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Application of Local Linking to Asymptotically Linear Wave Equations with Resonance

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Abstract. Existence of a time-periodic solution to a non-linear wave equation with resonance is established by a variational method. We consider the 2π-periodic weak solution to a wave equation $\Box u(x,t) = h(x,t,u(x,t))$ of space dimension 1, where $h(x,t,\xi)$ is asymptotically linear in ξ both as $\xi \to 0$ or $\xi \to \infty$, with the co-efficient as $\xi \to \infty$ belonging to $\sigma(\Box)$. It is proved that there are some cases, where the difference of $h(t,x,\xi)$ from its linear approximation is not bounded, that guarantee the existence of a non-trivial weak solutions. The proof is based on local linking theory and $(WPS)^*$ condition for the existence of a non-trivial critical point of a functional.

1. Introduction

The purpose of this paper is to prove the existence of a non-trivial time-periodic solution to the following non-linear wave equation (WE) with asymptotically linear non-linear term h ($\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$):

$$\text{(WE)} \qquad \begin{cases} \Box u(x,t) = h(x,t,u(x,t)), & (0 < x < \pi,\ t \in \mathbf{R}), \\ u(0,t) = u(\pi,t) = 0 & (t \in \mathbf{R}), \\ u(x,t+2\pi) = u(x,t) & (0 < x < \pi,\ t \in \mathbf{R}). \end{cases}$$

Many authors treated this problem by variational methods under various conditions on $h(x,t,\xi)$. For example, Rabinowitz [13] dealt with $h(x,t,\xi) = -f(\xi)$ where $f(\xi)$ is strictly monotonically increasing in ξ and is super-linear both at 0 and ∞. (Here $f(\xi)$ is said super-linear at ∞ if $\int_0^\xi f(\eta)d\eta \le \theta\xi f(\xi)$ for some constant $\theta \in [0,1/2)$ and for ξ with sufficiently large $|\xi|$.) Tanaka [14] discussed in detail the case where $h(x,t,\xi) = \pm|\xi|^{p-1}\xi$ for some $p > 1$. [5] and [9] are also concerned with $h(x,t,\xi)$ that is super-linear in ξ as $|\xi| \to \infty$. (We would like to note that the authors of these papers investigated the existence of infinitely many non-trivial solutions in the super-linear case.)

On the other hand, as for the case of $h(x,t,\xi)$ that satisfies $|h(x,t,\xi)| \le C_1|\xi| + C_2$, Brezis [3] gives a good survey of earlier fundamental results for this probrem. Chang, Wu and Li [4] is the pioneering work on the multiple existence of solutions to such probrems. Li

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and Szulkin [8], Kryszewski and Szulkin [7] and Bartsch and Ding [2] considered the case where $h(x,t,\xi)$ is asymptotically linear in ξ both as $\xi \to 0$ and $|\xi| \to \infty$. However, they all assume that if $h(x,t,\xi) = b\xi + o(\xi)$ as $|\xi| \to \infty$ (uniformly in (x,t)) and $b \in \sigma(\Box)$, then the "error" $h(x,t,\xi) - b\xi$ is bounded. *The main purpose of this paper is to show that this boundedness condition can be relaxed so as to allow the case like* $h(x,t,\xi) = b\xi + |\xi|^\alpha \mathrm{sgn}\xi$ *for* $0 < \alpha < 1$. (To be rigorous, $|\xi|^\alpha \mathrm{sgn}\xi$ should be deformed to a C^2 class function in a neighbourhood of 0.) Although the method of proof in this paper owes very much to that in [2], some generalizations and supplementations are necessary.

In the following Section 2, we prepare an abstract result on the line of local linking theory concerning the existence of a non-trivial critical point of a C^1 functional. Applying this result, we prove our main theorem in Section 3.

Although our notations are standard, we would like to note here that $B_r X$ denotes the closed r-ball of a Hilbert space X with center 0, while $B(u,r)$ denotes the open r-neighbourhood of u. ∂S is used to designate the topological boundary of S.

2. Abstract theory

Throughout this section, E denotes a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and Φ denotes a C^1 class functional on E. The gradient $\nabla\Phi(u)$ $(u \in E)$ is considered to be an element of E through the Riesz representation theorem. A subset $\tilde{E}$ is defined by $\tilde{E} := \{u \in E : \nabla\Phi(u) \neq 0\}$. Then recall that a map $V : \tilde{E} \to E$ is called a pseudo-gradient vector field for Φ if V satisfies the following conditions on $\tilde{E}$:

$$\begin{cases} \|V(u)\| \leq \frac{3}{2}\|\nabla\Phi(u)\|, \\ \langle\nabla\Phi(u), V(u)\rangle \geq \frac{1}{2}\|\nabla\Phi(u)\|^2. \end{cases}$$

It is well known that there exists a locally Lipschitz continuous pseudo-gradient vector field V for every C^1 class functional Φ ([12, lemma 6.1]). For such V, the ordinary differential equation

$$\frac{du(t)}{dt} = -V(u(t)), \quad u(0) = u_0 \qquad (u_0 \in \tilde{E})$$

has the unique solution which is maximally defined in the positive direction of t. This maximal solution will be called the pseudo-gradient flow defined by V and (starting from) u_0.

2.1. A region enclosing pseudo-gradient flows.

We start with the following simple observation about pseudo-gradient flows.

PROPOSITION 1. *Let* $f : E \to \mathbf{R}$ *be another* C^1 *class functional on E and let* $U := \{x \in E \mid f(x) < 0\}$ *be non-empty. Suppose that*

$$\langle\nabla\Phi(u), \nabla f(u)\rangle > 0 \quad \text{on } \partial U. \tag{1}$$

Then there exists a locally Lipschitz continuous pseudo-gradient vector field V for Φ for which every pseudo-gradient flow starting from a point in U remains in U as long as it is defined.

