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Hecke Modular Form Expansions for Eighth Order Mock Theta Functions

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Abstract. By using Hecke type modular forms, we relate the eighth order mock theta functions with theta functions.

1. Introduction

In his last letter to Hardy, Ramanujan wrote, "I discovered very interesting functions recently which I call Mock theta functions". Ramanujan listed seventeen such functions and assigned them orders three, five and seven. According to McIntosh, the order is analogous to the level of a modular form. Later Watson [9] found three more mock theta functions of order three.

Recently Gordon and McIntosh [4] considered eight mock theta functions and called them of order eight.

By using Bailey pair method, Andrews [1, 2] gave Hecke type modular form series for fifth order, sixth order and seventh order mock theta functions. Watson [9, pp. 64–66] gave Hecke type modular form series for the third order mock theta functions. Recently Choi [3] gave Hecke type modular form series for the tenth order mock theta functions.

In this paper we use Bailey pair method to derive Hecke type modular series for the eighth order mock theta functions. By using these Hecke type series, we give a relation between the eighth order mock theta functions $S_0(q)$, $S_1(q)$, and theta functions.

In our paper [8], by using these Hecke type modular series we have given these eighth order mock theta functions as coefficient of z^0 in the Laurent series expansion of rational expression of various theta series.

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2. Notation

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1}(1-q^i x)\,, \quad n \geq 0\,. \tag{2.1}$$

If $|q| < 1$

$$(x)_\infty = (x,q)_\infty = \lim_{n\to\infty}(x)_n = \prod_{i=0}^{\infty}(1-q^i x)\,, \tag{2.2}$$

$$(x_1,\cdots,x_r;q)_\infty = (x_1)_\infty,\cdots,(x_r)_\infty = \prod_{i=0}^{\infty}(1-q^i x_1)\cdots(1-q^i x_r)\,, \tag{2.3}$$

$$j(x,q) = (x,q/x,q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}} x^n \quad x \neq 0\,, \quad |q| < 1\,, \tag{2.4}$$

where $\binom{n}{2} = \frac{n^2-n}{2}$.

For $m \geq 1$

$$J_{a,m} = j(q^a, q^m)\,, \tag{2.5}$$

$$\bar{J}_{a,m} = j(-q^a, q^m)\,, \tag{2.6}$$

$$J_m = J_{m,3m} = j(q^m, q^m) = (q^m;q^m)_\infty\,. \tag{2.7}$$

DEFINITION 1.

$$\mathcal{S}_n = \begin{cases} \sum_{j=-n}^{n}(-1)^j q^{-j^2}\,, & n \geq 0 \\ \sum_{j=n+1}^{-n-1}(-1)^j q^{-j^2}\,, & n < 0\,. \end{cases} \tag{2.8}$$

DEFINITION 2. If n is an integer

$$sg(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ -1 & \text{if } n < 0\,. \end{cases} \tag{2.9}$$

With the help of $sg(n)$ we can convert the summation notation

$$\sum_{n=a}^{b} C_n = \sum_{\substack{n=-\infty \\ \mathrm{sgn}(b-n)=\mathrm{sgn}(n-a)}} sg(n-a)C_n\,. \tag{2.10}$$

We state the following theorem of Hickerson [6, Theorem 1.6, p. 646], which we shall be using in the text:

For $|q| < x < 1$, $|q| < y < 1$

$$\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r) q^{rs} x^r y^s = \frac{J_{2,4}\, j(qxy, q^2)\, j(-qxy^{-1}, q^2)\, j(x^2y^2, q^4)}{j(x^2, q^2)\, j(y^2, q^2)}. \tag{2.11}$$

3. Definition of the mock theta function of order eight

Gordon and McIntosh [4] defined the following mock theta function of order eight:

$$S_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}, \qquad T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}, \qquad V_0(q) := -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q:q^2)_n}$$

$$= -1 + 2\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},$$

and

$$S_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}, \qquad T_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}, \qquad V_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}.$$

4. Bailey pairs

In this section we give Bailey pairs which will be used in section 5 to give Hecke type identities.

