ.

PROOF. Since the surface Σ is projectively flat, the connection R defines a 2-density on Σ. Thus the operators of the above Lemma are globally defined. Let us prove that these operators depend only on the projective class of the connection Γ.

Let $\mathcal{I}_4 = R\nabla + \alpha\nabla R$. Denote by $\tilde{\mathcal{I}}_4$ the operator $\mathcal{I}_4$ written with respect to the connection $\tilde{\Gamma}$ which is projectively equivalent to Γ. After an easy calculation one has

$$\mathcal{I}_4 = \tilde{\mathcal{I}}_4 + (2\alpha + \lambda)(\tilde{\Gamma} - \Gamma)R .$$

Then $\mathcal{I}_4 = \tilde{\mathcal{I}}_4$ if and only if $\alpha = -\lambda/2$. The proof is analogous for the operators $\mathcal{I}_5, \mathcal{I}_6$ and $\mathcal{I}'_6$.

THEOREM 5.2. (i) *For every* λ, *there exist unique* (*up to constant*) 1-*cocycles*

$$\mathcal{J}_3 : \mathrm{Hol}(\Sigma) \to \mathcal{D}_{\lambda,\lambda+2}(\Sigma)$$
$$\mathcal{J}_4 : \mathrm{Hol}(\Sigma) \to \mathcal{D}_{\lambda,\lambda+3}(\Sigma)$$
$$\mathcal{J}_5 : \mathrm{Hol}(\Sigma) \to \mathcal{D}_{\lambda,\lambda+4}(\Sigma)$$

vanishing on $\mathrm{sl}_2(\mathbf{C})$. *They are given by the formulae*:

$$\mathcal{J}_3(X) = \nabla^3 X - L_X^{\lambda,\lambda+2}(\mathcal{I}_3) ,$$
$$\mathcal{J}_4(X) = \nabla^3 X\nabla - \frac{\lambda}{2}\nabla^4 X - L_X^{\lambda,\lambda+3}(\mathcal{I}_4) ,$$
$$\mathcal{J}_5(X) = \nabla^3 X\nabla^2 - \frac{2\lambda+1}{2}\nabla^4 X\nabla + \frac{\lambda(2\lambda+1)}{10}\nabla^5 X - L_X^{\lambda,\lambda+4}(\mathcal{I}_5) .$$

For $(\lambda, \mu) = (0, 5), (-4, 1)$, *respectively*, *the* 1-*cocycles vanishing on* $\mathrm{sl}_2(\mathbf{C})$ *are given by*

$$\mathcal{J}_6^0(X) = \nabla^3 X\nabla^3 - \frac{3}{2}\nabla^4 X\nabla^2 + \frac{3}{10}\nabla^5 X\nabla - L_X^{0,5}(\mathcal{I}_6) ,$$
$$\mathcal{J}_6^{-4} = \nabla^3 X\nabla^3 + \frac{9}{2}\nabla^4 X\nabla^2 + \frac{63}{10}\nabla^5 X\nabla + \frac{14}{5}\nabla^6 X - L_X^{-4,1}(\mathcal{I}'_6) .$$

(ii) *The* 1-*cocycles* $\mathcal{J}_3, \mathcal{J}_4$ *and* $\mathcal{J}_5$ *are non-trivial for every* λ *except* $\lambda = -1/2$, $\lambda = -1$ *and* $\lambda = -3/2$, *respectively. The* 1-*cocycles* $\mathcal{J}_6^0$ *and* $\mathcal{J}_6^{-4}$ *are non-trivial.*

(iii) *These* 1-*cocycles are independent on the choice of the projective structure.*

6. $\mathrm{sl}_2(\mathbf{R})$-invariant operators on S^1.

For almost all λ and μ, there exists unique (up to constant) $\mathrm{sl}_2(\mathbf{R})$-invariant bilinear differential operators $J_m^{\lambda,\mu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+m}$ given by

$$J_m^{\lambda,\mu}(\phi, \psi) = \sum_{i+j=m} (-1)^j m! \binom{2\lambda+m-1}{j}\binom{2\mu+m-1}{i}\phi^{(i)}\psi^{(j)} , \qquad (6.1)$$

called "Transvectants" (see [9], [11]).

