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Fourier Ultra-Hyperfunctions as Boundary Values of Smooth Solutions of the Heat Equation

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Abstract. We consider Fourier ultra-hyperfunctions and characterize them as boundary values of smooth solutions of the heat equation. Namely we show that the convolution of the heat kernel and a Fourier ultra-hyperfunction is a smooth solution of the heat equation with some exponential growth condition and, conversely that such smooth solution can be represented by the convolution of the heat kernel and a Fourier ultra-hyperfunction.

1. Introduction.

T. Matsuzawa characterized the spaces of distributions, ultradistributions and hyperfunctions by means of the boundary value of C^∞-solutions of the heat equation with appropriate growth rate condition [7], [8]. K. W. Kim, S.-Y. Chung and D. Kim characterized Fourier hyperfunctions [5] and M. Budinče-vić, Z. Lozanov-Crvenković and D. Perošić characterized tempered ultradistributions of Beurling and Roumieu type by similar idea [1].

In this paper we shall treat the space Q_0' of Fourier ultra-hyperfunctions. In §3 we introduce analytic functionals with unbounded carrier. In §4 we show that the convolution of the heat kernel and a Fourier ultra-hyperfunction is a smooth solution of the heat equation with some exponential growth condition (Lemma 4.3). In §5 we show that such smooth solution can be represented by the convolution of the heat kernel and a Fourier ultra-hyperfunction (Theorem 5.1) and in the middle of the proof we see that Fourier ultra-hyperfunction $T(z)$ has the form $T(z) = P(-\triangle)g(z) + h(z)$, where P is an ultradifferential operator, $g(z)$ and $h(z)$ are entire functions with some exponential growth. Thus we can characterize a space of *analytic functionals* by the same way.

REMARK 1.1. After completed the paper, the author recognized the almost same results had been obtained in [3]. Our space Q_0' of Fourier ultra-hyperfunctions was called in [2] and [3] the space $\mathcal{G}'$ of extended Fourier hyperfunctions. Our argument goes on in the complex space. For example in our main theorem (Theorem 5.1) the growth condition of $U(z,t)$ is given on $\mathbf{C}^n \times \{t; t > 0\}$ while in Theorem 2.5 of [3] it was given on $\mathbf{R}^n \times \{t; 0 < t < \varepsilon\}$.

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2. Notations.

First we define some notation used in this paper.

$$\mathbf{C}^n = \mathbf{R}^n + \imath \mathbf{R}^n .$$

$$z = x + \imath y , \quad \zeta = \xi + \imath \eta .$$

$$z = (z_1, z_2, \cdots, z_n) , \quad z_j = x_j + \imath y_j , \quad j = 1, 2, \cdots, n .$$

$$\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) , \quad \zeta_j = \xi_j + \imath \eta_j , \quad j = 1, 2, \cdots, n .$$

We denote by $\mathbf{C}_j = \mathbf{R}_j + \imath \mathbf{R}_j$ the z_j-plane, $j = 1, 2, \cdots, n$. We put

$$\langle \zeta, z \rangle = \sum_{j=1}^{n} \zeta_j z_j , \ z^2 = \langle z, z \rangle \quad \text{and} \quad |x|^2 = \sum_{j=1}^{n} |x_j|^2 .$$

$$\text{For } R \geq 0 , \quad B_R = \{x \in \mathbf{R}^n : |x_j| \leq R, \ j = 1, 2, \cdots, n\} .$$

Let K be a convex compact set in $\mathbf{R}^n$. Then we define the supporting function $h_K(x)$ by

$$h_K(x) = \sup_{\xi \in K} \langle \xi, x \rangle .$$

We denote the "*complex Laplacian*" by

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} .$$

We use the multi-index notations: for $m = (m_1, m_2, \cdots, m_n) \in \mathbf{N}^n$,

$$|m| = m_1 + \cdots + m_n ,$$

$$m! = m_1! m_2! \cdots m_n! .$$

Let L be a closed set in $\mathbf{C}^n$ and let $\overset{\circ}{L}$ be the interior of L. We denote by $\mathcal{H}(\overset{\circ}{L})$ the space of holomorphic functions on $\overset{\circ}{L}$ and by $\mathcal{C}(L)$ the space of continuous functions on L.

3. Analytic functionals with unbounded carrier.

Now we shall recall some definitions and facts about analytic functionals with unbounded carrier. For the details of this section, we refer the reader to [9] and [10].

DEFINITION 3.1. Let K and K' be convex compact sets in $\mathbf{R}^n$. Then we define $Q_b(\mathbf{R}^n + \imath K, K')$ as follows:

$$Q_b(\mathbf{R}^n + \imath K, K') := \{f \in \mathcal{H}(\mathbf{R}^n + \imath \overset{\circ}{K}) \cap \mathcal{C}(\mathbf{R}^n + \imath K) : \sup_{z \in \mathbf{R}^n + \imath K} |f(z) e^{h_{K'}(x)}| < +\infty\}.$$

$Q_b(\mathbf{R}^n + \imath K, K')$ is a Banach space. If $K_1 \subset K_2$ and $K_1' \subset K_2'$, we can define the canonical mapping

$$Q_b(\mathbf{R}^n + \imath K_2, K_2') \hookrightarrow Q_b(\mathbf{R}^n + \imath K_1, K_1').$$

DEFINITION 3.2. We define the space Q_0 as follows:

$$Q_0 := \varprojlim_{K, K' \subset\subset \mathbf{R}^n} Q_b(\mathbf{R}^n + \imath K, K'),$$

where $\varprojlim$ means the projective limit with respect to the canonical mappings.

DEFINITION 3.3. We denote by Q_0' the dual space of Q_0. An element of Q_0' is called a Fourier ultra-hyperfunction.

DEFINITION 3.4. We define the space $Q(\mathbf{R}^n + \imath K, K')$ as follows:

$$Q(\mathbf{R}^n + \imath K, K') := \varinjlim_{\varepsilon > 0, \varepsilon' > 0} Q_b(\mathbf{R}^n + \imath K_\varepsilon, K'_{\varepsilon'}).$$

where $K_\varepsilon = K + B_\varepsilon$, $K'_{\varepsilon'} = K' + B_{\varepsilon'}$ and $\varinjlim$ means the inductive limit with respect to the canonical mappings.

