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A Fixed Point Formula for 0-pseudofree S^1-actions on Kähler Manifolds of Constant Scalar Curvature

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Abstract. Let M be an m-dimensional compact complex manifold and Ω a Kähler class of M. Assume that M admits an Ω-preserving 0-pseudofree S^1-action and that Ω contains a Kähler metric of constant scalar curvature. Then using the fixed point formula for the Bando-Calabi-Futaki character obtained in [5], we can obtain information on the fixed point data of the S^1-action. Our main result is Theorem 2.

1. Introduction

Let M be an m-dimensional compact complex manifold, $\mathrm{Aut}(M)$ the complex Lie group consisting of all biholomorphic automorphisms of M and $\mathfrak{h}(M)$ the Lie algebra of $\mathrm{Aut}(M)$, which consists of all holomorphic vector fields on M. Let Ω be a Kähler class of M and $\omega \in \Omega$ a Kähler form, which is identified with the Kähler metric in this paper. Let s_ω be the scalar curvature of ω and μ_Ω a real number defined by

$$\mu_\Omega = \frac{\Omega^{m-1} \cup c_1(M)[M]}{\Omega^m[M]}$$

where $c_1(M)$ is the first Chern class of M and $[M]$ is the fundamental cycle of M. Then a Lie algebra character $f_\Omega : \mathfrak{h}(M) \to \mathbf{C}$ is defined by

$$f_\Omega(X) = \frac{1}{2\pi} \int_M X F_\omega \, \omega^m$$

where F_ω is a function which satisfies $\triangle_\omega F_\omega = s_\omega - m\mu_\Omega$. Then in [1], [2], [4], it is proved that f_Ω does not depend on the choice of the Kähler metrics $\omega \in \Omega$ and that $f_\Omega(X) = 0$ for any $X \in \mathfrak{h}(M)$ if Ω contains a Kähler metric of constant scalar curvature.

Assume that $\mathrm{Aut}(M)$ contains a positive dimensional compact connected subgroup G and let $\mathfrak{g}$ be the Lie algebra of G. Then, under the assumption that Ω is equal to the first Chern class $c_1(L)$ of a holomorphic G-line bundle L, Nakagawa [10] defined a group character $\widehat{f}_\Omega : G \to \mathbf{C}/(\mathbf{Z} + \mu_\Omega \mathbf{Z})$ which is a lift of $f_\Omega|\mathfrak{g}$ by using a Simons character of a certain

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foliation. Then $f_\Omega(X) = 0$ for any $X \in \mathfrak{g}$ implies that $\widehat{f_\Omega}(\sigma) = 0$ for any $\sigma \in G$ and hence $\widehat{f_\Omega}(\sigma) = 0$ for any $\sigma \in G$ if Ω contains a Kähler metric of constant scalar curvature.

In this paper, a faithful biholomorphic action of S^1 on M is called simply an S^1-action. An S^1-action is called 0-pseudofree when the action is not free and the fixed point set

$$M^{S^1} = \{x \in M \mid g \cdot x = x \text{ for all } g \in S^1\}$$

consists only of points (cf. [6], [9]). Let $R(S^1) = \mathbf{Z}[t, t^{-1}]$ be the representation ring of S^1 where t is the standard 1-dimensional representation of S^1 defined by the natural inclusion $S^1 \subset GL(1; \mathbf{C})$.

Now we assume that M admits a 0-pseudofree S^1-action. Suppose that the fixed point set M^{S^1} consists of r points $q_1, \ldots, q_r$ and that

$$T_{q_j}M = \sum_{s=1}^{m} t^{p_{js}} \in R(S^1) \quad (1 \le j \le r)$$

as an S^1-representation space where p_{js} are integers. Let β_j be an integer defined by

$$\beta_j = \sum_{s=1}^{m} p_{js} \quad (1 \le j \le r)\,.$$

Set $M^g = \{x \in M \mid g \cdot x = x\}$ for $g \in S^1$ and let P be the set defined by

$$P = \{\text{odd prime numbers } p \mid M^{\sigma_p} = M^{S^1}\}$$

where $\sigma_p \in S^1$ is the primitive p-th root of unity. Note that none of p_{js} $(1 \le j \le r,\ 1 \le s \le m)$ is a multiple of p if $p \in P$ and that the set of prime numbers p which are not contained in P is a finite set because the number of orbit types of an S^1-action on a compact manifold is finite.