To prove this proposition we prepare the following

LEMMA 2. *Let Φ, f satisfy* (1). *Then, there exists a locally Lipschitz continuous pseudo-gradient vector field V for Φ on $\tilde{E}$, for which*

$$\langle V(u)\,,\ \nabla f(u)\rangle > 0 \quad \text{on } \partial U \tag{2}$$

holds.

PROOF. Since Φ, f are C^1 class and satisfy (1), there exists a $\gamma_u > 0$ for every $u \in \partial U$ such that

$$\|u - v\| < \gamma_u \Rightarrow \begin{cases} \|\nabla\Phi(u) - \nabla\Phi(v)\| < 1/2\|\nabla\Phi(u)\|\,, \\ \langle\nabla\Phi(u), \nabla f(v)\rangle > 0\,. \end{cases}$$

Set $Y := \bigcup_{u\in\partial U} B(u, \gamma_u) \supset \partial U$. Then the covering $\{B(u, \gamma_u) \mid u \in \partial U\}$ of Y possesses a locally finite refinement $\{U_\alpha \mid \alpha \in \Lambda\}$ so that there exists $u_\alpha \in \partial U$ for each $\alpha \in U_\alpha$ satisfying $U_\alpha \subset B(u_\alpha, \gamma_{u_\alpha})$. Let $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ be a locally Lipschitz continuous partition of unity on Y w.r.t. $\{U_\alpha \mid \alpha \in \Lambda\}$, and set $V_1(u) := \sum_{\alpha\in\Lambda} \varphi_\alpha(u)\nabla\Phi(u_\alpha)$ for all $u \in Y$. Then V_1 is a locally Lipschitz continuous pseudo-gradient vector field for Φ on Y and satisfies (2). On the other hand, since the functional Φ is of C^1 class, there exists a locally Lipschitz continuous pseudo-gradient vector field V_2 for Φ on $\tilde{E}$. Let $\{\beta_1, \beta_2\}$ be a locally Lipschitz continuous partition of unity subordinate to the open covering $Y \cup (E \setminus \partial U)$ of E. Then $V(u) := \beta_1(u)V_1(u) + \beta_2(u)V_2(u)$ $(u \in \tilde{E})$ is a locally Lipschitz continuous pseudo-gradient vector field for Φ on $\tilde{E}$ and satisfies (2). □

Now we begin with the

PROOF OF PROPOSITION 1. By the preceding Lemma, we may take a locally Lipschitz continuous pseudo-gradient vector field V for Φ on $\tilde{E}$, for which (2) holds. Suppose $u_0 \in U$ and let $u(t)$ be the gradient flow starting from u_0 and defined for $t \in [0, T)$ (T may be ∞). If $f(u(t)) = 0$ for some $t \in (0, T)$, there exists a $t_0 \in (0, T)$ such that $f(u(t_0)) = 0$ and $f(u(t)) < 0$ for every $t \in (0, t_0)$. Since $t \mapsto f(u(t))$ is differentiable at t_0, this implies $0 \le \langle(\nabla f)(u(t_0)), u'(t_0)\rangle = -\langle(\nabla f)(u(t_0)), V(u(t_0))\rangle$. On the other hand, $f(u(t_0)) = 0$ means that $u(t_0) \in \partial U$ and hence $\langle V(u(t_0))\,,\ \nabla f(u(t_0))\rangle > 0$ by (2). Thus the existence of t such that $f(u(t)) = 0$ leads to a contradiction, so we are done. □

REMARK 3. After the authors finished this work, they noticed that P. Majer [11] considered a "repelling" condition that is more general than (1). However, under this general condition, the existence of a pseudo-gradient vector field (in the sense defined above) as in Lemma 2 is not necessarily guaranteed.

Although in Proposition 1 the region U is assumed to have a smooth boundary, similar results will hold for regions with piecewise smooth boundaries. For later use, we record one such occasion.

PROPOSITION 4. *Suppose that there exists an orthogonal decomposition* $E = V_\infty \oplus W_\infty$, *and* Φ *satisfies the following condition* (R). *Then there exists a locally Lipschitz continuous pseudo-gradient vector field* V *for* Φ *on* $\tilde{E}$, *for which the region*

$$U := \{(v_\infty, w_\infty) \mid \|v_\infty\| < \max\{R_1, \delta\|w_\infty\|^\lambda\}\} \tag{3}$$

encloses pseudo-gradient flows starting from its elements.

(R) *The following* (i), (ii) *hold for some* $\lambda \geq 0$, $\delta > 0$, *and* $R_1 > 0$, *where*

$$u = w_\infty + v_\infty \quad (w_\infty \in W_\infty\,,\ v_\infty \in V_\infty)\,.$$

(i) $$\left\langle \nabla\Phi(u)\,,\ v_\infty - \lambda\delta^2 \frac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0 \quad (\text{if } \|v_\infty\| = \delta\|w_\infty\|^\lambda\,,\ \|v_\infty\| \geq R_1)$$

(ii) $$\langle \nabla\Phi(u)\,,\ v_\infty\rangle > 0 \quad (\text{if } \|v_\infty\| \geq \delta\|w_\infty\|^\lambda\,,\ \|v_\infty\| = R_1)$$

PROOF. Note that the condition (R) roughly says that the inner product of the gradient vector $\nabla\Phi$ and the outward normal vector to U is positive on ∂U, hence ∂U "repels" the gradient flow starting from inside of U. (One should recognize that except for points with $\|v_\infty\| = R_1$, ∂U is smooth).