DEFINITION 3.5. We denote by $Q'(\mathbf{R}^n + \imath K, K')$ the dual space of $Q(\mathbf{R}^n + \imath K, K')$. An element of $Q'(\mathbf{R}^n + \imath K, K')$ is called an analytic functional with carrier $\mathbf{R}^n + \imath K$ and of type K'.

REMARK 3.6. The following fact is known:

$$Q_0' = \bigcup_{K, K' \subset\subset \mathbf{R}^n} Q'(\mathbf{R}^n + \imath K, K').$$

LEMMA 3.7 ([10]). *Let $T \in Q_0'$. Then there exist convex compact sets K and K' and a Radon measure μ such that*

$$\langle T, \varphi \rangle = \int_{\mathbf{R}^n + \imath K} \varphi(\zeta) e^{h_{K'}(\xi)} d\mu(\zeta), \quad \textit{for } \varphi \in Q_0,$$

and $|\mu(\mathbf{R}^n + \imath K)| < \infty$.

4. Preparations for Main theorem.

In this section, we shall prepare some lemmas for the main theorem.

LEMMA 4.1. *For $t > 0$ and $z \in \mathbf{C}^n$, we put*

$$E(z,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{z^2}{4t}} \quad (\textit{heat kernel})\,.$$

Then $E(z,t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \triangle\right) E(z,t) = 0\,, \tag{1}$$

$$\int_{\mathbf{R}^n} E(x,t)dx = 1, \tag{2}$$

$$E(z,t) \in Q_0 \quad \textit{for fixed } t > 0\,. \tag{3}$$

LEMMA 4.2. *For $\varphi(z) \in Q_0$, we put*

$$\varphi_t(\zeta) = \int_{\mathbf{R}^n} E(\zeta - x, t)\varphi(x)dx\,, \quad \zeta \in \mathbf{C}^n, \quad t > 0\,.$$

(i) *$\varphi_t(\zeta)$ belongs to Q_0.*
(ii) *$\lim_{t\to 0_+} \varphi_t(\zeta) = \varphi(\zeta)$ in Q_0.*

PROOF. (i) is obvious. Let us prove (ii): for $t > 0$ we have

$$\varphi_t(\zeta) = \int_{\mathbf{R}+\imath\eta_n} \cdots \int_{\mathbf{R}+\imath\eta_1} E(\zeta - z, t)\varphi(z)dz_1 \cdots dz_n\,.$$

Then

$$\begin{aligned}
\varphi_t(\zeta) - \varphi(\zeta) &= \int_{\mathbf{R}^n} E(\zeta - x, t)\varphi(x)dx - \varphi(\zeta) \\
&= \int_{\mathbf{R}+\imath\eta_n} \cdots \int_{\mathbf{R}+\imath\eta_1} E(\zeta - z, t)\varphi(z)dz_1 \cdots dz_n - \varphi(\zeta) \\
&= \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} E(\xi - x, t)\varphi(x + \imath\eta)dx_1 \cdots dx_n - \varphi(\zeta) \\
&= \int_{\mathbf{R}^n} E(u,t)(\varphi(\zeta - u) - \varphi(\zeta))du\,.
\end{aligned} \tag{4}$$

By the definition of the projective limit topology, we have to prove

$$\sup_{\zeta \in \mathbf{R}^n + \imath K} |\varphi_t(\zeta) - \varphi(\zeta)| e^{h_{K'}(\xi)} \to 0 \quad (\text{as } t \to 0_+)$$

for any convex compact sets K and K' in $\mathbf{R}^n$. Take $R > 0$ such that $K' \subset B_R$. Since $\varphi(\zeta) \in Q_0$, there exists a constant $C \geq 0$ such that $|\varphi(\zeta)| \leq Ce^{-R\sum_{j=1}^n |\xi_j|}$. On the other hand, by Cauchy's integral formula, we have $|\frac{\partial}{\partial \zeta_j}\varphi(\zeta)| \leq Ce^{-R\sum_{j=1}^n |\xi_j|}$. Therefore,

$$\begin{aligned}
|\varphi(\zeta - u) - \varphi(\zeta)| &= \left| \int_0^1 \frac{d}{dt}\varphi(t(\zeta - u) + (1-t)\zeta)dt \right| \\
&\leq \int_0^1 \sum_{j=1}^n |\frac{\partial}{\partial \zeta_j}\varphi(\zeta - tu)||u_j|dt \\
&\leq C\int_0^1 \sum_{j=1}^n e^{-R\sum_{j=1}^n |\xi_j| + tR\sum_{j=1}^n |u_j|}|u_j|dt \\
&\leq Ce^{-R\sum_{j=1}^n |\xi_j| + R\sum_{j=1}^n |u_j|} \times \sum_{j=1}^n |u_j| \,. \qquad (5)
\end{aligned}$$

If $|u_j| \leq 1$, then $|\varphi(\zeta - u) - \varphi(\zeta)| \leq Ce^{-R\sum_{j=1}^n |\xi_j|} \times \sum_{j=1}^n |u_j|$; that is,

$$|\varphi(\zeta - u) - \varphi(\zeta)|e^{R\sum_{j=1}^n |\xi_j|} \leq Ce^{nR}\sum_{j=1}^n |u_j| \,.$$

Therefore, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|u| < \delta \Rightarrow |\varphi(\zeta - u) - \varphi(\zeta)|e^{R\sum_{j=1}^n |\xi_j|} < \varepsilon \,.$$

