

Zeta Regularized Product Expressions for Multiple Trigonometric Functions

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Abstract. We introduce a multiple analogue of the gamma function which differs from the one defined by Barnes [B]. Using this function, we give expressions of the multiple sine and cosine functions in terms of zeta regularized products. The expression of the multiple sine function can be interpreted as a reflection formula of this new multiple analogue of the gamma function.

1. Introduction

The initial study of multiple trigonometric functions is due to Hölder [H] who treated the double sine function $\mathcal{S}_2(x)$ in 1886. It is observed that the multiple sine functions $\mathcal{S}_r(x)$ describe the values $\zeta(2m + 1)$ of the Reimann zeta function at odd integer points as their special values (see [KW1, KOW]). Recently we found that these special values appear also as the extremal values of multiple trigonometric functions [KW3]. On the other hand, due to the work by Lerch [L], we have a zeta regularized product expression of the classical sine function. Furthermore, quite recently in [KW4], we discovered similar expressions for the multiple trigonometric functions of small orders, that is, the double and the triple ones, in the course of the study of certain finite companions of such multiple trigonometric functions. Instead of the classical sine function, however, in order to obtain such expressions for the multiple ones it is necessary to introduce a new zeta regularization method. Namely, we need to give an extended interpretation to the original zeta regularized product, e.g. developed in [D, V]. With the help of this interpretation of the regularized product we can arrive at introducing a multiple analogue $\mathbf{G}_m(x)$ of the gamma function which differs from the one defined by Barnes [B]. We call it a **basic multiple gamma function**. We remark that the Barnes multiple gamma function $\Gamma_m(x)$ can be defined through the multiple Hurwitz zeta function and consequently, it is expressed as a zeta regularized product. Therefore the so-called *normalized* multiple sine function $S_m(x)$ is also defined by the same manner (see e.g. [KKo, KW3]). Although there is an explicit relation between $S_r(x)$'s and $\mathcal{S}_r(x)$'s, no such

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simple expression for $\mathcal{S}_r(x)$ has been expected so far. To avoid a possible confusion due to their names, we sometimes call $\mathcal{S}_r(x)$ and $\mathcal{C}_r(x)$ the **basic** multiple sine and the basic multiple cosine functions, respectively.

The main purpose of the present paper is to give expressions of the multiple trigonometric functions $\mathcal{S}_r(x)$ and $\mathcal{C}_r(x)$ of general order r in terms of zeta regularized products. In particular, we find that the expression of $\mathcal{S}_r(x)$ can be regarded as a reflection formula of the basic multiple gamma function $\mathbf{G}_m(x)$.

Recall the basic multiple sine function

$$\mathcal{S}_r(x) = e^{\frac{x^{r-1}}{r-1}} \prod_{n=-\infty, n \neq 0}^{\infty} P_r\left(\frac{x}{n}\right)^{n^{r-1}}$$

and the basic multiple cosine function

$$\mathcal{C}_r(x) = \prod_{n=-\infty, n: \text{odd}}^{\infty} P_r\left(\frac{x}{\frac{n}{2}}\right)^{\left(\frac{n}{2}\right)^{r-1}}$$

of order $r \geq 2$. Here we put

$$P_r(u) = (1-u) \exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right).$$

The zeta regularized product we use here is introduced in [KW4] and given as follows. Let $\mathbf{a} = \{a_n\}_{n=1,2,\dots}$ be a divergent series of non-zero complex numbers and $\mathbf{b} = \{b_n\}_{n=1,2,\dots}$ a series of complex numbers. Suppose that the Dirichlet series defined by

$$\phi_{\mathbf{a}, \mathbf{b}}(s) := \sum_{n=1}^{\infty} b_n \cdot a_n^{-s}$$

can be extended to a holomorphic function around $s = 0$. (We are assuming that the series converges absolutely for large enough $\text{Re}(s)$.) Then we define the zeta regularized product of the sequences “ $\{a_n^{b_n}\}_{n=1,2,\dots}$ ” by