We state only a sketch of proof. As in the proof of Lemma 2, we can construct a pseudo-gradient vector field V on $\tilde{E}$ which satisfies the condition (R) with $\nabla\Phi$ replaced by V. Then suppose $u_0 \in U$ and let $u(t)$ be the gradient flow determined by V that starts from u_0. If there exists a $\tilde{t} > 0$ for which $u(\tilde{t}) \notin U$, then there exists a $t_0 \in (0, \tilde{t}]$ such that $u(t_0) \in \partial U$ and $u(t) \in U$ for every $t \in [0, t_0)$. Recall that $u(t)$ can be written uniquely as $u(t) = v(t) + w(t)$, by a $v(t) \in V_\infty$ and a $w(t) \in W_\infty$. According as $\delta\|w(t_0)\|^\lambda > R_1$ or $\delta\|w(t_0)\|^\lambda < R_1$, consider $f(t) := \|v(t)\|^2 - \delta^2\|w(t)\|^{2\lambda}$ or $f(t) := \|v(t)\|^2 - R_1^2$, respectively (the case of $\delta\|w(t_0)\|^\lambda = R_1$ will be treated later). Then $f(t_0) = 0$ and $f(t) < 0$ if $t < t_0$ is sufficiently close to t_0. Hence $f'(t_0) \geq 0$, which contradicts the condition (R) with u and $\nabla\Phi(u)$ replaced by $u(t_0)$ and $V(u(t_0))$, respectively. (Indeed, for example, if $R_1 < \delta\|w(t_0)\|^\lambda$, $f'(t_0) = -2\langle V(u(t_0)), v(t_0) - \lambda\delta^2 w(t_0)/\|w(t_0)\|^{2-2\lambda}\rangle$.)

In the case where $R_1 = \delta\|w(t_0)\|^\lambda$, use the fact that there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ satisfying $t_k < t_0$ and $R_1 \leq \delta\|w(t_k)\|^\lambda$ for all k or $R_1 \geq \delta\|w(t_k)\|^\lambda$ for all k. By using this sequence we are led to the same contradiction. □

REMARK 5. Let γ be a positive smooth functional on $\tilde{E}$ and consider a differential equation

$$\frac{du(t)}{dt} = -\gamma(u(t))V(u(t))\,, \quad u(0) = u_0 \quad (u_0 \in \tilde{E}) \tag{4}$$

instead of

$$\frac{du(t)}{dt} = -V(u(t)), \quad u(0) = u_0 \quad (u_0 \in \tilde{E}).$$

Then by an essentially same argument, we can see that the conclusions of Propositions 1, 4 will still hold if the pseudo-gradient flow is replaced by the solution to (4).

For later use, we note the following.

COROLLARY 6 (Homotopy to a constant map in the negative region). *Suppose that* Φ *satisfies the condition* (R) *in Proposition 4 and let* U *be as in* (3)*. In addition, assume that the following conditions hold:*

(a) Φ *is bounded on every bounded sets.*

(b) *For every* $\varepsilon, M > 0$ *with* $\varepsilon < M < \infty$,

$$\inf\{\|\nabla\Phi(u)\| \mid u \in \Phi^{-1}([-M, -\varepsilon])\} > 0.$$

(c) *Under the notation that* Q_∞ *is the orthogonal projection onto* W_∞,

$$\lim_{R\to\infty} \inf\{\Phi(u) \mid u \in U,\ \|Q_\infty u\| \geq R\} = -\infty.$$

Then for each $r > 0$, *every continuous map* $\varphi\colon S^n \to U \cap \{u \in \tilde{E} \mid \Phi(u) < 0,\ \|u\| < r\}$ *with* $n < \dim W_\infty - 1$ (S^n : n*-dimensional usual sphere*) *is homotopic to a constant map in* $U \cap \{u \in E \mid \Phi(u) < 0\}$.

PROOF. Let φ be as in the statement of the corollary. Then there exists a constant $\varepsilon > 0$ satisfying $\Phi \circ \varphi \leq -\varepsilon$. On the other hand, from the assumption (c), there exists a constant $R_2 > 0$ such that $u \in U$, $\|Q_\infty u\| \geq R_2$ imply $\Phi(u) < 0$. By the definition of U, we can see that $u \in U$ and $\|Q_\infty u\| \leq R_2$ imply the boundedness of $\|u\|$. Hence $C_0 := \inf\{\Phi(u) \mid u \in U \text{ with } \|Q_\infty u\| \leq R_2\} \in \mathbf{R}$ is well-defined by the assumption (a) in the Corollary. By Proposition 4, there exists a pseudo-gradient vector field V for Φ, for which the maximal solution $\sigma(u, t)$ of the differential equation

$$\frac{d\sigma(u,t)}{dt} = -\frac{V(\sigma(u,t))}{\|V(\sigma(u,t))\|}, \quad \sigma(u,0) = u$$

starting from any point u of $\varphi(S^n)$ remains in U as long as it exists, and the value of Φ on this flow is strictly less than $\Phi(\varphi(u)) \leq -\varepsilon$ for $t \geq 0$. Moreover, by the assumption (a) and (b), $\sigma(u, t)$ is defined for every $u \in \varphi(S^n)$ and $t \geq 0$ with $\inf_{t\geq 0} \Phi(\sigma(u, t)) = -\infty$. Therefore, by rescaling the time variable t, we may assume that $\sigma : \varphi(S^n) \times [0, 1] \to U \cap \{u \in E \mid \Phi(u) < 0\}$ is continuous with $\sigma(u, 0) = u$ and $C_0 - 1 < \Phi(\sigma(u, 1)) < C_0$ ($\forall u \in \varphi(S^n)$). Hence $\|Q_\infty\sigma(u, 1)\| > R_2$ holds for all $u \in \varphi(S^n)$. By virtue of the assumption (c), we may also assume that the image $\sigma(\varphi(S^n) \times [0, 1])$ is contained in a ball $\{u \in E \mid \|u\| < C\}$ for some constant C.

Next we set for $t \in [1, 2]$ and $u \in \varphi(S^n)$,

$$\sigma(u, t) := (2 - t)\sigma(u, 1) + (t - 1)Q_\infty\sigma(u, 1).$$

Thus we have defined σ on $\varphi(S^n) \times [0,2]$ and it is easy to see that σ is continuous there. Moreover, since $\|Q_\infty \sigma(u,t)\| = \|Q_\infty \sigma(u,1)\| > R_2$ for $t \in [1,2]$, $\Phi(\sigma(u,t)) < 0$ for $(u,t) \in \varphi(S^n) \times [1,2]$, and $\sigma(u,2) \in W_\infty$ for $u \in \varphi(S^n)$.