For δ_1 with $0 < \delta_1 < \delta$,

$$\begin{aligned}
(4) &= \int_{|u|\leq\delta_1} E(u,t)(\varphi(\zeta - u) - \varphi(\zeta))du \\
&\quad + \int_{|u|\geq\delta_1} E(u,t)\varphi(\zeta - u)du - \int_{|u|\geq\delta_1} E(u,t)\varphi(\zeta)du \\
&= I_1 + I_2 + I_3 \,.
\end{aligned}$$

$$\begin{aligned}
|I_1|e^{h_{K'}(\xi)} &\leq |I_1|e^{R\sum_{j=1}^n |\xi_j|} \\
&\leq \int_{|u|\leq\delta_1} E(u,t)|\varphi(\zeta - u) - \varphi(\zeta)|e^{R\sum_{j=1}^n |\xi_j|}du \\
&< \varepsilon\int_{\mathbf{R}^n} E(u,t)du = \varepsilon \,.
\end{aligned}$$

$$
\begin{aligned}
|I_2|e^{h_{K'}(\xi)} &\le |I_2|e^{R\sum_{j=1}^n |\xi_j|} \\
&\le \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\delta_1^2}{8t}} e^{R\sum_{j=1}^n |\xi_j|} \int_{|u|\ge\delta_1} e^{-\frac{|u|^2}{8t}} |\varphi(\zeta - u)| du \\
&\le \frac{C}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\delta_1^2}{8t}} e^{R\sum_{j=1}^n |\xi_j|} \int_{|u|\ge\delta_1} e^{-\frac{|u|^2}{8t}} e^{-R\sum_{j=1}^n |\xi_j - u_j|} du \\
&\le \frac{C}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\delta_1^2}{8t}} \int_{\mathbf{R}^n} e^{-\frac{|u|^2}{8t} + R\sum_{j=1}^n |u_j|} du \\
&\le C e^{-\frac{\delta_1^2}{8t}} \to 0, \quad (t \to 0_+) .
\end{aligned}
$$

$$
\begin{aligned}
|I_3|e^{h_{K'}(\xi)} &\le |I_3|e^{R\sum_{j=1}^n |\xi_j|} \\
&\le \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\delta_1^2}{8t}} e^{R\sum_{j=1}^n |\xi_j|} \int_{|u|\ge\delta_1} e^{-\frac{|u|^2}{8t}} |\varphi(\zeta)| du \\
&\le \frac{C}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\delta_1^2}{8t}} e^{R\sum_{j=1}^n |\xi_j|} \int_{|u|\ge\delta_1} e^{-\frac{|u|^2}{8t}} e^{-R\sum_{j=1}^n |\xi_j|} du \\
&\le C e^{-\frac{\delta_1^2}{8t}} \to 0\,, \quad (t \to 0_+) .
\end{aligned}
$$

Therefore, we can conclude that $\varphi_t(\zeta)$ tends to $\varphi(\zeta)$ in Q_0 as $t \to 0_+$. □

LEMMA 4.3. *Let $T \in Q_0'$ and $t > 0$. We define $U(z,t)$ by the convolution of T and $E(z,t)$*:

$$U(z,t) := \langle T_\zeta, E(z-\zeta,t)\rangle .$$

Then $U(z,t)$ satisfies the following conditions:

$$U(z,t) \quad \text{is an entire function of } z,\ (t>0 \text{ fixed}) . \tag{6}$$

$$\left(\frac{\partial}{\partial t} - \triangle\right) U(z,t) = 0 . \tag{7}$$

There exist $R \ge 0$, $b > 0$ and $C \ge 0$ such that

$$|U(z,t)| \le C e^{\frac{1}{4t}\sum_{j=1}^n (b+|y_j|)^2 + R\sum_{j=1}^n |x_j| + nR^2 t} . \tag{8}$$

PROOF. By Lemma 3.7, there exist convex compact sets K_1, K_1' such that

$$\langle T, \varphi\rangle = \int_{\mathbf{R}^n + \imath K_1} \varphi(\zeta) e^{h_{K_1'}(\xi)} d\mu(\zeta) .$$

Therefore,

$$U(z,t) = \int_{\mathbf{R}^n + \imath K_1} E(z-\zeta,t) e^{h_{K_1'}(\xi)} d\mu(\zeta) . \tag{9}$$

We set $K_2' = B_{2R}$. By $|E(z-\zeta,t)| \le Ce^{-h_{K_2'}(x-\xi)} \leqq Ce^{2R\sum_{j=1}^n(|x_j|-|\xi_j|)}$, we have

$$|E(z-\zeta,t)e^{h_{K_1'}(\xi)}| \le Ce^{2R\sum_{j=1}^n(|x_j|-|\xi_j|)}e^{R\sum_{j=1}^n|\xi_j|}$$
$$\le Ce^{2R\sum_{j=1}^n|x_j|}e^{-R\sum_{j=1}^n|\xi_j|}. \tag{10}$$

By Lebesgue's dominated convergence theorem,

$$\lim_{z\to z_0} U(z,t) = \int_{\mathbf{R}^n+\imath K_1} \lim_{z\to z_0} E(z-\zeta,t)e^{h_{K_1'}(\xi)}d\mu(\zeta)$$
$$= U(z_0,t).$$

Therefore, $U(z,t)$ is a continuous function in $\mathbf{C}^n$. Let γ be a Jordan curve in $\mathbf{C}_j$. Then

$$\int_\gamma \int_{\mathbf{R}^n+\imath K_1} |E(z-\zeta,t)e^{h_{K_1'}(\xi)}||d\mu(\zeta)||dz_j| \le C\int_\gamma e^{2R\sum_{j=1}^n|x_j|}|dz_j|$$
$$< +\infty.$$

By Fubini's theorem we have

$$\int_\gamma U(z,t)dz_j = \int_{\mathbf{R}^n+\imath K_1}\int_\gamma E(z-\zeta,t)dz_j e^{h_{K_1'}(\zeta)}d\mu(\zeta) = 0.$$

Therefore, by Morera's theorem $U(z,t)$ is an entire function of z_j. By Hartogs' theorem $U(z,t)$ is an entire function in $\mathbf{C}^n$.

Let $K_1' \subset B_R$, $R > 0$, $K_2' = B_{2R}$. By $E(z-\zeta,t) \in Q_0$ and Cauchy's integral formula there exists a constant $C \ge 0$ such that

$$\left|\frac{\partial^2}{\partial {z_j}^2}E(z-\zeta,t)\right| \le Ce^{-h_{K_2'}(x-\xi)}$$
$$\le Ce^{2R\sum_{j=1}^n(|x_j|-|\xi_j|)}.$$

Therefore,

$$\left|\frac{\partial^2}{\partial {z_j}^2}E(z-\zeta,t)e^{h_{K_1'}(\xi)}\right| \le Ce^{2R\sum_{j=1}^n|x_j|-R\sum_{j=1}^n|\xi_j|}. \tag{11}$$

We can exchange the integration and the differentiation in (9):

$$\triangle U(z,t) = \int_{\mathbf{R}^n+\imath K_1} \triangle E(z-\zeta,t)e^{h_{K_1'}(\xi)}d\mu(\zeta).$$

Let $0 < a_0 < t < a_1$. Then

$$\left|\frac{\partial}{\partial t}E(\zeta,t)\right| \le \frac{(4\pi a_1)^{\frac{n}{2}}\dfrac{|\zeta^2+\cdots+\zeta^2|}{4{a_0}^2} + 2\pi n(4\pi a_1)^{\frac{n}{2}-1}}{(4\pi a_0)^n}\left|e^{\frac{\eta^2}{4a_0}-\frac{\xi^2}{4a_1}}\right|, \tag{12}$$

which implies

$$\sup_{\zeta\in\mathbf{R}^n+\imath K_1}\left|\frac{\partial}{\partial t}E(\zeta,t)e^{h_{K_1'}(\xi)}\right| < \infty\,.$$

Since $a_0 > 0$, $a_1 > 0$ are arbitrary, we can differentiate the right-hand side of (9) under the integral for $t > 0$. Therefore, we have (7).