BAILEY'S LEMMA. *If for $n \geq 0$, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are related by*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}} \tag{4.1}$$

then for $n \geq 0$

$$\beta'_n = \sum_{r=0}^{n} \frac{\alpha'_r}{(q)_{n-r}(aq)_{n+r}} \tag{4.2}$$

where

$$\beta_n' = \frac{1}{\left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n} \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j \left(\frac{aq}{\rho_1\rho_2}\right)_{n-j} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j}{(q)_{n-j}} \tag{4.3}$$

and

$$\alpha_r' = \frac{(\rho_1)_r(\rho_2)_r \left(\frac{aq}{\rho_1\rho_2}\right)^r \alpha_r}{\left(\frac{aq}{\rho_1}\right)_r \left(\frac{aq}{\rho_2}\right)_r}, \tag{4.4}$$

for any given numbers ρ_1 and ρ_2.

COROLLARY. *If $\{\alpha_n\}$ and $\{\beta_n\}$ form a Bailey pair relative to then,*

$$\sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n}{\left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n} = \frac{(aq)_\infty \left(\frac{aq}{\rho_1\rho_2}\right)_\infty}{\left(\frac{aq}{\rho_1}\right)_\infty \left(\frac{aq}{\rho_2}\right)_\infty} \sum_{n=0}^{\infty} (\rho_1)_n(\rho_2)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n, \tag{4.5}$$

provided both sums converge absolutely.

By letting $n \to \infty$ for β_n' we have the corollary.

We give eight Bailey pairs in the following lemmas.

LEMMA 4.1. *The sequences α_n, β_n, form a Bailey pair, where*

$$\alpha_n(a,b,c,q) = \frac{q^{n^2}(bc)^n(1-aq^{2n})(a/b)_n)a/c)_n}{(1-a)(qb)_n(qc)_n} \times \sum_{j=0}^{n} \frac{(-1)^j(1-aq^{2j-1})(a)_{j-1}(b)_j(c)_j}{q^{\binom{j}{2}}(bc)^j(q)_j(a/b)_j(a/c)_j} \tag{4.6}$$

and

$$\beta_n(a,b,c,q) = \frac{1}{(qb)_n(qc)_n}. \tag{4.7}$$

PROOF. See Andrews [2, Theorem 2.3, p. 72].

LEMMA 4.2. *The sequences α_n and β_n form a Bailey pair, where*

$$\alpha_n'(aq,b,q) = \frac{(-1)^n(1-aq^{2n+1})q^{\frac{3n^2+n}{2}}(aq/b)_n b^n}{(1-aq)(bq)_n} \times \left[1 + \sum_{j=0}^{n} \frac{(1-aq^{2j})(aq)_{j-1}(b)_j a^{-j} q^{-j^2} b^{-j}}{(q)_j(aq/b)_j}\right] \tag{4.8}$$

and

$$\beta'_n(aq, b, q) = \frac{1}{(qb)_n}\,. \tag{4.9}$$

PROOF. See Andrews [1, Theorem 4, p. 119 and Lemma 6, p. 120].

LEMMA 4.3.

$$\begin{aligned}\alpha_n(1, -1, q^2) &= q^{3n^2+n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2} - q^{3n^2-n} \sum_{-(n-1)}^{(n-1)} (-1)^j q^{-2j^2} \\ &= q^{3n^2+n} \mathcal{S}_n - q^{3n^2-n} \mathcal{S}_{n-1}\,, \quad n \geq 1\end{aligned} \tag{4.10}$$

$$\alpha_0(1, -1, q^2) = 1$$

and

$$\beta_n(1, -1, q^2) = \frac{1}{(-q^2; q^2)_n} \tag{4.11}$$

form a Bailey pair.

PROOF. By Andrews [1, Theorem 4, p. 119] and Andrews [1, Lemma 7, (5.6), p. 121].

LEMMA 4.4.

$$\begin{aligned}\alpha_n(q^2, -1, q^2) &= \frac{(1 - q^{4n+2}) q^{3n^2+n}}{(1 - q^2)} \left[1 + 2 \sum_{j=1}^{n} (-1)^j q^{-2j^2} \right] \\ &= \frac{(1 - q^{4n+2}) q^{3n^2+n}}{(1 - q^2)} \left[\sum_{j=-n}^{n} (-1)^j q^{-2j^2} \right]\end{aligned} \tag{4.12}$$

and

$$\beta_n(q^2, -1, q^2) = \frac{1}{(-q^2; q^2)_n} \tag{4.13}$$

form a Bailey pair.

PROOF. By Andrews [1, Lemma 7, (5.7), p. 121] and Andrews [1, Theorem 4, p. 119].