Next, we define $\sigma(u,t) := \{(t-2)R_2/\|\sigma(u,2)\| + (3-t)\}\sigma(u,2)$ for $(u,t) \in \varphi(S^n) \times [2,3]$. Then σ is continuous on $\varphi(S^n) \times [0,3]$, $\sigma(u,t) \in W_\infty$, $\Phi(\sigma(u,t)) < 0$ for $(u,t) \in \varphi(S^n) \times [2,3]$, and $\|\sigma(u,3)\| = R_2$.

Suppose now that $\dim W_\infty < \infty$. Then $\sigma(\cdot,3) \circ \varphi$ is a continuous map from S^n to a sphere of radius R_2 in the finite dimensional Euclidean space W_∞. Since we have assumed that $n < \dim W_\infty - 1$, this map is homotopic in this sphere to a constant map. (In the terminology of algebraic topology, this is expressed as the triviality of the homotopy group $\pi_n(S^k)$ $(n < k)$. See e.g., [6, Cor. 4.9].) Denoting this homotopy by $H(x,t)$ for $(x,t) \in S^n \times [0,1]$ with $H(x,0) = \sigma(\varphi(x),3)$ $(x \in S^n)$ and $H(x,1)$ a constant point in that sphere. Collecting the parts we have made so far, we set

$$\tau(x,t) := \begin{cases} \sigma(\varphi(x), 4t)\,, & (0 \le t \le 3/4)\,, \\ H(x, 4t-3)\,, & (3/4 \le t \le 1)\,. \end{cases}$$

Then τ gives the desired homotopy.

If $\dim W_\infty = \infty$, we use the compactness of $\sigma(\varphi(S^n),3)$ to obtain a finite dimensional subspace W of W_∞ with $\dim W - 1 > n$ for which the orthogonal projection P_W onto W makes $\sup_{w \in \sigma(\varphi(S^n),3)} \|P_W w - w\|$ sufficiently small so that the segment joining $\sigma(u,3)$ to $P_W \sigma(u,3)$ is contained in $U \cap \{u \in E \mid \|Q_\infty u\| > R_2\}$. Therefore we can construct a homotopy in $U \cap \{u \in E \mid \Phi(u) < 0\}$ from $\sigma(\cdot,3)$ to a continuous map with values in a finite dimensional subspace of W_∞. Thus we are led to the case already discussed. □

2.2. Local linking and the existence of a critical point. First let us recall the definition of local linking.

DEFINITION 7. Let Φ be a C^1 functional on E. Then we define as follows.

(i) If there exist an orthogonal decomposition $E = V_0 \oplus W_0$ and an $r > 0$ satisfying the following condition, Φ is said to have a local linking at 0 with respect to (V_0, W_0):

$$\begin{cases} \Phi(u) \ge 0 & (\forall u \in B_r V_0)\,, \\ \Phi(u) \le 0 & (\forall u \in B_r W_0)\,, \end{cases} \tag{5}$$

where, $B_r V_0 := \{u \in V_0 : \|u\| \le r\}$, $B_r W_0 := \{u \in W_0 : \|u\| \le r\}$.

(ii) Φ is said to have a strong local linking at 0 w.r.t. (V_0, W_0) if there exists an $r > 0$ satisfying (5), and the following properties hold for some $\varepsilon > 0$:

$$\begin{cases} \Phi(u) \ge \varepsilon & \text{on} \quad \partial B_r V_0\,, \\ \Phi(u) \le -\varepsilon & \text{on} \quad \partial B_r W_0\,. \end{cases} \tag{6}$$

Needless to say, 0 is a critical point of Φ if it has a local linking at 0.

The notion generalizing the so-called Palais–Smale condition has been introduced to obtain a useful conclusion from the local linking property.

DEFINITION 8. Suppose that a sequence $\{E_n\}_n$ of finite dimensional subspaces of E satisfies

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset E\,, \quad E = \overline{\bigcup_{n=1}^{\infty} E_n}\,, \tag{7}$$

and let P_n denote the orthogonal projection from E onto E_n. Then,

(i) a sequence $\{u_j\}_j$ is called a $(PS)^*_c$ sequence (w.r.t. Φ and $\{E_n\}_n$) provided $u_j \in E_{n_j}$, $n_j \to \infty$, $\Phi(u_j) \to c$ and $P_{n_j}\big(\nabla\Phi(u_j)\big) \to 0$ (as $j \to \infty$);

(ii) Φ is said to satisfy the $(WPS)^*_c$ condition if every $(PS)^*_c$ sequence has a subsequence weakly convergent to a critical point u of Φ with $\Phi(u) = c$.

The following easy paraphrase is useful in proving the existence of a non-trivial critical point.

LEMMA 9. *Let $\{E_n\}_n$ be as in Definition 8 and let Φ satisfy the $(WPS)^*_c$ condition for every $c \in \mathbf{R}$. Moreover, suppose that 0 is the only critical value of Φ. Then for any $\varepsilon > 0$, $M > 0$ and $n_0 \in \mathbf{N}$, there exist an $n_1 > n_0$ and $b > 0$ such that*

$$\|\nabla\Phi_n(u)\| \geq b\,, \quad \forall u \in \Phi_n^{-1}([-M, -\varepsilon]) \cup \Phi_n^{-1}([\varepsilon, M]) \tag{8}$$

holds for every $n \geq n_1$, where $\Phi_n := \Phi|_{E_n}$. (Note that $\nabla\Phi_n(u) = P_n(\nabla\Phi(u))$ for $u \in E_n$.)

Now let us collect the conditions relevant to our abstract theory about the critical points.

(Φ1) With respect to a sequence $\{E_n\}_n$ of finite dimensional subspaces satisfying (7), Φ satisfies $(WPS)^*_c$ condition for every $c \in \mathbf{R}$.