Let us recall the results of [2], [9]. The first group of cohomology $\mathrm{H}^1(\mathrm{Vect}(S^1), \mathrm{sl}_2(\mathbf{R}); \mathcal{D}_{\lambda,\mu}(S^1))$ is one-dimensional in the following cases:

(a) $\mu - \lambda = 2, \lambda \neq -1/2$,
(b) $\mu - \lambda = 3, \lambda \neq -1$,
(c) $\mu - \lambda = 4, \lambda \neq -1/2$,
(d) $(\lambda, \mu) = (-4, 1), (0, 5)$.

Otherwise, this cohomology group is trivial (see [2]).

This group of cohomology is generated by the following 1-cocycles which are particular cases of the Transvectants (6.1):

$$\begin{aligned}
J_3(X,\phi) &= X'''\phi\,,\\
J_4(X,\phi) &= X'''\phi' - \frac{\lambda}{2}X^{IV}\phi\,,\\
J_5(X,\phi) &= X'''\phi'' - \frac{2\lambda+1}{2}X^{IV}\phi' + \frac{\lambda(2\lambda+1)}{10}X^{V}\phi\,,\\
J_6(X,\phi) &= X'''\phi''' - \frac{3}{2}X^{IV}\phi'' + \frac{3}{10}X^{V}\phi'\,,\\
J_6'(X,\phi) &= X'''\phi''' + \frac{9}{2}X^{IV}\phi'' + \frac{63}{10}X^{V}\phi' + \frac{14}{5}X^{VI}\phi\,.
\end{aligned} \tag{6.2}$$

(see [2], [9]).

The 1-cocycles given in Theorem 5.2 are the complex analogue of the 1-cocycles (6.2).

7. Proof of the main Theorems.

In this section, we will prove Theorem 4.1 and Theorem 5.2.

7.1. Proof of the Theorem 5.2. To prove that the operators $\mathcal{J}_k$, for $k = 3, 4, 5, 6$, are 1-cocycles one has to check the 1-cocycle relation. It reads as follows

$$\mathcal{J}_k[X,Y] - L_X^{\lambda,\lambda+k-1}(\mathcal{J}_k(Y)) + L_Y^{\lambda,\lambda+k-1}(\mathcal{J}_k(X)) = 0\,, \tag{7.1}$$

where $X, Y \in \mathrm{Hol}(\Sigma)$.

Let us verify (7.1) for the operator $\mathcal{J}_3$. It is obvious that $L_X^{\lambda,\lambda+2}(\mathcal{R})$ is a 1-cocycle. It suffices then to verify the relation (7.1) for the 0-order operator $\nabla^3 X$:

$$\begin{aligned}
&\nabla^3([X,Y])\phi - L_X^{\lambda,\lambda+2}(\nabla^3 Y\phi) + L_Y^{\lambda,\lambda+2}(\nabla^3 X\phi)\\
&\quad = (2\nabla X\nabla^3 Y + X\nabla^4 Y - 2\nabla Y\nabla^3 X - Y\nabla^4 X)\phi - X\nabla(\nabla^3\phi)\\
&\qquad - (\lambda+2)\nabla X\nabla^3 Y\phi + \nabla^3 Y(X\nabla\phi + \lambda\nabla X\phi) + Y\nabla(\nabla^3 X\phi)\\
&\qquad + (\lambda+2)\nabla Y\nabla^3 X\phi - \nabla^3 X(Y\nabla\phi + \lambda\nabla Y\phi)\\
&\quad = 0\,.
\end{aligned}$$

Let us prove that the 1-cocycle $\mathcal{J}_3$ vanishes on $\mathrm{sl}_2(\mathbf{C})$. Let X be one of vector fields (2.2). After calculation one has $\nabla^3 X = 2R\nabla X + X\nabla R$. It is easy to see that $L_X^{\lambda,\lambda+2}(\mathcal{I}_3) = 2R\nabla X + X\nabla R$. Hence, one obtains $\mathcal{J}_3(X) = 0$.

In the same manner we prove that the operators $\mathcal{J}_k$, for $k = 4, 5, 6$, are 1-cocycles vanishing on $\mathrm{sl}_2(\mathbf{C})$. Theorem 5.2 (i) is proven.