LEMMA 4.5.

$$\alpha'_n(q^4, -q, q^2) = \frac{(1 - q^{4n+4}) q^{3n^2+4n+1}}{(1 - q)(1 - q^4)} \sum_{j=0}^{2n+1} (-1)^{\frac{j^2+j+2}{2}} q^{-\binom{j+1}{2}} \tag{4.14}$$

and

$$\beta'_n(q^4, -q, q^2) = \frac{1}{(-q^3; q^2)_n}\,, \tag{4.15}$$

form a Bailey pair.

PROOF. Letting $q \to q^2$ and putting $a = q^2$, $b = -q$, in Lemma 4.2, we have

$$\begin{aligned}
\alpha'_n(q^4, -q, q^2) &= \frac{(-1)^n(1-q^{4n+4})q^{3n^2+3n}(-q)^n(-q^3;q^2)_n}{(1-q^4)(-q^3;q^2)_n} \\
&\quad \times \left[1 + \sum_{j=1}^{n} \frac{(-1)^j q^{-2j^2-3j}(1-q^{4j+2})(q^4;q^2)_{j-1}(-q;q^2)_j}{(q^2;q^2)_j(-q^3;q^2)_j}\right] \\
&= \frac{(1-q^{4n+4})q^{3n^2+4n}}{(1-q^4)}\left[1 + \sum_{j=1}^{n} \frac{(-1)^j q^{-2j^2-3j}(1-q^{2j+1})}{(1-q)}\right] \\
&= \frac{(1-q^{4n+4})q^{3n^2+4n+1}}{(1-q)(1-q^4)}\left[\sum_{j=0}^{n}(-1)^j q^{-2^2-3j-1} - \sum_{j=0}^{n}(-1)^j q^{-2j^2-j}\right] \\
&= \frac{(1-q^{4n+4})q^{3n^2+4n+1}}{(1-q)(1-q^4)}\left[\sum_{j=0}^{n}(-1)^j q^{-\binom{2j+2}{2}} - \sum_{j=0}^{n}(-1)^j q^{-\binom{2j+1}{2}}\right] \\
&= \frac{(1-q^{4n+4})q^{3n^2+4n+1}}{(1-q)(1-q^4)}\left[\sum_{j=0}^{2n+1}(-1)^{\frac{j^2+j+2}{2}} q^{-\binom{j+1}{2}}\right]
\end{aligned}$$

and

$$\beta'_n(q^2, -q, q^2) = \frac{1}{(-q^3;q^2)_n}.$$

LEMMA 4.6.

$$\alpha'_n(q^2, -q, q^2) = \frac{(1-q^{2n+1})q^{3n^2+2n}}{(1-q)}\left[\sum_{j=0}^{2n}(-1)^{\frac{j^2+j}{2}} q^{-\binom{j+1}{2}}\right] \tag{4.16}$$

and

$$\beta'_n(q^2, -q, q^2) = \frac{1}{(-q^3;q^2)_n}, \tag{4.17}$$

form a Bailey pair.

PROOF. Letting $q \to q^2$ and putting $a = 1$, $b = -q$ in Lemma 4.2, we have

$$\begin{aligned}
\alpha'_n(q^2, -q, q^2) &= \frac{(1-q^{2n+1})q^{3n^2+2n}}{(1-q)}\left[1 + \sum_{j=1}^{n}(-1)^j(1+q^{2j})q^{-2j^2-j}\right] \\
&= \frac{(1-q^{2n+1})q^{3n^2+2n}}{(1-q)}\left[\sum_{j=0}^{n}(-1)^j q^{-2j^2-j} + \sum_{j=1}^{n}(-1)^j q^{-2j^2+j}\right] \\
&= \frac{(1-q^{2n+1})q^{3n^2+2n}}{(1-q)}\sum_{j=0}^{2n}(-1)^{\frac{j^2+j}{2}} q^{-\binom{j+1}{2}}
\end{aligned}$$

and

$$\beta'_n(q^2, -q, q^2) = \frac{1}{(-q^3; q^2)_n} .$$

LEMMA 4.7.

$$\alpha'_n(1, q^{-1}, q^2) = (-1)^n \left[q^{3n^2+2n} \sum_{j=0}^{2n} q^{-\binom{j+1}{2}} - q^{3n^2-2n} \sum_{j=0}^{2n-2} q^{-\binom{j+1}{2}} \right] \tag{4.18}$$

and

$$\beta_n(1, q^{-1}, q^2) = \frac{1}{(q; q^2)_n} , \tag{4.19}$$

form a Bailey pair.