$$\prod_{n=1}^{\infty} (a_n)^{b_n} = \exp(-\phi'_{\mathbf{a}, \mathbf{b}}(0)).$$

Note that if $b_n \equiv 1$, then it is immediate to see that $\prod_{n=1}^{\infty} ((a_n))^1$ agrees with the usual zeta regularized product $\prod_{n=1}^{\infty} a_n$ (see [D]), while $\prod_{n=1}^{\infty} ((a_n))^{b_n} \neq \prod_{n=1}^{\infty} a_n^{b_n}$ in general. Actually, even if we employ any generalized version of the zeta regularized product developed, e.g. in [I, KW2, KiKSW, KiW], for instance, the product $\prod_{n=1}^{\infty} n^{n^m}$ does not exist, though we can show that $\prod_{n=1}^{\infty} ((n))^{n^m}$ exists and indeed equals $\exp(-\zeta'(-m))$. More precisely, we verify that (in the proof of the main theorems in Section 3) the function defined by the regularized

product

$$\mathbf{G}_m(x)^{-1} := \prod_{n=1}^{\infty} ((n+x))^{n^m}$$

exists as an entire function of x and plays a role of a gamma function in the present treatment of the basic multiple trigonometric functions (see also Remark 2 in Section 3). Note that if all b_n are positive integers, then each b_n can be considered as a multiplicity of a_n . Throughout the paper we assume that $-\pi \leq \arg(a_n) < \pi$.

The following expressions (and its proof) of the basic multiple sine functions are the main result of this paper.

THEOREM 1.1. *For each positive integer m we have*

$$(1) \quad \mathcal{S}_{2m}(x) = \frac{\prod_{n=1}^{\infty} ((n-x))^{n^{2m-1}}}{\prod_{n=1}^{\infty} ((n+x))^{n^{2m-1}}},$$

$$(2) \quad \mathcal{S}_{2m+1}(x) = \exp\left((-1)^m \frac{\zeta(2m+1)(2m)!}{2^{2m}\pi^{2m}} \right) \prod_{n=1}^{\infty} ((n-x))^{n^{2m}} \cdot \prod_{n=1}^{\infty} ((n+x))^{n^{2m}},$$

where $\zeta(s)$ denotes the Riemann zeta function.

We have proved (1) and (2) of the theorem when $m = 1$ in [KW4]. We will find that the formula in the theorem gives an analogue of the reflection formula of the classical gamma function: $\Gamma(x)\Gamma(1-x) = \pi / \sin \pi x$ (see Corollary 3.3).

Since $\mathcal{C}_r(x)$ is a 2^{r-1} -multi-valued function, we treat the function $\tilde{\mathcal{C}}_r(x) = \mathcal{C}_r(x)^{2^{r-1}}$ in place of $\mathcal{C}_r(x)$. Obviously, $\tilde{\mathcal{C}}_r(x)$ defines a single valued function. We also give the expression of the multiple cosine functions $\tilde{\mathcal{C}}_r(x)$ in terms of the zeta regularized product.

THEOREM 1.2. *For each positive integer m we have*

$$(1) \quad \tilde{\mathcal{C}}_{2m}(x) = \frac{\prod_{n=1}^{\infty} \left(\left(n - \frac{1}{2} - x \right) \right)^{(2n-1)^{2m-1}}}{\prod_{n=1}^{\infty} \left(\left(n - \frac{1}{2} + x \right) \right)^{(2n-1)^{2m-1}}},$$

$$(2) \quad \tilde{\mathcal{C}}_{2m+1}(x) = \exp\left((1-2^{2m})(-1)^m \frac{\zeta(2m+1)(2m)!}{2^{2m}\pi^{2m}} \right) \\ \times \prod_{n=1}^{\infty} \left(\left(n - \frac{1}{2} - x \right) \right)^{(2n-1)^{2m}} \cdot \prod_{n=1}^{\infty} \left(\left(n - \frac{1}{2} + x \right) \right)^{(2n-1)^{2m}}.$$

We prove this theorem from the result in Theorem 1.1 with the help of the duplication formula of the basic multiple sine function ([KOW, KKo]).