By Remark 3.6 and the continuity of T, $T \in Q_b'(\mathbf{R}^n + \imath K, K')$ for some K, K'. Therefore, we can find a constant $C \geq 0$ such that

$$\begin{aligned}|U(z,t)| &= |< T_\zeta, E(z-\zeta,t) >| \\ &\leq C \sup_{\zeta\in\mathbf{R}^n+\imath K}|E(z-\zeta,t)e^{h_{K'}(\xi)}|\,. \end{aligned} \tag{13}$$

Let $K \subset \mathring{B}_b$, $b > 0$, $K' \subset B_R$, $R \geq 0$. Then we have

$$\begin{aligned}&\sup_{\zeta\in\mathbf{R}^n+\imath K}|E(z-\zeta,t)e^{h_{K'}(\xi)}| \\ &\leq \frac{C}{(4\pi t)^{\frac{n}{2}}}\sup_{\zeta\in\mathbf{R}^n+\imath K} e^{-\frac{1}{4t}\sum_{j=1}^n\{(x_j-\xi_j)^2-(y_j-\eta_j)^2\}+R\sum_{j=1}^n|\xi_j|} \\ &\leq \frac{C}{(4\pi t)^{\frac{n}{2}}}e^{nR^2t+R\sum_{j=1}^n|x_j|}\sup_{\zeta\in\mathbf{R}^n+\imath K} e^{\frac{1}{4t}\sum_{j=1}^n(y_j-\eta_j)^2} \\ &\leq \frac{C}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{\varepsilon n}{4t}}e^{\frac{\varepsilon n}{4t}+nR^2t+R\sum_{j=1}^n|x_j|}\sup_{\zeta\in\mathbf{R}^n+\imath K} e^{\frac{1}{4t}\sum_{j=1}^n(y_j-\eta_j)^2} \\ &\leq Ce^{nR^2t+\frac{\varepsilon n}{4t}+R\sum_{j=1}^n|x_j|}\sup_{\zeta\in\mathbf{R}^n+\imath K} e^{\frac{1}{4t}\sum_{j=1}^n(y_j-\eta_j)^2} \\ &\leq Ce^{\frac{1}{4t}\sum_{j=1}^n(b+|y_j|)^2+R\sum_{j=1}^n|x_j|+nR^2t}\,.\end{aligned}$$

So we have

$$|U(z,t)| \leq Ce^{\frac{1}{4t}\sum_{j=1}^n(b+|y_j|)^2+R\sum_{j=1}^n|x_j|+nR^2t}\,. \qquad \square$$

LEMMA 4.4. *Let* $P(\partial) = \sum_{|\alpha|=0}^\infty a_\alpha\partial^\alpha$ *be a partial differential operator of infinite order with constant coefficients satisfying the following condition*:

$$\exists L_1 \geq 0\,,\quad \exists C > 0\,,\quad s.t.\ |a_\alpha| \leq \frac{CL_1^{|\alpha|}}{\alpha!^2}\quad for\ \forall\alpha\in\mathbf{N}^n\,.$$

Then the operators

$$P(\partial): Q_0 \to Q_0 \quad and \quad P(\partial): Q_0' \to Q_0' \tag{14}$$

are continuous. For $T \in Q_0'$, (14) *is defined as follows*:

$$\langle P(\partial)T,\varphi\rangle = \langle T, P(-\partial)\varphi\rangle \quad for\ \ \varphi\in Q_0\,.$$

PROOF. Let $\varphi \in Q_0$, K, K', convex compact sets, $K \subset B_b$, $b > 0$, $K_1 = B_{b+2L_1}$. By Cauchy's integral theorem,

$$\left|\frac{\partial^\alpha}{\partial z^\alpha}\varphi(z)\right| \leq \alpha! \sup_{0\leq\theta_j\leq 2\pi} |\varphi(z_1 + 2L_1 e^{\imath\theta_1}, \cdots, z_n + 2L_1 e^{\imath\theta_n})| \times \left(\frac{1}{2L_1}\right)^{|\alpha|}.$$

Therefore, for $z \in \mathbf{R}^n + \imath K$, we have

$$\begin{aligned}\left|\sum_{|\alpha|=0}^{\infty} a_\alpha \frac{\partial^\alpha}{\partial z^\alpha}\varphi(z)e^{h_{K'}(x)}\right| &\leq C \sup_{z\in\mathbf{R}^n+\imath K_1} |\varphi(z)|e^{h_{K'}(x)} \sum_{|\alpha|=0}^{\infty}\left(\frac{1}{2}\right)^{|\alpha|} \\ &\leq C \sup_{z\in\mathbf{R}^n+\imath K_1} |\varphi(z)|e^{h_{K'}(x)}.\end{aligned}$$

It follows that $P(\partial) : Q_0 \to Q_0$ is continuous. The continuity of $P(\partial) : Q_0' \to Q_0'$ is obtained by the duality. □

The following Lemma 4.5 and Lemma 4.6 are very useful later. For the details of the proof we refer the reader to [6]:

LEMMA 4.5 ([6]). *If a function $f(t)$, defined for $t > 0$, satisfies the condition*:

$$\exists L > 0,\ \exists C > 0, \quad s.t.\ |f(t)| \leq Ce^{\frac{L}{t}}, \quad t > 0,$$

then there exists $l > 0$ such that $f(t)e^{-N^*(\frac{l}{t})}$ is bounded, where

$$N^*(t) = \sup_p \log\left(\frac{l^p t^p}{p!}\right). \tag{15}$$

LEMMA 4.6 ([6]). *For any $l > 0$ and $\varepsilon_1 > 0$ there exists a function $v(t) \in C_0^\infty(\mathbf{R})$, an ultradifferential operator $P(d/dt)$ and $L_1 \geq 0$ such that*

$$\operatorname{supp} v(t) \subset [0, \varepsilon_1],$$

$$|v(t)| \leq \frac{1}{2}e^{-N^*(\frac{l}{t})}, \quad t > 0,$$

where $N^*(t)$ is given by (15),

$$P\left(\frac{d}{dt}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{d}{dt}\right)^n, \quad |a_n| \leq \frac{CL_1^n}{n!^2}, \quad P\left(\frac{d}{dt}\right)v(t) = \delta + w(t),$$

$$w(t) \in C_0^\infty(\mathbf{R}), \quad \operatorname{supp} w(t) \subset \left[\frac{\varepsilon_1}{2}, \varepsilon_1\right].$$

LEMMA 4.7. *Let $f(x)$ be a measurable function on $\mathbf{R}^n$ satisfying the following condition*:

$$\exists R \geq 0, \quad \exists C \geq 0 \quad s.t.\ |f(x)| \leq Ce^{R\sum_{j=1}^n |x_j|}.$$

Then we can consider $f(x)$ as an element of Q_0' in the following manner:

$$\langle T_f, \varphi\rangle = \int_{\mathbf{R}} f(x)\varphi(x)dx\,, \quad \varphi \in Q_0\,.$$

PROOF. The linearity of T_f is trivial. We prove only the continuity. Let K and K' be convex compact sets in $\mathbf{R}^n$ such that $\mathring{K}' \supset B_R$. Since $\varphi \in Q_0$, we have $\sup_{z\in\mathbf{R}^n+\imath K} |\varphi(z)e^{h_{K'}(x)}| < \infty$. Therefore,

$$\begin{aligned}|\langle T_f, \varphi\rangle| &\le \int_{\mathbf{R}^n} |f(x)||\varphi(x)|dx \\ &\le C \sup_{z\in\mathbf{R}^n+\imath K} |\varphi(z)e^{h_{K'}(x)}| \int_{\mathbf{R}^n} e^{R\sum_{j=1}^n |x_j| - h_{K'}(x)}dx \\ &\le C \sup_{z\in\mathbf{R}^n+\imath K} |\varphi(z)e^{h_{K'}(x)}|\,.\end{aligned}$$

The proof is complete. □

5. Main Theorem.

THEOREM 5.1. *Let $T \in Q_0'$ and $U(z,t) = \langle T_\zeta, E(z-\zeta,t)\rangle$. Then $U(z,t)$ is an entire function of z and C^∞-function of t, $t > 0$ satisfying the following conditions*:

$$\left(\frac{\partial}{\partial t} - \triangle\right) U(z,t) = 0\,, \tag{16}$$

$$U(z,t) \to T \quad in \quad Q_0' \quad (t \to 0_+)\,. \tag{17}$$

There exist $R \ge 0$, $b \ge 0$ and $C \ge 0$ such that

$$|U(z,t)| \le Ce^{\frac{1}{4t}\sum_{j=1}^n (b+|y_j|)^2 + R\sum_{j=1}^n |x_j| + nR^2 t} \quad for \ z \in \mathbf{C}^n\,, \quad t > 0\,. \tag{18}$$

Conversely, for a function $U(z,t)$, $t > 0$, entire in z, C^∞ in $t > 0$, satisfying (16) *and* (18)*, there exists a unique $T \in Q_0'$ such that*

$$\langle T_\zeta, E(z-\zeta,t)\rangle >= U(z,t)\,.$$

PROOF. By Lemma 4.3, we have (16) and (18).

Let $\varphi \in Q_0$. By Lemma 3.7,

$$\begin{aligned}\int_{\mathbf{R}^n} U(z,t)\varphi(x)dz &= \int_{\mathbf{R}^n} \langle T_\zeta, E(z-\zeta,t)\rangle\varphi(z)dz \\ &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n+\imath K} E(z-\zeta,t)e^{h_{K'}(\xi)}d\mu(\zeta)\varphi(z)dz\,.\end{aligned}$$

Take $R > 0$ such that $K' \subset B_R$, and put $K'_2 = B_{2R}$ and $K'_3 = B_{3R}$. By $E(z-\zeta, t) \in Q_0$ and $\varphi(z) \in Q_0$, there exist $C_2 \geq 0$ and $C_3 \geq 0$ such that

$$|E(z-\zeta,t)| \leq C_2 e^{-h_{K'_2}(x-\xi)} = C_2 e^{-2R\sum_{j=1}^n |x_j-\xi_j|},$$

$$|\varphi(z)| \leq C_3 e^{-h_{K'_3}(x)} = C_3 e^{-3R\sum_{j=1}^n |x_j|}.$$

Since we have

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}^n+\imath K} |E(z-\zeta,t)| e^{h_{K'}(\xi)} |d\mu(\zeta)||\varphi(z)||dz| \\
&\leq C_2C_3 \int_{\mathbf{R}^n}\int_{\mathbf{R}^n+\imath K} e^{-2R\sum_{j=1}^n |x_j-\xi_j| + R\sum_{j=1}^n |\xi_j|} |d\mu(\zeta)| e^{-3R\sum_{j=1}^n |x_j|} dx \\
&\leq C \int_{\mathbf{R}^n} e^{-R\sum_{j=1}^n |x_j|} dx \int_{\mathbf{R}^n+\imath K} e^{-2R\sum_{j=1}^n |\xi_j|} |d\mu(\zeta)| < +\infty\,.
\end{aligned}$$

By Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} U(z,t)\varphi(z)dz &= \int_{\mathbf{R}^n+\imath K} \left\{ \int_{\mathbf{R}} E(z-\zeta,t)\varphi(z)dz \right\} e^{h_{K'}(\xi)} d\mu(\zeta) \\
&= \langle T_\zeta, \varphi_t(\zeta)\rangle\,.
\end{aligned}$$