(Φ2) Φ is bounded on every bounded set.

(Φ3) There exists an orthogonal decomposition $E = V_0 \oplus W_0$ that satisfies one of the following conditions:

  (i) Φ has a strong local linking at 0 w.r.t. (V_0, W_0).

  (ii) Φ has a local linking at 0 w.r.t. (V_0, W_0), and for some $r > 0$ with the property (5), every $(PS)^*_0$ sequence in $B_{2r}E$ has a strongly convergent subsequence.

($\widetilde{\Phi}$4) There exists an orthogonal decomposition $E = V_\infty \oplus W_\infty$ that satisfies the following (i)~(iii) for some $\lambda \geq 0$, $\delta > 0$, $R_1 > 0$:

  (i) $\left\langle \nabla\Phi(u)\,,\; v_\infty - \lambda\delta^2 \dfrac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0$, (if $\|v_\infty\| = \delta\|w_\infty\|^\lambda\,,\; \|v_\infty\| \geq R_1$),

  (ii) $\langle \nabla\Phi(u)\,,\; v_\infty \rangle > 0$, (if $\|v_\infty\| \geq \delta\|w_\infty\|^\lambda\,,\; \|v_\infty\| \geq R_1$),

  (iii) for every $c < 0$ there exists an $R > 0$ such that $\Phi(u) < c$ provided $\|v_\infty\| \leq \delta\|w_\infty\|^\lambda$ and $\|w_\infty\| \geq R$, where

$$u = w_\infty + v_\infty \quad (w_\infty \in W_\infty\,,\; v_\infty \in V_\infty)\,.$$

REMARK 10. The conditions $(\Phi 1)$ to $(\Phi 3)$ are the same as those in [2], while $(\widetilde{\Phi 4})$ is a generalization of $(\Phi 4)$ in [2]. (i) and (ii) in $(\widetilde{\Phi 4})$ is the same as in condition (R) appearing in Proposition 4.

When we assume the conditions $(\Phi 1)$ and $(\Phi 3)$, we adopt the following notations:

$$
\begin{aligned}
&\Phi_n := \Phi|_{E_n}\,, \\
&\Phi_n^c := \{u \in E_n \,:\, \Phi(u) \le c\}\,, \qquad && (\Phi_n)_c := \{u \in E_n \,:\, \Phi(u) \ge c\}\,, \\
&E_n^1 := E_n \cap V_0\,, && E_n^2 := E_n \cap W_0\,, \\
&B_n^j := B_r E \cap E_n^j \quad (j = 1, 2)\,, && S_n^j := \partial B_n^j \quad (j = 1, 2)\,.
\end{aligned}
$$

We say that the sequence $\{E_n\}_n$ in $(\Phi 1)$ is *compatible* with the orthogonal decomposition $V_0 \oplus W_0$ [*resp.* $V_\infty \oplus W_\infty$] $(\Phi 1)$ if

$$E_n = (E_n \cap V_0) \oplus (E_n \cap W_0) \quad [\mathit{resp.}\ E_n = (E_n \cap V_\infty) \oplus (E_n \cap W_\infty)] \tag{9}$$

holds for every n.

The following lemma finishes the preparation for our main result in this section.

LEMMA 11. *If Φ satisfies $(\widetilde{\Phi 4})$ with $\{E_n\}_n$ being compatible w.r.t. $(V_\infty \oplus W_\infty)$, then $\Phi|_{E_n}$ satisfies $(\widetilde{\Phi 4})$ with $(V_\infty \cap E_n, W_\infty \cap E_n)$ instead of $(V_\infty \oplus W_\infty)$ for every $n \in \mathbf{N}$.*

PROOF. We consider only the condition (i) of $(\widetilde{\Phi 4})$, because the condition (ii) can be proved similarly. We set $f(u) := \|v_\infty\|^2 - \delta^2 \|w_\infty\|^{2\lambda}$ where $u = v_\infty + w_\infty$, $v_\infty \in V_\infty$, $w_\infty \in W_\infty$. We can show that $\nabla f(u) \in E_n$ for every $n \in \mathbf{N}$ if $u \in E_n$. Indeed, $\langle \nabla f(u), z\rangle = 2\langle v_\infty^n, \tilde{v}_\infty\rangle - 2\lambda\delta^2 \|w_\infty^n\|^{2\lambda-2}\langle w_\infty^n, \tilde{w}_\infty\rangle$ for every $z \in E_n^\perp$ where $u = v_\infty^n + w_\infty^n \in E_n$ and $z = \tilde{v}_\infty + \tilde{w}_\infty$. Since $\{E_n\}_n$ in $(\widetilde{\Phi 4})$ is compatible w.r.t. $(V_\infty \oplus W_\infty)$, $P_n P_\infty = P_\infty P_n$ where $P_n \colon E \to E_n$ and $P_\infty \colon E \to V_\infty$ are orthogonal projections. Hence we have $\langle v_\infty^n, \tilde{v}_\infty\rangle = \langle P_\infty P_n u, P_\infty(1 - P_n)z\rangle = \langle P_n P_\infty u, (1 - P_n)P_\infty z\rangle = \langle (1 - P_n)P_n P_\infty u, P_\infty z\rangle = 0$ and $\langle w_\infty^n, \tilde{w}_\infty\rangle = 0$ similarly. Therefore $\nabla f(u) \in E_n$ for every $n \in \mathbf{N}$ if $u \in E_n$.

On the other hand since $\langle \nabla\Phi(u), \nabla f(u)\rangle > 0$ if $f(u) = 0$ and $\|P_\infty u\| \ge R_1$ by the condition (i), we obtain $\langle P_n \nabla\Phi(u), \nabla f(u)\rangle = \langle \nabla\Phi(u), P_n \nabla f(u)\rangle = \langle \nabla\Phi(u), \nabla f(u)\rangle > 0$ if $u \in E_n$, $f(u) = 0$ and $\|P_\infty u\| \ge R_1$. □

Now we can state and prove our main abstract result. The line of arguments is similar to that for Theorem 2.1 of [2].