2. Preparation of the proof

In order to prove the theorems, we recall the fundamental results in [KW4] for functions defined by the present zeta regularized product.

LEMMA 2.1.

$$(A) \quad \prod_{n=1}^{\infty} ((a_n))^{wb_n} = \left\{ \prod_{n=1}^{\infty} ((a_n))^{b_n} \right\}^w \quad \text{for any } w \in \mathbf{C}.$$

$$(B) \quad \prod_{n=1}^{\infty} ((a_n))^{b_n+c_n} = \prod_{n=1}^{\infty} ((a_n))^{b_n} \cdot \prod_{n=1}^{\infty} ((a_n))^{c_n}$$

whenever all of the appearing regularized products exist.

$$(C) \quad \prod_{n=1}^{\infty} ((\lambda a_n))^{b_n} = \lambda^{\phi_{\mathbf{a},\mathbf{b}}(0)} \prod_{n=1}^{\infty} ((a_n))^{b_n} \quad \text{for any } \lambda > 0. \quad \square$$

Define also the Dirichlet series attached to the data (\mathbf{a}, \mathbf{b}) by

$$\phi_{\mathbf{a},\mathbf{b}}(s, x) := \sum_{n=1}^{\infty} b_n \cdot (a_n - x)^{-s}.$$

Denote by μ the exponent of convergence of the series $\sum_{n=1}^{\infty} |b_n| \cdot |a_n|^{-t}$, that is, the series converges for $\text{Re}(t) = \mu + \varepsilon$ and diverges for $\text{Re}(t) = \mu - \varepsilon$ for any $\varepsilon > 0$. Then the Dirichlet series $\phi_{\mathbf{a},\mathbf{b}}(s, x)$ converges absolutely in the region $\text{Re}(s) > \mu$ and uniformly for each compact subset in x -space \mathbf{C} which does not meet any a_n . Thus we see that the function $\phi_{\mathbf{a},\mathbf{b}}(s, x)$ defines a holomorphic function in the region $\text{Re}(s) > \mu$. Let p be the integer part of μ . We assume that $\phi_{\mathbf{a},\mathbf{b}}(s, x)$ can be extended to a holomorphic function at $s = 0$. Then we define

$$\prod_{n=1}^{\infty} ((a_n - x))^{b_n} = \exp \left(- \frac{\partial}{\partial s} \phi_{\mathbf{a},\mathbf{b}}(0, x) \right).$$

As in the cases [V, I, KiW], we see that this zeta regularized product defines an entire function with zeros of indicated order as follows.

LEMMA 2.2. *Suppose that b_n are all positive integers. Then the function $\prod_{n=1}^{\infty} ((a_n - x))^{b_n}$ is analytically extended to the whole complex plane as an entire function whose zeros are exactly given by $x = a_n$ with multiplicity b_n . More precisely, there exists a polynomial $P(x)$ of degree at most p such that*

$$\prod_{n=1}^{\infty} ((a_n - x))^{b_n} = e^{P(x)} \prod_{n=1}^{\infty} \left(1 - \frac{x}{a_n} \right)^{b_n} \exp \left(b_n \sum_{\ell=1}^p \frac{1}{\ell} \left(\frac{x}{a_n} \right)^{\ell} \right). \quad \square$$

We also recall the periodicity of the multiple sine function. By the fact

$$\frac{S'_r}{S_r}(x) = \pi x^{r-1} \cot(\pi x) \quad \text{with } S_r(0) = 1$$

for $r \geq 2$, the binomial expansion shows the following results (for details, see [KKo]).