Assume moreover that a Kähler class Ω is equal to the first Chern class $c_1(L)$ of a holomorphic S^1-line bundle L and suppose that $L|_{q_j} = t^{\gamma_j} \in R(S^1)$ for $\gamma_j \in \mathbf{Z}$, $1 \le j \le r$. Then μ_Ω is a rational number and there exists an integer q such that $q\mu_\Omega$ is an integer. Let $\alpha \in \mathbf{C}$ denote the primitive p-th root of unity.

Then using Theorem 2.1, Lemma 2.3 and Theorem 2.5 in [5], we have the next theorem.

THEOREM 1. *For any $p \in P$, set*

$$\begin{aligned} F_\Omega(\sigma_p) = (m+1)\sum_{i=0}^{m}(-1)^i \binom{m}{i} (S_{+1}(m-2i) - S_{-1}(m-2i)) \\ -m\mu_\Omega \sum_{i=0}^{m+1}(-1)^i \binom{m+1}{i} S_0(m+1-2i) \end{aligned}$$

where

$$S_\varepsilon(n) = \frac{1}{p}\sum_{k=1}^{p-1}\sum_{j=1}^{r}\frac{1}{1-\alpha^k}(\alpha^{\varepsilon\beta_j k + n\gamma_j k} - 1)^{m+1}\prod_{s=1}^{m}\frac{1}{1-\alpha^{-p_{js}k}}$$

for $\varepsilon = +1, -1, 0$. *Then* $\widehat{f}_\Omega(\sigma)$ *is equal to* $F_\Omega(\sigma_p) \bmod \mathbf{Z} + \mu_\Omega \mathbf{Z}$.

2. Main result

For any $p \in P$, let $\vec{u}_p$ be an element of $\mathbf{Z}^r$ defined by

$$\vec{u}_p = \left(\overline{\prod_{s=1}^{m} p_{1s}}, \dots, \overline{\prod_{s=1}^{m} p_{rs}}\right)$$

where $\overline{n}$ $(1 \leq \overline{n} \leq p-1)$ denotes the mod p inverse of the integer n which is not a multiple of p. Then we have the next theorem.

THEOREM 2. *Assume that* $\Omega = c_1(L)$ *contains a Kähler metric of constant scalar curvature and suppose that* $q\mu_\Omega \in \mathbf{Z}$ *for* $q \in \mathbf{Z}$. *Let* $\vec{v}_q$ *be an element of* $\mathbf{Z}^r$ *defined by*

$$\vec{v}_q = q((m+1)\beta_1\gamma_1^m - m\mu_\Omega\gamma_1^{m+1}, \dots, (m+1)\beta_r\gamma_r^m - m\mu_\Omega\gamma_r^{m+1}).$$

Then the inner product $\vec{u}_p \cdot \vec{v}_q \in \mathbf{Z}$ *is a multiple of* p *for any* $p \in P$ *such that* $p > m+1$.

We need the following lemmas to prove Theorem 2.

LEMMA 1. *Let* p *be an odd prime number*, ρ_j, λ_j *integers and* μ_j *an integer which is not a multiple of* p. *Then we have*

$$\frac{1}{p}\sum_{k=1}^{p-1}\prod_{j=1}^{N}\alpha^{k\rho_j}\frac{\alpha^{k\lambda_j}-1}{\alpha^{k\mu_j}-1} \equiv -\frac{1}{p}\prod_{j=1}^{N}\lambda_j\overline{\mu_j} \pmod{\mathbf{Z}}.$$