PROOF. By Andrews [1, Lemma 7, (5.8), p. 121].

LEMMA 4.8.

$$\alpha'_n(q^2, q, q^2) = \frac{(-1)^n (1 + q^{2n+1}) q^{3n^2+2n}}{1 + q} \sum_{j=0}^{2n} q^{-\binom{j+1}{2}} \tag{4.20}$$

and

$$\beta'_n(q^2, q, q^2) = \frac{1}{(q^3; q^2)_n} , \tag{4.21}$$

form a Bailey pair.

PROOF. By Andrews [1, Lemma 7, (5.9) p. 121].

5. Hecke type identities

Using Bailey pair method, given by Andrews [1], we give Hecke type identities for the eighth order mock theta functions. We shall be using the Bailey pairs, given as Lemmas in the previous section.

THEOREM 1.

$$J_{1,4} S_0(q) = \sum_{n=-\infty}^{\infty} q^{4n^2+n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2} . \tag{5.1}$$

PROOF. Making $q \to q^2$ and letting $\rho_1 \to \infty$, $\rho_2 = -q$, $a = 1$ in (4.5), we have

$$\sum_{n=0}^{\infty} q^{n^2} \alpha_n = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \sum_{n=0}^{\infty} q^{n^2} (-q; q^2)_n \beta_n .$$

Taking the Bailey pair in Lemma 4.3 i.e. $\alpha_n = q^{3n^2+n}\mathcal{S}_n - q^{3n^2-n}\mathcal{S}_{n-1}$ and $\beta_n = \frac{1}{(-q^2;q^2)_n}$, $n \geq 1$ and since we have defined $\alpha_0 = 1$, in Lemma 4.3, we have

$$1 + \sum_{n=1}^{\infty} [q^{4n^2+n}\mathcal{S}_n - q^{4n^2-n}\mathcal{S}_{n-1}] = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}\,.$$

Now

$$\text{Right side} = J_{1,4}S_0(q)$$

and

$$\text{Left side} = 1 + \sum_{n=1}^{\infty} [q^{4n^2+n\mathcal{S}_n - q^{4n^2-n}\mathcal{S}_{n-1}}]\,.$$

But $\mathcal{S}_{n-1} = -\mathcal{S}_{-n}$ by definition 1. So

$$\begin{aligned}
\text{Left side} &= 1 + \sum_{n=1}^{\infty} [q^{4n^2+n}\mathcal{S}_n + q^{4n^2-n}\mathcal{S}_{-n}] \\
&= 1 + \sum_{n=1}^{\infty} q^{4n^2+n}\mathcal{S}_n + \sum_{n=-1}^{-\infty} q^{4n^2+n}\mathcal{S}_n \\
&= \sum_{n=-\infty}^{\infty} q^{4n^2+n}\mathcal{S}_n = \sum_{n=-\infty}^{\infty} q^{4n^2+n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2}\,.
\end{aligned}$$

Hence the Theorem.

THEOREM 2.

$$J_{1,4}S_1(q) = \sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-2j^2}\,. \tag{5.2}$$

PROOF. Making $q \to q^2$ and letting $\rho_1 \to \infty$, $\rho_2 = -q$, $a = q^2$ in (4.5) we have,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(-q^3;q^2)_n}\alpha_n = \frac{(q^4;q^2)_\infty}{(-q^3;q^2)_\infty} \sum_{n=0}^{\infty} q^{n^2+2n}(-q;q^2)_n\beta_n\,.$$

Taking the Bailey pair in Lemma 4.4 i.e.

$$\alpha_n = \frac{(1-q^{4n+2})q^{3n^2+n}}{(1-q^2)} \sum_{j=-n}^{n} (-1)^j q^{-2j^2}$$

and

$$\beta_n = \frac{1}{(-q^2;q^2)_n}\,,$$

we have

$$\sum_{n=0}^{\infty} \frac{q^{4n^2+3n}(1-q^{4n+2})(-q;q^2)_n}{(1-q^2)(-q^3;q^2)_n} \sum_{j=-n}^{n} (-1)^j q^{-2j^2} = \frac{(q^4;q^2)_\infty}{(-q^3;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(-q^2;q^2)_n},$$

and therefore

$$\begin{aligned}\sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-2j^2} &= \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(-q^2;q^2)_n} \\ &= J_{1,4} S_1(q).\end{aligned}$$