LEMMA 2.3. *We have*

$$S_r(x + 1) = \frac{S'_r(1)}{2\pi} \prod_{k=1}^r S_k(x)^{\binom{r-1}{k-1}}, \tag{1}$$

where $S'_r(1)$ is given by

$$S'_r(1) = -2\pi \exp\left(-2 \sum_{1 < \ell < r, \ell: \text{odd}} \binom{r-1}{\ell-1} \zeta'(1-\ell)\right) \tag{2}$$

$$= -2\pi \exp\left(- (r-1)! \sum_{1 < \ell < r, \ell: \text{odd}} \frac{(-1)^{\frac{\ell-1}{2}}}{(r-\ell)!(2\pi)^{\ell-1}} \zeta(\ell)\right). \tag{3}$$

Note here that

$$\zeta(m) = \frac{2(2\pi)^{m-1}(-1)^{\frac{m-1}{2}}}{(m-1)!} \zeta'(1-m). \quad \square \tag{4}$$

We put for convenience

$$S_1(x) = 2\pi x \prod_{n=-\infty, n \neq 0}^{\infty} P_1\left(\frac{x}{n}\right) = 2\pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = 2 \sin(\pi x)$$

and

$$C_1(x) = 2 \prod_{n=-\infty, n: \text{odd}}^{\infty} P_1\left(\frac{x}{\frac{n}{2}}\right) = 2 \prod_{n=1, n: \text{odd}}^{\infty} \left(1 - \frac{x^2}{\left(\frac{n}{2}\right)^2}\right) = 2 \cos(\pi x).$$

Then, by the formula due to Lerch [L] (see [KW2]), we have

$$S_1(x) = x \prod_{n=1}^{\infty} (n-x) \prod_{n=1}^{\infty} (n+x) \tag{5}$$

and

$$C_1(x) = \prod_{n=1}^{\infty} \left(n - \frac{1}{2} - x\right) \prod_{n=1}^{\infty} \left(n - \frac{1}{2} + x\right). \tag{6}$$

We note here that Euler regarded the divergent series $\sum_{n=1}^{\infty} n^2 \log n$ as $-\zeta'(2)$ ($= \zeta(3)/4\pi^2$) in [E]. In our present notation, this implies $\prod_{n=1}^{\infty} (n)^{n^2} = \exp(-\zeta'(2))$.

3. Proof of the main theorems

Define

$$\mathbf{G}_m(x) := \prod_{n=1}^{\infty} ((n+x))^{-n^m}. \tag{7}$$

We call $\mathbf{G}_m(x)$ a *basic multiple gamma function* of order m . We first study the behavior of this $\mathbf{G}_m(x)$ under the translation $x \rightarrow x + 1$. Since

$$\mathbf{G}_m(x+1)^{-1} = \prod_{n=1}^{\infty} ((n+x+1))^{n^m} = \prod_{\ell=1}^{\infty} ((\ell+x))^{(\ell-1)^m}, \tag{8}$$

by the property (B) of Lemma 2.1, the binomial theorem shows

$$\mathbf{G}_m(x+1) = \prod_{k=0}^m \mathbf{G}_k(x) \binom{m}{k} (-1)^{m-k}. \tag{9}$$

We put temporarily $f_m(x) = \mathbf{G}_m(x)^{-1}$ and $g_m(x) = f_m(-x)$. Then, similarly to the translation formula (8), we obtain

$$g_m(x+1) = (-x) \prod_{k=0}^m g_k(x) \binom{m}{k}. \tag{10}$$

First note that we have proved the expressions of $\mathcal{S}_2(x)$ and $\mathcal{S}_3(x)$ in [KW4]. Thus we show the assertions of Theorem 1.1 by induction. Assume the formulas (1) and (2) in Theorem 1.1 are true for k less than m . Look at the ratio $g_{2m-1}(x)/f_{2m-1}(x)$. Then by (9) and (10), we see that

$$\begin{aligned} \frac{g_{2m-1}(x+1)}{f_{2m-1}(x+1)} &= (-x) \prod_{k=0}^{2m-1} \left\{ g_k(x) f_k(x)^{-(-1)^{2m-1-k}} \right\} \binom{2m-1}{k} \\ &= (-x) \cdot \frac{g_{2m-1}(x)}{f_{2m-1}(x)} \cdot \prod_{j=0}^{m-1} \left\{ g_{2j}(x) f_{2j}(x) \right\} \binom{2m-1}{2j} \cdot \prod_{j=1}^{m-1} \left\{ \frac{g_{2j-1}(x)}{f_{2j-1}(x)} \right\} \binom{2m-1}{2j-1}. \end{aligned}$$