THEOREM 12. *Let Φ be a C^1 class functional on a Hilbert space E and let the conditions $(\Phi 1)$ to $(\widetilde{\Phi 4})$ be satisfied with $\{E_n\}_n$ in $(\Phi 1)$ compatible with the decomposition $V_0 \oplus W_0$ in $(\Phi 3)$ and $V_\infty \oplus W_\infty$ in $(\widetilde{\Phi 4})$ (cf. (9)). Moreover, suppose that*

$$\limsup_{n\to\infty} [\dim E_n \cap W_\infty - \dim E_n \cap W_0] > 0 \tag{10}$$

holds, and the set U defined by (3) contains $B_{2r}E$, where r is the number appearing in $(\Phi 3)$. Then Φ has at least one non-zero critical point.

PROOF. We prove this theorem by contradiction. So suppose that there exist no critical points other than the origin. First we consider the easier case where (i) of ($\Phi 3$) holds. Then there exist an $r > 0$ and an $\varepsilon > 0$ satisfying (6). Suppose $\dim W_0 > 0$. Then $\dim E_n \cap W_0 > 0$ holds for large n because of the compatibility of $\{E_n\}_n$ with the orthogonal decomposition $V_0 \oplus W_0$. By the assumption (10), there exists an increasing sequence $\{n_j\}_j$ of natural numbers satisfying $\dim E_{n_j} \cap W_\infty - \dim E_{n_j} \cap W_0 > 0$. We may also assume that $\dim E_{n_j} \cap W_0 > 0$. Then, by virtue of Lemma 9 and Lemma 11, we may apply Corollary 6 with E replaced by $E_{n_j} = (E_{n_j} \cap V_\infty) \oplus (E_{n_j} \cap W_\infty)$ and Φ with Φ_{n_j} for sufficiently large j. Therefore, by considering a homeomorphism φ from a standard sphere to $S^2_{n_j}$, we can obtain a homotopy $\tau_j \colon S^2_{n_j} \times [0, 1] \to E_{n_j}$ from the identity map $S^2_{n_j} \to E_{n_j}$ to a constant map. Note that because of our assumption ($\widetilde{\Phi 4}$) and the proof of Corollary 6, we may suppose that there exists a constant C independent of j such that the image of each τ_j is contained in the ball $\{u \in E \mid \|u\| \le C\}$ and hence $M := \sup\{\Phi(u) \mid \|u\| \le C\}$ is well defined. Indeed, first choose an $R_2 > 0$ such that $u \in U$ with $\|Q_\infty u\| \ge R_2$ satisfies $\Phi(u) < 0$, and then set $C_0 := \inf\{\Phi(u) \mid u \in U,\ \|Q_\infty u\| \le R_2\}$ (the infinum exists by ($\Phi 2$)). Finally, by (iii) of ($\widetilde{\Phi 4}$), there exists a constant $C_1 > 0$ such that $u \in U$ and $\Phi(u) > C_0 - 1$ imply $\|Q_\infty u\| \le C_1$, hence $\|u\| \le C_1 + \delta C_1^\lambda$.

Now, since $\partial(B^2_{n_j} \times [0, 1]) = (B^2_{n_j} \times \{0\}) \cup (S^2_{n_j} \times (0, 1)) \cup (B^2_{n_j} \times \{1\})$, the map τ_j determines a continuous map $\gamma_j \colon \partial(B^2_{n_j} \times [0, 1]) \to E_{n_j}$ by the following definition, where a_j denotes the constant value $\tau_j(u, 1)$ ($u \in S^2_{n_j}$):

$$\gamma_j(u,t) := \begin{cases} u & (u \in B^2_{n_j},\ t = 0)\,, \\ \tau_j(u,t) & (u \in S^2_{n_j},\ t \in (0,1))\,, \\ a_j & (u \in B^2_{n_j},\ t = 1)\,. \end{cases}$$

Set $\Gamma_j := \{\rho \mid \rho \in C(B^2_{n_j} \times [0, 1], E_{n_j}),\ \rho|_{\partial(B^2_{n_j} \times [0,1])} = \gamma_j\}$. Note that by the well known Dugundij extension theorem, there exists a $\rho \in \Gamma_j$ with values in the ball $\{u \in E_{n_j} \mid \|u\| \le C\}$. Therefore,

$$c := \inf_{\rho \in \Gamma_j} \sup\{\Phi(u) \mid u \in \rho(B^2_{n_j} \times [0, 1])\} \le M$$

holds by the definition of M. By a standard argument (*cf.* [2, Lemma 3.2]), it can be proved that $\rho(B^2_{n_j} \times [0, 1]) \cap S^1_{n_j} \ne \emptyset$ for any $\rho \in \Gamma_j$. (Note that $\dim E_{n_j} \cap W_0 < \dim E_{n_j} \cap W_\infty \le \dim E_{n_j}$ implies $\dim E_{n_j} \cap V_0 > 0$.) Hence $c \ge \varepsilon$. On the other hand, $c_0 := \sup\{\Phi(u) \mid u \in \gamma_j(\partial(B^2_{n_j} \times [0, 1]))\} \le 0$ holds by construction. So, by Ekeland's mini-max theorem ([12, Theorem 4.3]), there exists a point $u_j \in E_{n_j}$ such that $\varepsilon \le \Phi_{n_j}(u_j) < M + 1$ and $\|\nabla \Phi_{n_j}(u_j)\| < 1/j$. However, this contradicts Lemma 9.

In the case where $W_0 = \{0\}$, then $\Phi(0) = 0$ and $\Phi(u) \geq \varepsilon$ for u with $\|u\| = r$. Therefore, together with (iii) of $(\widetilde{\Phi}4)$, applying Ekeland's mini-max theorem ([12, Theorem 4.3]) (in the situation of mountain pass Theorem), we obtain a point $u_j \in E_{n_j}$ such that $\varepsilon \leq \Phi_{n_j}(u_j)$, $\sup_j \Phi_{n_j}(u_j) < \infty$ and $\|\nabla \Phi_{n_j}(u_j)\| < 1/j$. Hence the same contradiction occurs.