By Lemma 4.2, $\varphi_t(\zeta) \to \varphi(\zeta)$ in Q_0 $(t \to 0_+)$, we have

$$\langle T_\zeta, \varphi_t(\zeta)\rangle \to \langle T_\zeta, \varphi(\zeta)\rangle \quad (t \to 0_+)\,.$$

It shows that

$$U(z,t) \to T \quad \text{in } Q'_0 \quad (t \to 0_+)\,.$$

Now we shall prove the converse. Let $U(z,t)|_{\mathbf{R}^n+\imath B_{b_1}} = c(z,t)$, $b_1 \geq 0$, and $F(z,t) = c(z,t)e^{-R\sum_{j=1}^n |x_j| - nR^2 t}$. Then we have

$$\left(\frac{\partial}{\partial t} - \triangle\right) c(z,t) = 0\,,$$

$$|c(z,t)| \leq C e^{\frac{1}{4t}\sum_{j=1}^n (b+b_1)^2 + R\sum_{j=1}^n |x_j| + nR^2 t},$$

$$|F(z,t)| \leq C e^{\frac{1}{4t}\sum_{j=1}^n (b+b_1)^2}\,.$$

By Lemma 4.5, there exists $l > 0$ such that $|F(z,t)|e^{-N^*(\frac{1}{t})}$ is bounded for $t > 0$. For this $l > 0$, by Lemma 4.6, we construct $P\left(\frac{d}{dt}\right)$ and $v(t)$. Let

$$\tilde{c}(z,t) = \int_0^\infty c(z,t+s)v(s)ds\,.$$

We need two lemmas.

LEMMA 5.2. *$\tilde{c}(z,t)$ satisfies the following conditions*:

$$\left(\frac{\partial}{\partial t} - \triangle\right)\tilde{c}(z,t) = 0\,, \quad t > 0\,, \tag{19}$$

there exist $R \geq 0$, $C \geq 0$, such that

$$|\tilde{c}(z,t)| \leq Ce^{R\sum_{j=1}^{n}|x_j| + nR^2 t}\,, \quad t \geq 0 \tag{20}$$

$$\tilde{c}(z,t) \text{ is an entire function of } z \text{ for } t \geq 0\,. \tag{21}$$

PROOF. By assumption,

$$\begin{aligned} |c(z,t+s)|e^{-R\sum_{j=1}^{n}|x_j| - nR^2(t+s)} &= |F(z,t+s)| \\ &\leq Ce^{N^*(\frac{1}{t+s})} \leq Ce^{N^*(\frac{1}{s})}\,. \end{aligned}$$

By the above inequality, we have

$$|c(z,t+s)| \leq Ce^{N^*(\frac{1}{s}) + R\sum_{j=1}^{n}|x_j| + nR^2(t+s)}\,.$$

Since $|v(s)| \leq \frac{1}{2}e^{-N^*(\frac{1}{s})}$, $s > 0$,

$$|c(z,t+s)v(s)| \leq Ce^{R\sum_{j=1}^{n}|x_j| + nR^2(t+s)}\,. \tag{22}$$

By supp $c(z,t+s)v(s) \subset [0,\varepsilon_1]$,

$$c(z,t+s)v(s) = \begin{cases} c(z,t+s)v(s)\,, & s > 0 \\ 0\,, & s = 0\,. \end{cases} \tag{23}$$

Therefore, $\lim_{t\to 0_+} c(z,t+s)v(s) := c(z,s)v(s)$ exists for $s \geq 0$ and it follows from (22) that $|c(z,t+s)v(s)| \in L^1([0,\varepsilon_1])$ as function of s. By Lebesgue's dominated theorem, we have

$$\begin{aligned} \lim_{t\to 0_+} \tilde{c}(z,t) &= \int_0^{\varepsilon_1} \lim_{t\to 0_+} c(z,t+s)v(s)ds \\ &= \int_0^{\varepsilon_1} c(z,s)v(s)ds\,. \end{aligned}$$

Therefore, $\tilde{c}(z,t)$ is a continuous function in $t \geq 0$. Furthermore

$$\begin{aligned} \lim_{z\to z_0} \tilde{c}(z,t) &= \int_0^{\infty} \lim_{z\to z_0} c(z,t+s)v(s)ds \\ &= \tilde{c}(z_0,t)\,, \end{aligned}$$

which proves the continuity of $\tilde{c}(z,t)$ in z, $t \geq 0$.

It is obvious that $\tilde{c}(z,t)$ is an entire function of z for $t > 0$. We see that $\tilde{c}(z,t)$ is an entire functions of z for $t = 0$. Let γ be a Jordan curve in $\mathbf{C}_j$, $\varepsilon_2 > 0$,

$$\begin{aligned} M &:= C\int_\gamma e^{R\sum_{j=1}^n |z_j|}|dz_j|\,,\\ m &:= e^{nR^2},\\ 0 < \delta &< \min\left\{\frac{\varepsilon_2}{Mm}, \varepsilon_1, 1\right\}\,. \end{aligned}$$

Then by

$$\begin{aligned} \int_\gamma \lim_{t\to 0_+} \tilde{c}(z,t)dz_j &= \int_\gamma\int_0^\delta c(z,s)v(s)dsdz_j + \int_\gamma\int_\delta^{\varepsilon_1} c(z,s)v(s)dsdz_j\\ &= \int_\gamma\int_0^\delta c(z,s)v(s)dsdz_j\,, \end{aligned}$$

we have

$$\begin{aligned} \left|\int_\gamma \lim_{t\to 0_+} \tilde{c}(z,t)dz_j\right| &\le C\int_\gamma e^{R\sum_{j=1}^n |x_j|}|dz_j|\int_0^\delta e^{nR^2 s}ds\\ &= \delta Mm < \varepsilon_2\,. \end{aligned}$$

Since $\varepsilon_2 > 0$ is arbitrary, we have $\int_\gamma \tilde{c}(z,0)dz_j = 0$. By Morera's theorem $\tilde{c}(z,0)$ is an entire function of z_j. By Hartogs' theorem, $\tilde{c}(z,0)$ is an entire function in $\mathbf{C}^n$.