THEOREM 3.

$$\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} T_0(q) = \sum_{n=0}^{\infty} q^{4n^2+7n+3}(1-q^{2n+2}) \sum_{j=0}^{2n+1} (-1)^{\frac{j^2+j+2}{2}} q^{-\binom{j+1}{2}}. \tag{5.3}$$

PROOF. Making $q \to q^2$ and letting $\rho_1 \to \infty$, $\rho_2 = -q^2$, $a = q^4$ in (4.5) we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2+3n}(-q^2;q^2)_n}{(-q^4;q^2)_n} \alpha_n = \frac{(q^6;q^2)_\infty}{(-q^4;q^2)_\infty} \sum_{n=0}^{\infty} q^{n^2+3n}(-q^2;q^2)_n \beta_n .$$

Taking the Bailey pair in Lemma 4.5 i.e.

$$\alpha_n(q^4,-q,q^2) = \frac{(1-q^{4n+4})q^{3n^2+4n+1}}{(1-q)(1-q^4)} \sum_{j=0}^{2n+1} (-1)^{\frac{j^2+j+2}{2}} q^{-\binom{j+1}{2}}$$

and

$$\beta_n = \frac{1}{(-q^3;q^2)_n},$$

we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{q^{4n^2+7n+3}(1-q^{4n+4})(-q^2;q^2)_n}{(1-q)(1-q^4)(-q^4;q^2)_n} \sum_{j=0}^{2n+1} (-1)^{\frac{j^2+j+2}{2}} q^{-\binom{j+1}{2}} \\ &= \frac{(q^6;q^2)_\infty}{(-q^4;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+3n+2}(-q^2;q^2)_n}{(-q^3;q^2)_n}\end{aligned}$$

or

$$\sum_{n=0}^{\infty} q^{4n^2+7n+3}(1-q^{2n+2)} \sum_{j=0}^{2n+1} (-1)^{\frac{j^2+j+2}{2}} q^{-\binom{j+1}{2}}$$

$$= \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+3n+2}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} .$$

Now right side is $\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} T_0(q)$.

Hence the Theorem.

THEOREM 4.

$$\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} T_1(q) = \sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=0}^{2n} (-1)^{\frac{j^2+j}{2}} q^{\binom{j+1}{2}} . \tag{5.4}$$

PROOF. Making $q \to q^2$ and letting $\rho_1 \to \infty$, $\rho_2 = -q^2$, $a = q^2$ in (4.5) , we have

$$\sum_{n=0}^{\infty} q^{n^2+n} \alpha_n = \frac{(q^4;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{n^2+n}(-q^2;q^2)_n \beta_n .$$

Taking the Bailey pair in Lemma 4.6 i.e. $\alpha_n = \frac{(1-q^{2n+1})q^{3n^2+2n}}{(1-q)} \sum_{j=0}^{2n} (-1)^{\frac{j^2+j}{2}} q^{-\binom{j+1}{2}}$ and $\beta_n = \frac{1}{(-q^3;q^2)_n}$, we have

$$\sum_{n=0}^{\infty} \frac{q^{4n^2+3n}(1-q^{2n+1})}{(1-q)} \sum_{j=0}^{2n} (-1)^{\frac{j^2+j}{2}} q^{-\binom{j+1}{2}}$$
$$= \frac{(q^4;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^2;q^2)_n}{(-q^3;q^2)_n} ,$$

and therefore

$$\sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=0}^{2n} (-1)^{\frac{j^2+j}{2}} q^{-\binom{j+1}{2}}$$
$$= \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} .$$

Now right side is $\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} T_1(q)$.

Hence the Theorem.