The case where (ii) of assumption $(\Phi 3)$ holds could be treated similarly as in the proof of Theorem 1 in [10] (*cf.* also [2, Lemma 3.2]) by using a deformation. In more detail, due to $(\Phi 1)$, $(\Phi 2)$ and the assumption that 0 is the only critical point, there exists a continuous deformation $\eta_j : E^1_{n_j} \times [0, 1] \to E_{n_j}$ for large j, possessing the following properties where $\varepsilon > 0$ is independent of j (*cf.* Lemma 9, [10, Lemma 2], [2, Lemma 3.1]):

— $\|\eta_j(u, t) - u\| < r/2$ for all $(u, t) \in B_r E^2_{n_j} \times [0, 1]$;

— For some constant $\varepsilon > 0$, $\Phi(\eta_j(u, 1)) > \varepsilon$ for all $u \in B_r E^2_{n_j} \setminus B_{r/3} E$.

Then, instead of $S^1_{n_j}$ in the proof above, we obtain $\rho(B^2_{n_j} \times [0, 1]) \cap \eta_j(S^1_{n_j}, 1) \neq \emptyset$ for any $\rho \in \Gamma_j$, and the proof similarly goes on. □

2.3. Almost quadratic, hyperbolic functional. In this subsection, we prepare an abstract framework suitable for the treatment of wave equations. The following conditions stating that Φ is "nearly hyperbolic" are of our primary concern:

$(\Phi 5)$ There exist a self-adjoint bounded linear operator $L : E \to E$ and a C^1 functional Φ_1 on E:

- (a) $\Phi(u) := \frac{1}{2}\langle Lu, u\rangle + \Phi_1(u)$,
- (b) $\|\nabla \Phi_1(u)\| = o(\|u\|)$ $\quad (as\ \|u\| \to \infty)$, and $\|\nabla \Phi_1(u)\|$ is bounded on every bounded set.
- (c) $0 \notin \sigma(L)$ or $0 \in \sigma(L)$ is an isolated point of the spectrum.

$(\Phi 6)$ In addition to $(\Phi 5)$, $\nabla \Phi_1$ is bounded.

The following two lemmas show that the conditions $(\Phi 5)$ and $(\Phi 6)$ are nearly sufficient for $(\widetilde{\Phi}4)$.

LEMMA 13. *Suppose that Φ satisfies $(\Phi 5)$ with $0 \notin \sigma(L)$ and $\sigma(L) \cap \mathbf{R}^{\pm} \neq \emptyset$. Then Φ satisfies $(\widetilde{\Phi}4)$ with $\lambda = 1$, $V_\infty := X^+$, and $W_\infty := X^-$, where $X^{\pm}$ is the spectral subspace w.r.t. L and $\sigma(L) \cap \mathbf{R}^{\pm}$. Moreover, $-\Phi$ satisfies $(\widetilde{\Phi}4)$ with $\lambda = 1$, $V_\infty := X^-$ and $W_\infty := X^+$.*

PROOF. In correspondence with the decomposition $E = X^+ \oplus X^-$, we write $u = u^+ + u^-$ for $u \in E$. Since $0 \notin \sigma(L)$, there exists $\gamma > 0$ such that $\pm\langle Lu^{\pm}, u^{\pm}\rangle \geq \gamma \|u^{\pm}\|^2$ for every $u^{\pm} \in X^{\pm}$. Let us fix a $\delta > 0$ with $0 < \delta^2 < \frac{\gamma}{\|L\|} \leq 1$, and let $\lambda = 1$. Then by (b) of $(\Phi 5)$, for every ε_1 with $0 < \varepsilon_1 < \frac{\delta\gamma}{1+\delta}$ there exists an $R_1 > 0$ such that $\|\nabla \Phi_1(u)\| < \varepsilon_1 \|u\|$ for every $u \in E$ with $\|u\| \geq R_1$. Therefore, if u satisfies $\|u^+\| = \delta\|u^-\|$ and $\|u^+\| \geq R_1$, it holds that

$$\langle \nabla \Phi(u), u^+ - \delta^2 u^-\rangle \geq \langle Lu^+, u^+\rangle - \delta^2\langle Lu^-, u^-\rangle + \langle \nabla \Phi_1(u), u^+ - \delta^2 u^-\rangle$$

$$\begin{aligned}&\geq \gamma\|u^+\|^2 + \gamma\delta^2\|u^-\|^2 - \varepsilon_1\|u\|(\|u^+\| + \delta^2\|u^-\|)\\ &\geq 2\gamma\|u^+\|^2 - \varepsilon_1(1+\delta)\left(1+\frac{1}{\delta}\right)\|u^+\|^2\\ &\geq 2\|u^+\|^2\left(\gamma - \varepsilon_1\left(1+\frac{1}{\delta}\right)\right).\end{aligned}$$

Similarly for u with $\|u^+\| \geq \delta\|u^-\|$ and $\|u^+\| \geq R_1$,

$$\begin{aligned}\langle\nabla\Phi(u), u^+\rangle &= \langle Lu^+, u^+\rangle + \langle\nabla\Phi_1(u), u^+\rangle \geq \gamma\|u^+\|^2 - \varepsilon_1\|u\|\|u^+\|\\ &\geq \gamma\|u^+\|^2 - \varepsilon_1\left(1+\frac{1}{\delta}\right)\|u^+\|^2 \geq \|u^+\|^2\left(\gamma - \varepsilon_1\left(1+\frac{1}{\delta}\right)\right).\end{aligned}$$

Hence

$$\langle\nabla\Phi(u), u^+ - \delta^2 u^-\rangle > 0 \quad \text{if } \|u^+\| = \delta\|u^-\|, \quad \|u^+\| \geq R_1;$$
$$\langle\nabla\Phi(u), u^+\rangle > 0 \quad \text{if } \|u^+\| \geq \delta\|u^-\|, \quad \|u^+\| \geq R_1,$$

and hence the condition (i) and (ii) of $(\widetilde{\Phi}4)$ hold.