Furthermore by Cauchy's integral formula,

$$|\triangle c(z,t+s)| \le C \sup_{0\le\theta\le 2\pi} |c(z+e^{\iota\theta}, t+s)|\,.$$

By $\sup_{0\le\theta\le 2\pi} |c(z+e^{\iota\theta},t+s)||v(s)| \in L^1([0,\varepsilon_1])$ for s, we have

$$\triangle\tilde{c}(z,t) = \int_0^\infty \triangle c(z,t+s)v(s)ds\,,$$

By $\left(\frac{\partial}{\partial t} - \triangle\right)c(z,t+s) = 0$, and

$$\frac{\partial}{\partial t}\tilde{c}(z,t) = \int_0^\infty \frac{\partial}{\partial t}c(z,t+s)v(s)ds\,,$$

we have (19).

By (22), (23) and supp $c(z, t+s)v(s) \subset [0, \varepsilon_1]$,

$$\begin{aligned}
|\tilde{c}(z,t)| &= \left|\int_0^\infty c(z,t+s)v(s)ds\right| \\
&= \left|\int_0^{\varepsilon_1} c(z,t+s)v(s)ds\right| \\
&\le \int_0^{\varepsilon_1} |c(z,t+s)v(s)|ds \\
&\le Ce^{R\sum_{j=1}^n |x_j| + nR^2 t}\int_0^{\varepsilon_1} e^{nR^2 s}ds \\
&\le C_1 e^{R\sum_{j=1}^n |x_j| + nR^2 t} \text{ for } t \ge 0\,.
\end{aligned}$$

Therefore, we have (20). □

LEMMA 5.3. *Let $t > 0$ and P be an ultradifferential operator given in Lemma* 4.6. *Then we have*

$$P(-\triangle)\int_0^\infty c(z,t+s)v(s)ds = \int_0^\infty P(-\triangle)c(z,t+s)v(s)ds\,.$$

PROOF. Since $c(z, t+s)$ is an entire function, by Cauchy's integral formula, we have

$$\frac{\partial^{2m}}{\partial z^{2m}}c(z,t+s) = \frac{(2m)!}{(2\pi\imath)^n}\int_{\partial B(z,4L_1)} \frac{c(\zeta,t+s)}{(\zeta-z)^{2m+1}}d\zeta\,,$$

where L_1 is of Lemma 4.6. Therefore,

$$\begin{aligned}
\left|\frac{\partial^{2m}}{\partial z^{2m}}c(z,t+s)\right||v(s)| &\le \frac{(2m)!}{(4L_1)^{2m}}\sup_{0\le\theta\le 2\pi}|c(z+4L_1e^{\imath\theta},t+s)||v(s)| \\
&\le \frac{(2m)!}{(4L_1)^{2m}}Ce^{R\sum_{j=1}^n |x_j| + nR^2(t+s)}\,.
\end{aligned}$$

Since

$$\int_0^{\varepsilon_1} \frac{(2m)!}{(4L_1)^{2m}}e^{R\sum_{j=1}^n |x_j| + nR^2(t+s)}ds < +\infty\,,$$

we have

$$\frac{\partial^{2m}}{\partial z^{2m}}\int_0^\infty c(z,t+s)v(s)ds = \int_0^\infty \frac{\partial^{2m}}{\partial z^{2m}}c(z,t+s)v(s)ds\,.$$

Furthermore by

$$\sum_{m=0}^{\infty}\int_0^{\infty}|a_m|\left|\frac{\partial^{2m}}{\partial z^{2m}}c(z,t+s)\right||v(s)|ds$$
$$\leq Ce^{R\sum_{j=1}^{n}|x_j|+nR^2t}\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^m<+\infty\,,$$

we have

$$P(-\triangle)\int_0^{\infty}c(z,t+s)v(s)ds=\int_0^{\infty}P(-\triangle)c(z,t+s)v(s)ds\,. \qquad \square$$

Now we resume the proof of Theorem 5.1.

We notice that $v(s)\in\mathcal{E}'[0,\varepsilon_1]$ since supp $v(s)\subset[0,\varepsilon_1]$, where $\mathcal{E}'[0,\varepsilon_1]$ is the space of Schwartz distributions with support in $[0,\varepsilon_1]$ and for $t>0$, $c(z,t+s)\in C^{\infty}((-\varepsilon_3,\infty))$ of s, $0<\varepsilon_3<t$. Therefore, since $c(z,t+s)$ is a test function, for $t>0$, by Lemma 5.3 and (16), we have

$$\begin{aligned}P(-\triangle)\tilde{c}(z,t)&=P(-\triangle)\int_0^{\infty}c(z,t+s)v(s)ds\\&=\int_0^{\infty}P(-\triangle)c(z,t+s)v(s)ds\\&=\langle v(s),\,P(-\triangle)c(z,t+s)\rangle\\&=\left\langle v(s),\,P(-\frac{\partial}{\partial t})c(z,t+s)\right\rangle\\&=\left\langle P\left(\frac{\partial}{\partial s}\right)v(s),\,c(z,t+s)\right\rangle\\&=\langle\delta(s)+w(s),\,c(z,t+s)\rangle\\&=c(z,t)+\int_0^{\infty}c(z,t+s)w(s)ds\,.\end{aligned}$$

Namely we obtain

$$P(-\triangle)\tilde{c}(z,t)=c(z,t)+\int_0^{\infty}c(z,t+s)w(s)ds\,,\quad t>0\,.$$

Now, we put

$$H(z,t)=-\int_0^{\infty}c(z,t+s)w(s)ds\,, \tag{24}$$

$$g(z)=\tilde{c}(z,0)\,,\quad h(z)=-\int_0^{\infty}c(z,s)w(s)ds\,, \tag{25}$$

$$T(z)=P(-\triangle)g(z)+h(z)\,. \tag{26}$$

Then by Lemma 5.2 and supp $w(s) \subset [\varepsilon_1/2, \varepsilon_1]$, $H(z,t)$, $g(z)$ and $h(z)$ are entire functions and

$$\left(\frac{\partial}{\partial t} - \triangle\right) H(z,t) = 0\,, \quad t > 0\,, \tag{27}$$

$$|H(z,t)| \le Ce^{R\sum_{j=1}^{n}|x_j| + nR^2 t}\,, \tag{28}$$

$$|g(z)| \le Ce^{R\sum_{j=1}^{n}|x_j|}\,, \quad |h(z)| \le Ce^{R\sum_{j=1}^{n}|x_j|}\,, \tag{29}$$

and by Lemma 4.7 and Lemma 4.4, $T(z) \in Q_0'$. We define $A(z,t)$ and $B(z,t)$, $t > 0$ by

$$A(z,t) = g(z) * E(z,t) := \int_{\mathbf{R}^n} g(\zeta)E(z-\zeta,t)d\zeta\,,$$

$$B(z,t) = h(z) * E(z,t) := \int_{\mathbf{R}^n} h(\zeta)E(z-\zeta,t)d\zeta\,.$$

Then we can see that

$$\left(\frac{\partial}{\partial t} - \triangle_x\right) A(x,t) = 0\,, \tag{30}$$

$$\left(\frac{\partial}{\partial t} - \triangle_x\right) B(x,t) = 0\,. \tag{31}$$