PROOF. For any integers n, ℓ, we have

$$\sum_{k=1}^{p-1} n(\alpha^k)^\ell = \begin{cases} n\dfrac{1-\alpha^{p\ell}}{1-\alpha^\ell} - n = -n & (\text{if } \ell \text{ is not a multiple of } p) \\ n(p-1) = -n + np & (\text{if } \ell \text{ is a multiple of } p) \end{cases}$$

and hence it follows that

$$\sum_{k=1}^{p-1}\Phi(\alpha^k, \alpha^{-k}) \equiv -\Phi(1,1) \pmod{p}$$

for any polynomial $\Phi(x, y)$ with integer coefficients. Therefore we have

$$\frac{1}{p}\sum_{k=1}^{p-1}\prod_{j=1}^{N}\alpha^{k\rho_j}\frac{\alpha^{k\lambda_j}-1}{\alpha^{k\mu_j}-1} = \frac{1}{p}\prod_{j=1}^{N}\sum_{k=1}^{p-1}\alpha^{k\rho_j}\frac{(\alpha^{k\mu_j})^{\overline{\mu_j}\lambda_j}-1}{\alpha^{k\mu_j}-1}$$

$$\equiv -\frac{1}{p}\prod_{j=1}^{N}\lim_{x\to 1}\frac{x^{\overline{\mu_j}\lambda_j}-1}{x-1}$$

$$= -\frac{1}{p}\prod_{j=1}^{N}\lambda_j\overline{\mu_j} \pmod{\mathbf{Z}}. \qquad \square$$

LEMMA 2. *Let λ be a positive integer and μ a non-negative integer. Then we have*

$$\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}(\lambda-2i)^{\mu} = \begin{cases} 0 & \text{if } \mu<\lambda \text{ or } \mu=\lambda+1 \\ 2^{\lambda}\lambda! & \text{if } \mu=\lambda \end{cases}$$

PROOF. Set

$$N(\lambda,\mu) = \frac{1}{(-1)^{\lambda}\lambda!}\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}i^{\mu}.$$

Then since $f^{(\mu)}(-1) = 0$ for $f(x) = (1+x)^{\lambda}$, $0\le\mu<\lambda$, it follows from the binomial theorem that

$$\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}i(i-1)\dots(i-\mu+1) = 0$$

if $0\le\mu<\lambda$. Using the equality above, we can prove that $N(\lambda,\mu)=0$ for $0\le\mu<\lambda$ by induction. Hence we have

$$\begin{aligned} N(\lambda,\lambda) &= \frac{1}{(-1)^{\lambda}\lambda!}(-\lambda)\sum_{i=1}^{\lambda}(-1)^{i-1}\binom{\lambda-1}{i-1}i^{\lambda-1} \\ &= \frac{1}{(-1)^{\lambda-1}(\lambda-1)!}\sum_{j=0}^{\lambda-1}(-1)^j\binom{\lambda-1}{j}(j+1)^{\lambda-1} = N(\lambda-1,\lambda-1) \end{aligned}$$

and therefore it follows that

$$N(\lambda,\lambda) = N(1,1) = 1 \iff \sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}i^{\lambda} = (-1)^{\lambda}\lambda!.$$

Moreover we have

$$\begin{aligned} N(\lambda,\lambda+1) &= \frac{1}{(-1)^{\lambda-1}(\lambda-1)!}\sum_{j=0}^{\lambda-1}(-1)^j\binom{\lambda-1}{j}(j+1)^{\lambda} \\ &= \frac{1}{(-1)^{\lambda-1}(\lambda-1)!}\sum_{j=0}^{\lambda-1}(-1)^j\binom{\lambda-1}{j}(j^{\lambda}+\lambda j^{\lambda-1}) \end{aligned}$$

$$= N(\lambda-1,\lambda) + \lambda N(\lambda-1,\lambda-1) = N(\lambda-1,\lambda) + \lambda$$

and therefore it follows that

$$N(\lambda,\lambda+1) = N(\lambda-1,\lambda) + \lambda = \cdots = N(1,2) + 2 + \cdots + (\lambda-1) + \lambda = \frac{\lambda(\lambda+1)}{2}$$

$$\iff \sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i} i^{\lambda+1} = \frac{(-1)^\lambda \lambda(\lambda+1)!}{2}\,.$$

Using equalities above, we have

$$\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}(\lambda-2i)^{\mu} = 0 \quad \text{if } \mu < \lambda\,,$$

$$\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}(\lambda-2i)^{\lambda} = \sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}(-2i)^{\lambda} = (-2)^\lambda(-1)^\lambda\lambda! = 2^\lambda\lambda!\,,$$

$$\begin{aligned}\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}&(\lambda-2i)^{\lambda+1}\\ &= \sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}(-2i)^{\lambda+1} + (\lambda+1)\lambda\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}(-2i)^{\lambda}\\ &= (-2)^{\lambda+1}\frac{(-1)^\lambda\lambda(\lambda+1)!}{2} + (\lambda+1)\lambda(-2)^\lambda(-1)^\lambda\lambda! = 0\,. \qquad \square\end{aligned}$$