Hence the induction hypothesis asserts

$$\begin{aligned} \frac{g_{2m-1}(x+1)}{f_{2m-1}(x+1)} &= -\frac{g_{2m-1}(x)}{f_{2m-1}(x)} \cdot \mathcal{S}_1(x) \\ &\quad \times \prod_{j=1}^{m-1} \left\{ \mathcal{S}_{2j+1}(x) \exp \left(-\frac{(-1)^j \zeta(2j+1)(2j)!}{2^{2j} \pi^{2j}} \right) \right\} \binom{2m-1}{2j} \cdot \prod_{j=1}^{m-1} \mathcal{S}_{2j}(x) \binom{2m-1}{2j-1} \\ &= -\exp \left(-\sum_{j=1}^{m-1} \frac{(-1)^j (2j)!}{(2\pi)^{2j}} \binom{2m-1}{2j} \zeta(2j+1) \right) \end{aligned}$$

$$\begin{aligned} & \times \frac{g_{2m-1}(x)}{f_{2m-1}(x)} \cdot \prod_{k=1}^{2m-1} \mathcal{S}_k(x)^{\binom{2m-1}{k-1}} \\ & = \frac{\mathcal{S}'_{2m}(1)}{2\pi} \prod_{k=1}^{2m-1} \mathcal{S}_k(x)^{\binom{2m-1}{k-1}} \times \frac{g_{2m-1}(x)}{f_{2m-1}(x)}. \end{aligned}$$

In the last equality, we used the expression of the value of $\mathcal{S}'_{2m}(1)$ described in Lemma 2.3. Therefore, using Lemma 2.3 again, we find that the functions $\mathcal{S}_{2m}(x)$ and $g_{2m-1}(x)/f_{2m-1}(x)$ have exactly the same periodicity as meromorphic functions of order $2m$. On the other hand, since the both meromorphic functions have the same zeros and poles (counting with their multiplicity) by Lemma 2.2, there exists a polynomial $P(x)$ of degree at most $2m$ with real coefficients such that the identity

$$\frac{g_{2m-1}(x)}{f_{2m-1}(x)} = e^{P(x)} \mathcal{S}_{2m}(x).$$

holds. The aforementioned periodicity shows hence that the polynomial $P(x)$ should be of the form $2k\pi ix + c$ with an integer k while the real valuedness $P(x)$ on \mathbf{R} implies $k = 0$, that is, $P(x)$ should be a constant c . Noting $\frac{g_{2m-1}(0)}{f_{2m-1}(0)} = \mathcal{S}_{2m}(0) = 1$, we obtain $e^{P(x)} = 1$. Hence the desired expression $\mathcal{S}_{2m}(x)$ in (1) for m follows.

We next derive the expression of $\mathcal{S}_{2m+1}(x)$ in (2). The calculation similar to the one we did shows that

$$g_{2m+1}(x+1)f_{2m+1}(x+1) = \frac{\mathcal{S}'_{2m+1}(1)}{2\pi} \prod_{k=1}^{2m} \mathcal{S}_k(x)^{\binom{2m}{k-1}} \times g_{2m+1}(x)f_{2m+1}(x),$$

that is, the meromorphic function $g_{2m+1}(x)f_{2m+1}(x)$ has the same periodicity of $\mathcal{S}_{2m+1}(x)$. Therefore, employing again the same discussion made above, we prove the assertion for $\mathcal{S}_{2m+1}(x)$ by Lemma 2.2. This completes the proof of Theorem 1.1. \square