By (29), we have

$$A(z,t) \to g(z)\,, \quad t \to 0_+, \ \text{ uniformly in } \mathbf{C}^n, \tag{32}$$

$$B(z,t) \to h(z)\,, \quad t \to 0_+, \ \text{ uniformly in } \mathbf{C}^n\,. \tag{33}$$

Let $0 < t < T$. For $\delta > 0$,

$$\begin{aligned} |A(x,t)| &\le \int_{|\xi|\le\delta} |g(x-\xi)E(\xi,t)|d\xi + \int_{|\xi|\ge\delta} |g(x-\xi)E(\xi,t)|d\xi \\ &= I_1 + I_2\,. \end{aligned}$$

Then we have

$$\begin{aligned} I_1 &\le Ce^{R\sum_{j=1}^{n}|x_j|}\int_{\mathbf{R}^n} E(\xi,t)d\xi \le Ce^{R\sum_{j=1}^{n}|x_j|}\,. \\ I_2 &\le \int_{|\xi|\ge\delta} Ce^{R\sum_{j=1}^{n}|x_j-\xi_j|}E(\xi,t)d\xi \\ &= \frac{Ce^{R\sum_{j=1}^{n}|x_j|}}{(4\pi t)^{\frac{n}{2}}}\int_{|\xi|\ge\delta} e^{R\sum_{j=1}^{n}|\xi_j| - \frac{\xi^2}{4t}}d\xi \\ &= \frac{Ce^{R\sum_{j=1}^{n}|x_j|}}{(4\pi t)^{\frac{n}{2}}}\int_{|\xi|\ge\delta} e^{-\frac{1}{4t}(|\xi|-2Rt)^2 + R^2 t}d\xi\,. \end{aligned} \tag{34}$$

For $\delta > 0$, sufficiently large, there exists $\delta' > 0$ such that

$$\begin{aligned}
&\frac{Ce^{R\sum_{j=1}^{n}|x_j|}}{(4\pi t)^{\frac{n}{2}}}\int_{|\xi|\geq\delta} e^{-\frac{1}{4t}(|\xi|-2Rt)^2+R^2t}d\xi \\
&\quad= \frac{Ce^{R\sum_{j=1}^{n}|x_j|-\frac{\delta'^2}{8t}}}{(4\pi t)^{\frac{n}{2}}}\int_{|\xi|\geq\delta} e^{-\frac{1}{8t}(|\xi|-2Rt)^2}d\xi \\
&\quad\leq Ce^{R\sum_{j=1}^{n}|x_j|}\int_{|\xi|\geq\delta} e^{-\frac{1}{8T}(|\xi|-2Rt)^2}d\xi \\
&\quad\leq Ce^{R\sum_{j=1}^{n}|x_j|}\,.
\end{aligned}$$

It follows from (34) that $I_2 \leq Ce^{R\sum_{j=1}^{n}|x_j|}$. Therefore, we have

$$|A(z,t)| \leq Ce^{R\sum_{j=1}^{n}|x_j|}, \quad 0 < t < T\,. \tag{35}$$

Similarly,

$$|B(z,t)| \leq Ce^{R\sum_{j=1}^{n}|x_j|}, \quad 0 < t < T\,. \tag{36}$$

From (19), (20), (25), (30), (32) and (35), it follows that $\tilde{c}(z,t)$ and $A(z,t)$ satisfy the heat equation and the same estimates and have the same initial values. So by the uniqueness theorem for the initial value problem of the heat equation [4], it follows that

$$A(z,t) = \tilde{c}(z,t)\,. \tag{37}$$

Similarly,

$$B(z,t) = -\int_0^\infty c(z,t+s)w(s)ds\,. \tag{38}$$

By (26), (37) and (38), we have

$$\begin{aligned}
\langle T_\zeta, E(z-\zeta,t)\rangle &= T(z) * E(z,t) \\
&= P(-\triangle)g(z) * E(z,t) + h(z) * E(z,t) \\
&= P(-\triangle)\tilde{c}(z,t) - \int_0^\infty c(z,t+s)w(s)ds \\
&= U(z,t)\,.
\end{aligned}$$

□

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References

[1] M. BUDINČEVIĆ, Z. LOZANOV-CRVENKOVIĆ and D. PEROŠIĆ, Representation theorems for tempered ultradistributions, Publ. Inst. Math. (Beograd) **65** (1999), 142–160.

[2] S.-Y.CHUNG, D. KIM and S. K. KIM, Solvability of Mizohata and Lewy operators, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **40** (1993), 63–71.

[3] S.-Y. CHUNG, D. KIM and S. K. KIM, Structure of the extended Fourier hyperfunctions, Japan. J. Math. **19**, (1994), 217–226.

[4] F. JOHN, *Partial Differential Equations*, Springer (1982).

[5] K. W. KIM, S.-Y. CHUNG and D. KIM, Fourier hyperfunctions as the boundary values of smooth solutions of heat equations, Publ. RIMS, Kyoto Univ. **29** (1993), 289–300.

[6] H. KOMATSU, *Introduction to the Theory of Distributions* (in Japanese), Iwanami (1978).

[7] T. MATSUZAWA, A calculus approach to the hyperfunctions I, Nagoya Math. J. **108** (1987), 53–66.

[8] T. MATSUZAWA, A calculus approach to the hyperfunctions II, Trans. Amer. Math. Soc. **313** (1989), 619–654.

[9] M. MORIMOTO, Analytic functionals with non-compact carriers, Tokyo J. Math. **1** (1978), 77–103.

[10] P. SARGOS and M. MORIMOTO, Transformation des fonctionnelles analytiques à porteurs non compacts, Tokyo J. Math. **4** (1981), 457–492.

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