Using the lemmas above, we can prove Theorem 2 as follows. Since $\overline{n}\,\overline{n'} \equiv \overline{nn'} \pmod p$, it follows from Lemma 1 that

$$\begin{aligned}S_\varepsilon(n) &= \frac{1}{p}\sum_{k=1}^{p-1}\sum_{j=1}^{r}\frac{1}{1-\alpha^k}\left(\alpha^{k(\varepsilon\beta_j+n\gamma_j)}-1\right)^{m+1}\prod_{s=1}^{m}\frac{1}{1-\alpha^{-kp_{js}}}\\ &= -\frac{1}{p}\sum_{j=1}^{r}\sum_{k=1}^{p-1}\frac{\alpha^{k(\varepsilon\beta_j+n\gamma_j)}-1}{\alpha^k-1}\prod_{s=1}^{m}\alpha^{kp_{js}}\frac{\alpha^{k(\varepsilon\beta_j+n\gamma_j)}-1}{\alpha^{kp_{js}}-1}\\ &\equiv \frac{1}{p}\sum_{j=1}^{r}(\varepsilon\beta_j+n\gamma_j)^{m+1}\overline{\prod_{s=1}^{m}p_{js}} \quad (\operatorname{mod} \mathbf{Z})\,.\end{aligned}$$

Hence it follows from Lemma 2 that

$$
\begin{aligned}
& qF_{\Omega}(\sigma_p) \\
& \equiv \frac{q}{p}\sum_{j=1}^{r}\left\{\begin{array}{l}(m+1)\sum_{i=0}^{m}(-1)^i\binom{m}{i} \\ \left\{((m-2i)\gamma_j+\beta_j)^{m+1}-((m-2i)\gamma_j-\beta_j)^{m+1}\right\} \\ -m\mu_{\Omega}\sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}(m+1-2i)^{m+1}\gamma_j^{m+1}\end{array}\right\}\overline{\prod_{s=1}^{m}p_{js}} \\
& = \frac{q}{p}\sum_{j=1}^{r}\left\{2(m+1)^2\beta_j\gamma_j^m 2^m m! - m\mu_{\Omega}\gamma_j^{m+1}2^{m+1}(m+1)!\right\}\overline{\prod_{s=1}^{m}p_{js}} \\
& = \frac{1}{p}2^{m+1}(m+1)!\,\vec{u}_p\cdot\vec{v}_q \quad (\mathrm{mod}\ \mathbf{Z})\,,
\end{aligned}
$$

which is contained in $q(\mathbf{Z}+\mu_{\Omega}\mathbf{Z})\subset\mathbf{Z}$ because S^1 is connected and Ω contains a Kähler metric of constant scalar curvature. Here since p is prime to $2^{m+1}(m+1)!$, the equality above implies that $\vec{u}_p\cdot\vec{v}_q$ is a multiple of p. This completes the proof of Theorem 2. □

Let $\vec{w}$ be an element of $\mathbf{Z}^r$ defined by

$$
\vec{w} = \left(\beta_1^{m+1},\ldots,\beta_r^{m+1}\right) = \left(\left(\sum_{s=1}^{m}p_{1s}\right)^{m+1},\ldots,\left(\sum_{s=1}^{m}p_{rs}\right)^{m+1}\right).
$$

COROLLARY 1. *Assume that the first Chern class $c_1(M)$ of M contains a Kähler metric of constant scalar curvature. Then for any $p\in P$ such that $p>m+1$, the inner product $\vec{u}_p\cdot\vec{w}$ is a multiple of p.*

PROOF. Let L be the anticanonical bundle K_M^{-1} of M. Then $c_1(L)$ is equal to $c_1(M)$ and the S^1 action naturally lifts to an action on L. Since $\mu_{\Omega}=1$ and $\gamma_j=\beta_j$, we have $\vec{v}_q=\vec{w}$ for $q=1$ and hence it follows from Theorem 2 that $\vec{u}_p\cdot\vec{w}$ is a multiple of p. □