REMARK 1. It follows from Theorem 1.1 that

$$\prod_{n=1}^{\infty} ((n))^{n^{2m}} = \exp\left((-1)^{m-1} \frac{\zeta(2m+1)(2m)!}{2^{2m+1}\pi^{2m}} \right). \tag{11}$$

In other words, the value of $\zeta(s)$ at the odd integer point can be written as

$$\zeta(2m+1) = \frac{(-1)^{m-1} 2^{2m+1} \pi^{2m}}{(2m)!} \log\left(\prod_{n=1}^{\infty} ((n))^{n^{2m}} \right). \tag{12}$$

REMARK 2. We can describe a generalization of the formula given in the remark above. Let $\zeta(s, x) := \sum_{n=0}^{\infty} (n+x)^{-s}$ be the Hurwitz zeta function. Then for $m \geq 1$, we prove the following formula which is also considered as an analogue of the Lerch formula

[L]: $\Gamma(x)^{-1} = (2\pi)^{-\frac{1}{2}} \prod_{n=0}^{\infty} (n+x) = (2\pi)^{-\frac{1}{2}} e^{-\zeta'(0,x)}$.

$$\mathbf{G}_m(x)^{-1} = \prod_{n=1}^{\infty} ((n+x))^{n^m} = \prod_{\ell=0}^m e^{-\binom{m}{\ell} \zeta'(-\ell, x) x^{m-\ell}}. \quad (13)$$

Here the differentiation $\zeta'(-\ell, x)$ indicates the one with respect to s . In fact, by definition we obtain $\mathbf{G}_m(x)^{-1} = \exp\left(-\frac{\partial}{\partial s} \phi'_{\{n\}, \{n^m\}}(s, -x)|_{s=0}\right)$. Here, since

$$\begin{aligned} \phi_{\{n\}, \{n^m\}}(s, x) &= \sum_{n=1}^{\infty} n^m \cdot (n-x)^{-s} = \sum_{n=0}^{\infty} (n-x+x)^m \cdot (n-x)^{-s} \\ &= \sum_{n=0}^{\infty} \sum_{\ell=0}^m \binom{m}{\ell} (n-x)^{\ell} x^{m-\ell} \cdot (n-x)^{-s} \\ &= \sum_{\ell=0}^m \binom{m}{\ell} x^{m-\ell} \zeta(s-\ell, -x) \end{aligned}$$

we have

$$\phi'_{\{n\}, \{n^m\}}(0, -x) = \sum_{\ell=0}^m \binom{m}{\ell} x^{m-\ell} \zeta'(-\ell, x),$$

whence we get the formula (13).

The theorem shows that the basic multiple gamma function $\mathbf{G}_m(x)$ is a half zeta function of the corresponding basic multiple sine function $\mathcal{S}_{m+1}(x)$ in the sense of [HKW]. Actually, the formula (8), that is, the equation

$$\mathbf{G}_m(x+1) = \left\{ \prod_{k=0}^{m-1} \mathbf{G}_k(x) \binom{m}{k} (-1)^{m-k} \right\} \cdot \mathbf{G}_m(x) \quad (14)$$

is regarded as a generalization of the translation property of the gamma function $\Gamma(x+1) = x\Gamma(x)$. Moreover the theorem yields the reflection formulas of the basic gamma function as follows:

COROLLARY 3.1. *For $m \geq 1$ we have*

$$\mathcal{S}_{2m}(x) = \mathbf{G}_{2m-1}(x) \mathbf{G}_{2m-1}(-x)^{-1}, \quad (15)$$

$$\mathcal{S}_{2m+1}(x) = \exp\left((-1)^m \frac{\zeta(2m+1)(2m)!}{2^{2m}\pi^{2m}}\right) \mathbf{G}_{2m}(x)^{-1} \mathbf{G}_{2m}(-x)^{-1}. \quad \square \quad (16)$$

We will show the generalization of the duplication formula of $\mathbf{G}_m(x)$ in the end of the section (see Corollary 3.3).