Let us prove the non-triviality of the 1-cocycle $\mathcal{J}_3$ for $\lambda \neq -1/2$. Suppose that the 1-cocycle $\mathcal{J}_3(\Sigma)$ is trivial, then there exists an operator $A \in \mathcal{D}_{\lambda,\lambda+2}(\Sigma)$ such that

$$\mathcal{J}_3(X) = L_X^{\lambda,\lambda+2}(A)\,. \tag{7.2}$$

In a neighborhood of a point z, one can choice a local coordinates such that the connection $R = 0$. In these coordinates, the 1-cocycle J_3 coincides with the 1-cocyle J_3 of (6.2). Hence the relation (7.2) implies that the 1-cocycle J_3 is trivial which is absurd (see section 6). For $\lambda = -1/2$, take $A = -2\nabla^2 \in \mathcal{D}_{-\frac{1}{2},\frac{3}{2}}(\Sigma)$. One can easily check that $\mathcal{J}_3(X) = L_X^{-1/2.3/2}(A)$.

With the same arguments we prove the non-triviality of the 1-cocyles $\mathcal{J}_k$, for $k = 4, 5, 6$. Theorem 5.2 (ii) is proven.

7.2. Proof of Theorem 4.1. Let us prove that the dimension of the group of cohomology $\mathrm{H}^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbf{C}); \mathcal{D}_{\lambda,\mu}(\Sigma))$ is bounded by the dimension of the group of cohomology $\mathrm{H}^1(\mathrm{Vect}(S^1), \mathrm{sl}_2(\mathbf{R}); \mathcal{D}_{\lambda,\mu}(S^1))$. Let C and C' be two 1-cocycles in $\mathrm{H}^1(\mathrm{Hol}(\Sigma), \mathrm{sl}_2(\mathbf{C}); \mathcal{D}_{\lambda,\mu}(\Sigma))$. We will prove that C and C' are cohomologous. Denote $\tilde{C}$ and $\tilde{C}'$ the restriction of C and C' on a neighborhood of a point of Σ. The operators $\tilde{C}$ and $\tilde{C}'$ define 1-cocycle in $\mathrm{H}^1(\mathrm{Vect}(S^1), \mathrm{sl}_2(\mathbf{R}); \mathcal{D}_{\lambda,\mu}(S^1))$. These 1-cocycles are equal (up to constant); since the unique $\mathrm{sl}_2(\mathbf{R})$-invariant linear differential operators are given as in (6.2). It follows that the 1-cocyles C and C' are cohomologous. Now from the construction of the 1-cocyles given in Theorem 5.2 follows Theorem 4.1.

8. Final remark.

The group of cohomology $\mathrm{H}^1(\mathrm{Diff}(S^1), \mathrm{PSL}_2(\mathbf{R}); \mathcal{D}_{\lambda,\mu}(S^1))$ is one-dimensional, for generic λ, generated by the following 1-cocycles

$$
\begin{aligned}
\mathcal{S}_\lambda(f) &= S(f)\,,\\
\mathcal{T}_\lambda(f) &= S(f)\frac{d}{dx} - \frac{\lambda}{2}S(f)'\,,\\
\mathcal{U}_\lambda(f) &= S(f)\frac{d^2}{dx^2} - \frac{2\lambda+1}{2}S(f)'\frac{d}{dx} + \frac{\lambda(2\lambda+1)}{10}S(f)'' - \frac{\lambda(\lambda+3)}{5}S(f)^2\,,\\
\mathcal{V}_0(f) &= S(f)\frac{d^3}{dx^3} - \frac{3}{2}S(f)'\frac{d^2}{dx^2} + \Big(\frac{3}{10}S(f)'' + \frac{4}{5}S(f)^2\Big)\frac{d}{dx}\,,\\
\mathcal{V}_{-4}(f) &= S(f)\frac{d^3}{dx^3} + \frac{9}{2}S(f)'\frac{d^2}{dx^2} + \Big(\frac{63}{10}S(f)'' + \frac{4}{5}S(f)^2\Big)\frac{d}{dx} + \frac{14}{5}S(f)'''\\
&\quad + \frac{8}{5}S(f)S(f)'\,,
\end{aligned}
\tag{8.1}
$$

where $S(f)$ is the Schwarzian derivative (see [2]). It is a remarkable fact to see that the 1-cocycles (8.1) on the group $\mathrm{Diff}(S^1)$ have the same expression (up to change of sign) than the operators of the Lemma 5.1 if one replaces the connection R by -$S(f)$.

Since the group of biholomorphic maps on a Riemann surface is finite dimension (see [12]), this group does not integrate the Lie algebra of holomorphic vector fields. In some sense, the cohomology group (1.1) contains informations coming from the cohomology of the group of diffeomorphisms of S^1 and the cohomology of the Lie algebra of vector fields on S^1.

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