3. Examples

EXAMPLE 1. Let $M=\mathbf{CP}^m$ be the m-dimensional complex projective space. Let $[z_0:z_1:\cdots:z_m]$ be the homogeneous coordinate of M and H the hyperplane bundle over $M=\mathbf{CP}^m=(\mathbf{C}^{m+1}-\{0\})/\mathbf{C}^*$ which is defined by

$$
H=(\mathbf{C}^{m+1}-\{0\})\times_{(\mathbf{C}^*,\,\rho)}\mathbf{C}
$$

where ρ is a representation of $\mathbf{C}^*$ on $\mathbf{C}$ defined by $\rho(z)w=z^{-1}w$. Set $\Omega=c_1(H)$. Then Ω is the positive generator of $H^2(M;\mathbf{Z})$, $c_1(M)=(m+1)\Omega$ and hence we have $\mu_{\Omega}=m+1$.

Moreover Ω contains a Kähler metric of constant scalar curvature associated to a positive constant multiple of the standard metric (Fubini-Study metric) of M. S^1-actions on M, H are defined by

$$
\begin{aligned}
&g \cdot [z_0 : z_1 : \cdots : z_m] = [z_0 : gz_1 : \cdots : g^m z_m]\,,\\
&g \cdot [(z_0, z_1, \ldots, z_m), h] = [(z_0, gz_1, \ldots, g^m z_m), h] \quad (g \in S^1)\,.
\end{aligned}
$$

Then the fixed point set M^{S^1} of this action consists of following $m + 1$ points

$$
q_0 = [1 : 0 : \cdots : 0],\ q_1 = [0 : 1 : \cdots : 0],\ \ldots,\ q_m = [0 : \cdots : 0 : 1]
$$

and P consists of all odd prime numbers which are greater than m. Then since

$$
\begin{aligned}
&g \cdot [(\tau_1, \ldots, \tau_j, 1, \tau_{j+1}, \ldots, \tau_m), h] = [(\tau_1, \ldots, g^{j-1}\tau_j, g^j, g^{j+1}\tau_{j+1}, \ldots, g^m\tau_m), h]\\
&\quad = [(g^{-j}\tau_1, \ldots, g^{-1}\tau_j, 1, g\tau_{j+1}, \ldots, g^{m-j}\tau_m), g^{-j}h]\,,
\end{aligned}
$$

we have

$$
T_{q_j}\mathbf{CP}^m = t^{-j} + \cdots + t^{-1} + t + \cdots + t^{m-j} \in R(S^1)\,, \qquad H|q_j = t^{-j}\,.
$$

Hence it follows that

$$
\begin{aligned}
&\beta_j = \sum_{s=1}^{m} p_{js} = (-j) + \cdots + (-1) + 1 + \cdots + m - j = \frac{1}{2}(m+1)(m-2j)\,,\\
&\prod_{s=1}^{m} p_{js} = (-1)^j j!(m-j)!\,, \qquad \gamma_j = -j
\end{aligned}
$$

and therefore we have

$$
\begin{aligned}
\vec{u}_p \cdot \vec{v}_1 &= \sum_{j=0}^{m} \overline{(-1)^j j!(m-j)!} \left\{ (m+1)\frac{1}{2}(m+1)(m-2j)(-j)^m - m\mu_\Omega(-j)^{m+1} \right\}\\
&= (-1)^m \sum_{j=0}^{m} \overline{(-1)^j j!(m-j)!} \left\{ \frac{1}{2}(m+1)^2(m-2j)j^m + m(m+1)j^{m+1} \right\}\,.
\end{aligned}
$$

Since μ_Ω is an integer, it follows from Theorem 2 that $\vec{u}_p \cdot \vec{v}_1$ is a multiple of p for any prime number $p > m + 1$. In fact, we have the following results.

If $m = 2$ and $p = 5$, we have

$$
\frac{1}{p}\vec{u}_p \cdot \vec{v}_1 = \frac{1}{5}\left(\overline{2} \cdot 0 + \overline{-1} \cdot 6 + \overline{2} \cdot 12\right) = \frac{1}{5}(3 \cdot 0 + 4 \cdot 6 + 3 \cdot 12) = 12 \in \mathbf{Z}\,.
$$

Further computation shows that the values of $\vec{u}_p \cdot \vec{v}_1/p$ are as follows:

	$p=5$	$p=7$	$p=11$	$p=13$	$p=17$	$p=19$
$m=2$	12	12	12	12	12	12
$m=3$	-344	-128	-344	-128	-344	-128
$m=4$		6080	6720	4480	5120	2240
$m=5$		-220080	-70080	-92580	-205080	-55080

All values in the table above are integers.