The proof of Theorem 1.2 can be done in the following two ways. The first one is the same as the above for $\mathcal{S}_{2m}(x)$ by the periodicity of the multiple cosine function obtained from

the following characterization of $\mathcal{C}_r(x)$:

$$\frac{\mathcal{C}'_r(x)}{\mathcal{C}_r(x)} = -\pi x^{r-1} \tan(\pi x) \quad \text{with } \mathcal{C}_r(0) = 1.$$

The second one, which we take here, is to employ the duplication formula (see [KOW], [KKo]) of the multiple sine functions given by

$$\tilde{\mathcal{C}}_r(x) = \mathcal{S}_r(2x)\mathcal{S}_r(x)^{-2^{r-1}}. \tag{17}$$

In fact, using the property (C) in Lemma 2.1, we have the following result.

PROPOSITION 3.2. *Let $\zeta(s, x)$ be the Hurwitz zeta function. Then the duplication procedure of the function $\mathbf{G}_m(x)^{-1} = \prod_{n=1}^{\infty} (n+x)^{n^m}$ is described as*

$$\prod_{n=1}^{\infty} ((n+2x))^{n^m} = 2^{z(m,2x)} \cdot \left\{ \prod_{\ell=1}^{\infty} ((\ell+x))^{2^m} \right\} \cdot \prod_{\ell=1}^{\infty} \left(\left(\ell - \frac{1}{2} + x \right) \right)^{(2\ell-1)^m}. \tag{18}$$

Here we put

$$z(m, x) = \sum_{j=0}^m \binom{m}{j} (-x)^{m-j} \zeta(-j, x). \tag{19}$$

PROOF. Put

$$\rho_m(s, x) = \sum_{n=1}^{\infty} n^m \cdot (n+x)^{-s} \tag{20}$$

and

$$\tilde{\rho}_m(s, x) = \sum_{\ell=1}^{\infty} (2\ell-1)^m \left(\ell - \frac{1}{2} + x \right)^{-s}. \tag{21}$$

By the property (C) in Lemma 2.1 we have

$$\begin{aligned} \prod_{n=1}^{\infty} ((n+2x))^{n^m} &= \prod_{\ell=1}^{\infty} ((2\ell+2x))^{(2\ell)^m} \cdot \prod_{\ell=1}^{\infty} ((2\ell-1+2x))^{(2\ell-1)^m} \\ &= \left\{ 2^{\rho_m(0,x)} \prod_{\ell=1}^{\infty} ((\ell+x))^{2^m} \right\} \cdot 2^{\tilde{\rho}_m(0,x)} \prod_{\ell=1}^{\infty} \left(\left(\ell - \frac{1}{2} + x \right) \right)^{(2\ell-1)^m} \\ &= 2^{2^m \rho_m(0,x) + \tilde{\rho}_m(0,x)} \left\{ \prod_{\ell=1}^{\infty} ((\ell+x))^{2^m} \right\} \cdot \prod_{\ell=1}^{\infty} \left(\left(\ell - \frac{1}{2} + x \right) \right)^{(2\ell-1)^m}. \end{aligned}$$

Here we note (as in the calculation in Remark 2) that

$$\rho_m(0, x) = \sum_{j=0}^m \binom{m}{j} (-x)^{m-j} \zeta(-j, x) = z(m, x). \tag{22}$$

Also, we obtain

$$\begin{aligned}\tilde{\rho}_m(s, x) &= 2^s \sum_{\ell=1}^{\infty} (2\ell - 1)^m (2\ell - 1 + 2x)^{-s} \\ &= 2^s \left\{ \sum_{k=1}^{\infty} k^m (k + 2x)^{-s} - \sum_{\ell=1}^{\infty} (2\ell)^m (2\ell + 2x)^{-s} \right\} \\ &= 2^s \rho_m(s, 2x) - 2^m \rho_m(s, x).\end{aligned}$$