THEOREM 5.

$$J_{1,4}[1 + V_0(q)] = 2 \sum_{n=0}^{\infty} \sum_{j=0}^{2n} (-1)^n q^{4n^2+2n-\binom{j+1}{2}} (1 + q^{4n+2}) . \tag{5.5}$$

PROOF. Letting $q \to q^2$ and $\rho_1 \to \infty$, $\rho_2 = -q$, $a = 1$, in (4.5), we have

$$\sum_{n=0}^{\infty} q^{n^2}\alpha_n = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty} q^2(-q;q^2)_n\beta_n .$$

Taking the Bailey pair in Lemma 4.7 i.e.

$$\alpha_n = (-1)^n\left[q^{3n^2+2n}\sum_{j=0}^{2n} q^{-\binom{j+1}{2}} - q^{3n^2-2n}\sum_{j=0}^{2n-2} q^{-\binom{j+1}{2}}\right]$$

and

$$\beta_n = \frac{1}{(q;q^2)_n}\,,$$

we have

$$2\sum_{n=0}^{\infty}(-1)^n\left[q^{4n^2+2n}\sum_{j=0}^{2n} q^{-\binom{j+1}{2}} - q^{4n^2-2n}\sum_{j=0}^{2n+1} q^{-\binom{j+1}{2}}\right]$$
$$= 2\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}\,.$$

Right side $= J_{1,4}[1 + V_0(q)]$.

$$\text{Left side} = 2\left[\sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{4n^2+2n-\binom{j+1}{2}} + \sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{4n^2+6n+2-\binom{j+1}{2}}\right]$$
$$= 2\sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{4n^2+2n-\binom{j+1}{2}}(1+q^{4n+2})\,.$$

Hence the Theorem.

THEOREM 6.

$$J_{1,4}V_1(q) = \sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{4n^2+4n+1-\binom{j+1}{2}}\,. \tag{5.6}$$

PROOF. Letting $q \to q^2$, $\rho_1 \to \infty$, $\rho_2 = -q$, $a = q^2$ in (4.5), we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(-q^3;q^2)_n}\alpha_n = \frac{(q^4;q^2)_\infty}{(-q^3;q^2)_\infty}\sum_{n=0}^{\infty} q^{n^2+2n}(-q;q^2)_n\beta_n\,.$$

Taking the Bailey pair in Lemma 4.8 i.e.

$$\alpha_n = \frac{(-1)^n(1+q^{2n+1})q^{3n^2+2n}}{(1+q)}\sum_{j=0}^{2n} q^{-\binom{j+1}{2}}$$

and

$$\beta_n = \frac{1}{(q^3;q^2)_n}\,,$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{4n^2+4n+1-\binom{j+1}{2}} &= \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+2n+1}(-q;q^2)_n}{(q;q^2)_{n+1}}\,,\\ &= \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}V_1(q)\\ &= J_{1,4}V_1(q)\,.\end{aligned}$$

Hence the Theorem.

We will apply the constant term method to $J_{1,4}S_0(q)$ and $J_{1,4}S_1(q)$ and shall use (5.1) and (5.2) to obtain identities for them. For this we prove two relations which we will require in section 6.

$$J_{1,4}S_0(q) = \sum_{\substack{sg(r)=sg(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} sg(r)(-1)^{\frac{r-s}{2}}q^{2rs}q^{\frac{r^2+2rs+s^2+r+s}{2}}\,, \tag{5.7}$$

and

$$J_{1,4}S_1(q) = \sum_{\substack{sg(r)=sg(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} sg(r)(-1)^{\frac{r-s}{2}}q^{2rs}q^{\frac{r^2+2rs+s^2+3r+3s}{2}}\,. \tag{5.8}$$

PROOF OF (5.7). Apply (2.11) to the inner sum of (5.1) and put $r=n+j$, $s=n-j$, to get

$$\begin{aligned}J_{1,4}S_0(q) &= \sum_{n=-\infty}^{\infty}\sum_{j=-n}^{n}(-1)^j q^{4n^2+n-2j^2}\\ &= \sum_{\substack{n,j\\ sg(n-j)=sg(n+j)}} sg(n+j)(-1)^j q^{4n^2+n-2j^2}\\ &= \sum_{\substack{sg(r)=sg(s)\\ r\equiv s\,(\mathrm{mod}\,2)}} sg(r)(-1)^{\frac{r-s}{2}}q^{2rs}q^{\frac{r^2+2rs+s^2+r+s}{2}}\end{aligned}$$

which proves (5.7).

Similarly we can prove (5.8).

6. Mock theta functions as coefficients

In this section we will define theta functions $B(z,q)$, $C(z,q)$ and give a relation with mock theta functions $S_0(q)$, $S_1(q)$.