EXAMPLE 2. Let $\mathbf{CP}^2$, $\mathbf{CP}^3$ be complex projective spaces and set $M = \mathbf{CP}^2 \times \mathbf{CP}^3$, $\Omega = c_1(K_M^{-1})$. Then Ω contains a Kähler metric of constant scalar curvature associated to the product of standard metrics. An S^1-action on M is defined by

$$\begin{aligned} &g \cdot ([z_0 : z_1 : z_2], [w_0 : w_1 : w_2 : w_3]) \\ &\quad = ([z_0 : gz_1 : g^2 z_2], [w_0 : gw_1 : g^2 w_2 : g^3 w_3]) \ \ (g \in S^1) . \end{aligned}$$

Then the fixed point set M^{S^1} consists of following twelve points.

$$\begin{aligned} q_1 &= ([1:0:0],[1:0:0:0]) , & q_2 &= ([0:1:0],[1:0:0:0]) , \\ q_3 &= ([0:0:1],[1:0:0:0]) , & q_4 &= ([1:0:0],[0:1:0:0]) , \\ q_5 &= ([0:1:0],[0:1:0:0]) , & q_6 &= ([0:0:1],[0:1:0:0]) , \\ q_7 &= ([1:0:0],[0:0:1:0]) , & q_8 &= ([0:1:0],[0:0:1:0]) , \\ q_9 &= ([0:0:1],[0:0:1:0]) , & q_{10} &= ([1:0:0],[0:0:0:1]) , \\ q_{11} &= ([0:1:0],[0:0:0:1]) , & q_{12} &= ([0:0:1],[0:0:0:1]) \end{aligned}$$

and P consists of all prime numbers which are greater than 3. Since

$$([\tau_1 : g\tau_2 : g^2], [\tau_3 : g\tau_4 : g^2\tau_5 : g^3]) = ([g^{-2}\tau_1 : g^{-1}\tau_2 : 1], [g^{-3}\tau_3 : g^{-2}\tau_4 : g^{-1}\tau_5 : 1]) ,$$

$g \in S^1$ acts on the tangent space $T_{q_{12}}M$ via multiplication by the diagonal matrix of diagonal entries $g^{-2}, g^{-1}, g^{-3}, g^{-2}, g^{-1}$ and hence we have

$$\{p_{12\,1}, p_{12\,2}, p_{12\,3}, p_{12\,4}, p_{12\,5}\} = \{-2, -1, -3, -2, -1\} .$$

It follows from the same argument that

$$\begin{aligned} \{p_{11}, p_{12}, p_{13}, p_{14}, p_{15}\} &= \{1, 2, 1, 2, 3\} , \\ \{p_{21}, p_{22}, p_{23}, p_{24}, p_{25}\} &= \{-1, 1, 1, 2, 3\} , \\ \{p_{31}, p_{32}, p_{33}, p_{34}, p_{35}\} &= \{-2, -1, 1, 2, 3\} , \\ \{p_{41}, p_{42}, p_{43}, p_{44}, p_{45}\} &= \{1, 2, -1, 1, 2\}, \\ \{p_{51}, p_{52}, p_{53}, p_{54}, p_{55}\} &= \{-1, 1, -1, 1, 2\} \\ \{p_{61}, p_{62}, p_{63}, p_{64}, p_{65}\} &= \{-2, -1, -1, 1, 2\} , \end{aligned}$$