Thus, in particular,

$$2^m \rho_m(0, x) + \tilde{\rho}_m(0, x) = \rho_m(0, 2x).$$

Hence the result follows immediately from (22). \square

EXAMPLE. Since $B_1(x) = x - \frac{1}{2}$ and $B_2(x) = x^2 - x + \frac{1}{6}$, we have

$$z(1, m) = -x\zeta(0, x) + \zeta(-1, x) = \frac{1}{2}x^2 - \frac{1}{12}.$$

PROOF OF THEOREM 1.2. Now we recall the formula

$$\zeta(-j, x) = -\frac{B_{j+1}(x)}{j+1},$$

where $B_{j+1}(x)$ is the Bernoulli polynomial given by

$$B_{j+1}(x) = \sum_{k=0}^{j+1} \binom{j+1}{k} B_k x^{j+1-k}.$$

Note that the Bernoulli numbers satisfy $B_{2m+1} = 0$ for $m \geq 1$ while $B_1 = -\frac{1}{2} \neq 0$. By this fact, $z(m, x)$ can be calculated as

$$\begin{aligned}z(m, x) &= -\sum_{j=0}^m \binom{m}{j} \frac{(-x)^{m-j}}{j+1} \sum_{k=0}^{j+1} \binom{j+1}{k} B_k x^{j+1-k} \\ &= (-1)^{m+1} x^{m+1} \sum_{j=0}^m \sum_{k=0}^{j+1} \binom{m}{j} \binom{j+1}{k} \frac{(-1)^j}{j+1} B_k x^{-k}.\end{aligned}$$

Now looking at the coefficient of $B_1 x^m$, we have

$$\sum_{j=0}^m \binom{m}{j} \binom{j+1}{1} \frac{(-1)^j}{j+1} = \sum_{j=0}^m \binom{m}{j} (-1)^j = (1-1)^m = 0.$$

Hence we see that if m is odd (resp. even), then the polynomial $z(m, x)$ becomes even (resp. odd). Therefore, by the proposition above together with the duplication formula (17) of $\mathcal{S}_r(x)$

and Theorem 1.1, the proof of Theorem 1.2 can be accomplished. The detailed calculation is easy and is left to the reader. \square

REMARK 3. Similarly to the formula in Remark 1, by Theorem 1.2 we have

$$\prod_{n=1}^{\infty} \left(\left(n - \frac{1}{2} \right) \right)^{(2n-1)2^m} = \exp \left((-1)^{m-1} (1 - 2^m) \frac{\zeta(2m+1)(2m)!}{2^{2m+1}\pi^{2m}} \right). \quad (23)$$

A proof similar to the proposition gives the duplication formula of $\mathbf{G}_m(x)$.

COROLLARY 3.3. Put $\rho_m(s, x) = \sum_{n=1}^{\infty} n^m \cdot (n+x)^{-s}$. Then the duplication formula of $\mathbf{G}_m(x)$ holds:

$$\mathbf{G}_m(2x) = 2^{-w(m,x)} \cdot \mathbf{G}_m(x)^{2^m} \cdot \prod_{j=0}^m \left\{ \mathbf{G}_m \left(x - \frac{1}{2} \right) \right\}^{\binom{m}{j} 2^j (-1)^{m-j}}. \quad (24)$$

Here,

$$w(m, x) = 2^m \rho_m(0, x) + \sum_{j=0}^m \binom{m}{j} 2^j (-1)^{m-j} \rho_j \left(0, x - \frac{1}{2} \right)$$

and each $\rho_j(0, x)$ is calculated as

$$\rho_m(0, x) = \sum_{\ell=0}^m \binom{m}{\ell} (-1)^{m-\ell} x^{m-\ell} \zeta(-\ell, x). \quad \square$$

REMARK 4. An analogue of the Gauss-Legendre multiplication formula of the gamma function can be established similarly.

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