RELATION FOR $S_0(q)$. The Hecke type identity for $S_0(q)$ given in Theorem 1 of section 5 is

$$J_{1,4}S_0(q) = \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{t-s}{2}} q^{2rs} q^{\frac{r^2+2rs+s^2+r+s}{2}} . \tag{6.1}$$

DEFINITION. For $|q|<1$ and z not an integral power of q^4, let

$$B(z) = B(z,q) = \frac{z^2 J_z J_{1,2} J_4^2 j(-z,q^2)}{J_{1,4}^3 j(z,q^4)} j(zq,q^4) . \tag{6.2}$$

THEOREM 7. *In the annulus $|q^4|<|z|$, the coefficient of z^2 in $B(z)$ is $S_0(q)$.*

PROOF. Letting $q \to q^2$ and putting $x = -z^{\frac{1}{2}}$, $y = z^{\frac{1}{2}}$ in [6, Theorem 1, (2.12), p. 646], we have

$$\begin{aligned}\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^r q^{2rs} z^{\frac{r+s}{2}} &= \frac{J_{2,4} j(-zq^2,q^4) j(q^2,q^4) j(z^2,q^8)}{j(z,q^4) j(z,q^4)} \\ &= \frac{J_1 J_2 j(-z,q^2)}{j(z,q^4)} .\end{aligned} \tag{6.3}$$

Hence

$$\begin{aligned}J_{1,4}B(z) &= \frac{z^2 J_1 J_{1,2} J_4^2 j(-z,q^2)}{J_{1,4}^2 j(z,q^4)} j(qz,q^4) \\ &= \frac{z^2 J_1 J_2 j(-z,q^2)}{j(z,q^4)} j(qz,q^4) \\ &= z^2 \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^r q^{2rs} z^{\frac{r+s}{2}} \sum_{t=-\infty}^{\infty} (-1)^t q^{2t^2-t} z^t .\end{aligned} \tag{6.4}$$

From (6.4) the coefficient of z^2 is obtained by putting $t = -\frac{r+s}{2}$ and it is

$$\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-)^{\frac{r-s}{2}} q^{2rs} q^{\frac{r^2+2rs+s^2+r+s}{2}}$$

which equals $J_{1,4}S_0(q)$ by (6.1). Dividing by $J_{1,4}$ we have the Theorem.

RELATION FOR $S_1(q)$. The Hecke type identity for $S_1(q)$ given in Theorem 2 of section 5 is

$$J_{1,4}S_1(q) = \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s}{2}} q^{2rs} q^{\frac{r^2+2rs+s^2+3r+3s}{2}} . \tag{6.5}$$

DEFINITION. For $|q| < 1$, $z \neq q$ and z not an integral power of q^4, let

$$C(z) = \frac{z^2 J_1 J_{1,2} J_4^2 j(-z, q^2)}{J_{1,4}^3 j(z, q^4)} j(z/q, q^4) .$$

THEOREM 8. *In the annulus $|q^4| < |z|$, the coefficient of z^2 in $C(z)$ is $S_1(q)$*

PROOF. Letting $q \to q^2$ and putting $x = -z^{\frac{1}{2}}$, $y = z^{\frac{1}{2}}$ in [6, Theorem 1. p. 646], we have

$$\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s}{2}} q^{2rs} z^{\frac{r+s}{2}} = \frac{J_{2,4} j(-zq^2, q^4) j(q^2, q^4) j(z^2, q^8)}{J(z, q^4) j(z, q^4)}$$

$$= \frac{J_1 J_2 j(-z, q^2)}{j(z, q^4)} .$$

Hence

$$\begin{aligned} J_{1,4}C(z) &= \frac{z^2 J_1 J_{1,2} J_4^2 j(-z, q^2)}{J_{1,4}^2 j(z, q^4)} j(z/q, q^4) \\ &= \frac{z^2 J_1 J_2 j(-z, q^2)}{j(z, q^4)} j(z/q, q^4) \\ &= z^2 \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^r q^{2rs} z^{\frac{r+s}{2}} \sum_{t=-\infty}^{\infty} (-1)^t q^{2t^2-3t} z^t . \end{aligned} \tag{6.6}$$

From (6.6) the coefficient of the coefficient of z^2 is obtained by putting $t = -\frac{r+s}{2}$ and it is

$$\sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} sg(r)(-1)^{\frac{r-s}{2}} q^{2rs} q^{\frac{r^2+2rs+s^2+3r+3s}{2}}$$

which equals $J_{1,4}S_1(q)$ by (6.1). Dividing by $J_{1,4}$ we have the Theorem.

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