$$\{p_{71}, p_{72}, p_{73}, p_{74}, p_{75}\} = \{1, 2, -2, -1, 1\}\,,$$
$$\{p_{81}, p_{82}, p_{83}, p_{84}, p_{85}\} = \{-1, 1, -2, -1, 1\}\,,$$
$$\{p_{91}, p_{92}, p_{93}, p_{94}, p_{95}\} = \{-2, -1, -2, -1, 1\}\,,$$
$$\{p_{10\,1}, p_{10\,2}, p_{10\,3}, p_{10\,4}, p_{10\,5}\} = \{1, 2, -3, -2, -1\}\,,$$
$$\{p_{11\,1}, p_{11\,2}, p_{11\,3}, p_{11\,4}, p_{11\,5}\} = \{-1, 1, -3, -2, -1\}\,,$$
$$\{p_{12\,1}, p_{12\,2}, p_{12\,3}, p_{12\,4}, p_{12\,5}\} = \{-2, -1, -3, -2, -1\}$$

and therefore we have

$$\begin{aligned}\vec{w} &= \left(\left(\sum_{s=1}^{5} p_{1s}\right)^6, \dots, \left(\sum_{s=1}^{5} p_{12\,s}\right)^6\right) \\ &= (531441, 46656, 729, 15625, 64, 1, 1, 64, 15625, 729, 46656, 531441)\end{aligned}$$

$$\left(\prod_{s=1}^{5} p_{1s}, \dots, \prod_{s=1}^{5} p_{rs}\right) = (12, -6, 12, -4, 2, -4, 4, -2, 4, -12, 6, -12)\ .$$

Since $\overline{-n} \equiv -\overline{n} \pmod{p}$, we have

$$\vec{u}_p = \left(\overline{\prod_{s=1}^{5} p_{1s}}, \dots, \overline{\prod_{s=1}^{5} p_{rs}}\right) \equiv \left(\overline{12}, -\overline{6}, \overline{12}, -\overline{4}, \overline{2}, -\overline{4}, \overline{4}, -\overline{2}, \overline{4}, -\overline{12}, \overline{6}, -\overline{12}\right) \pmod{p}\,,$$

and hence we can see that $\vec{u}_p \cdot \vec{w} \equiv 0 \pmod{p}$ for any $p \in P$.

EXAMPLE 3. Let M be the surface obtained from $\mathbf{CP}^2$ by blowing up two points $[1:0:0]$, $[0:1:0]$ and $\pi : M \to \mathbf{CP}^2$ the canonical projection. Then since $\mathfrak{g}$ is not reductive (see [3] p.100), it follows from the result of Lichnerowicz [7], [8] that M does not admit any Kähler metric of constant scalar curvature, and therefore $c_1(K_M^{-1})$ does not contain any Kähler metric of constant scalar curvature in particular. An S^1-action on M is naturally induced by the action

$$g \cdot [z_0 : z_1 : z_2] = [z_0 : gz_1 : g^2 z_2] \quad (g \in S^1)$$

on $\mathbf{CP}^2$. Then the fixed point set M^{S^1} consists of five points q_1, q_2, q_3, q_4, q_5 where $q_1 = \pi^{-1}([0:0:1])$, $q_2 \in \pi^{-1}([1:0:0])$ is the point in M defined by the line $z_1 = 0$ through the point $[1:0:0]$ in $\mathbf{CP}^2$, $q_3 \in \pi^{-1}([1:0:0])$ is the point in M defined by the line $z_2 = 0$ through the point $[1:0:0]$ in $\mathbf{CP}^2$, $q_4 \in \pi^{-1}([0:1:0])$ is the point in M defined by the line $z_0 = 0$ through the point $[0:1:0]$ in $\mathbf{CP}^2$ and $q_5 \in \pi^{-1}([0:1:0])$ is the point in M defined by the line $z_2 = 0$ through the point $[0:1:0]$ in $\mathbf{CP}^2$. Then we can see that

$$(p_{11}, p_{12}) = (-2, -1)\,, \quad (p_{21}, p_{22}) = (-1, 2)\,, \quad (p_{31}, p_{32}) = (1, 1)\,,$$
$$(p_{41}, p_{42}) = (-2, 1)\,, \quad (p_{51}, p_{52}) = (-1, 2)\,.$$

Hence we have

$$\vec{u}_p \cdot \vec{w} = (\overline{2}, -\overline{2}, \overline{1}, -\overline{2}, -\overline{2}) \cdot (-27, 1, 8, -1, 1) = -14 \cdot 2 \cdot \overline{2} + 8 \cdot \overline{1} \equiv -6 \pmod{p}\,,$$

which is not a multiple of p unless $p = 3$.

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