Next for every ε_2 satisfying $0 < \varepsilon_2 < \frac{\gamma-\delta^2\|L\|}{1+\delta^2}$ there exists a $C_2 > 0$ such that

$$\Phi_1(u) = \int_0^1 \langle\nabla\Phi_1(su), u\rangle ds \leq C_2 + \frac{\varepsilon_2}{2}\|u\|^2$$

because of (b) of $(\Phi 5)$ and $\Phi_1(0) = 0$. Thus if $\|u^+\| \leq \delta\|u^-\|$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\langle Lu, u\rangle + \Phi_1(u)\\ &\leq \frac{1}{2}\|L\|\|u^+\|^2 - \frac{\gamma}{2}\|u^-\|^2 + \frac{\varepsilon_2}{2}\|u\|^2 + C_2\\ &\leq \frac{1}{2}\|L\|\delta^2\|u^-\|^2 - \frac{\gamma}{2}\|u^-\|^2 + \frac{\varepsilon_2}{2}(1+\delta^2)\|u^-\|^2 + C_2\\ &\leq \frac{1}{2}\|u^-\|^2\{\|L\|\delta^2 - \gamma + \varepsilon_2(1+\delta^2)\} + C_2.\end{aligned}$$

Hence for every $c < 0$ there exists an $R > 0$ such that $\Phi(u) < c$ provided $\|u^+\| \leq \delta\|u^-\|$, $\|u^-\| \geq R$. □

The following condition deals with a special case treated in [2].

LEMMA 14 ([2, Theorem 2.4]). *Suppose that Φ satisfies $(\Phi 6)$, $0 \in \sigma(L)$ and*

$$\Phi_1(u) \to -\infty \quad (u \in \ker L,\ \|u\| \to \infty).$$

Then, Φ satisfies $(\widetilde{\Phi}4)$ with $\lambda = 0$, $V_\infty = X^+$, and $W_\infty = X^- \oplus X^0$, where $X^\pm$ is the spectral subspace w.r.t. L and $\sigma(L) \cap \mathbf{R}^\pm$, $X^0 := \ker L$.

Now we look into the structure of Φ more closely. Specifically, we consider the following situations:

(Φ7) The Hilbert space E is densely and continuously embedded in another Hilbert space H and there exists an orthogonal decomposition $E = E^+ \oplus E^0 \oplus E^-$ so that the following conditions hold:

(a) The embeddings of $E^\pm$ into H are compact, and E^0 is a closed subspace of H.

(b) By using the orthogonal projections $P^\pm$ from E onto $E^\pm$, Φ can be written as

$$\Phi(u) = \frac{1}{2}(\|P^+u\|^2 - \|P^-u\|^2) - \Psi(u), \quad (u \in E)$$

where Ψ is a convex C^1 functional on the Hilbert space H, and $\nabla_H \Psi$ is bounded on every bounded subset of H ($\nabla_H \Psi$ denotes the gradient of Ψ on H).

(c) There exists a sequence $\{E_n\}_n$ of finite dimensional subspaces of E satisfying (7) and $P^\pm P_n = P_n P^\pm$ ($\forall n$), where P_n is the orthogonal projection onto E_n.

(Φ8) In addition to (Φ7), for a self-adjoint bounded linear operator S_∞ on H, Ψ is written as

$$\Psi(u) := \frac{1}{2}\langle S_\infty u, u\rangle + \Psi_1(u), \quad (u \in H), \tag{11}$$

where Ψ_1 is a C^1 functional on H satisfying

$$\nabla_H \Psi_1(u) = o(\|u\|_H) \quad (\text{as } \|u\|_H \to \infty). \tag{12}$$

Moreover, the bounded self-adjoint operator L on E defined by $L := P^+ - P^- - S$ satisfies the following conditions (a) and (b), where $S\colon E \to E$ is the self-adjoint operator on E determined by

$$\langle Su, v\rangle_E := \langle S_\infty u, v\rangle_H \quad (u, v \in E).$$

($\langle\cdot,\cdot\rangle_E$, $\langle\cdot,\cdot\rangle_H$ denote the inner product in E and H, respectively.)

(a) $L(E_n) \subset E_n$ ($n \in \mathbf{N}$) holds for the sequence $\{E_n\}_n$ in (Φ7),

(b) $0 \notin \sigma(L)$ or $0 \in \sigma(L)$ is an isolated point of the spectrum,

(Φ8$'$) In addition to (Φ8), $\nabla_H \Psi_1$ is bounded on H and

$$|\Psi_1(u)| \to \infty \quad (u \in \ker L, \|u\|_E \to \infty).$$

Let us recall the following facts concerning the $(WPS)^*$ condition.

LEMMA 15 ([2, Proposition 2.5]). *Let Φ satisfy (Φ7), and suppose that $\{u_j\}_{j\in\mathbf{N}}$ is a bounded $(PS)^*_c$ sequence for Φ. Then, there exist a critical point u of Φ with $\Phi(u) = c$ and a subsequence $\{u_{j_k}\}_{k\in\mathbf{N}}$ of $\{u_j\}_{j\in\mathbf{N}}$ such that*

$$u^\pm_{j_k} \to u^\pm \quad (k \to \infty) \quad \text{in } E,$$

$$u^0_{j_k} \rightharpoonup u^0 \quad (k \to \infty) \quad \text{in} \ \ E\,,$$

where $u = u^+ + u^0 + u^-$ $\quad (u^\pm \in E^\pm, u^0 \in E^0)$, $u_{j_k} = u^+_{j_k} + u^0_{j_k} + u^-_{j_k} \quad (u^\pm_{j_k} \in E^\pm, u^0_{j_k} \in E^0)$

LEMMA 16 ([2, Proposition 2.6]). *Let Φ satisfy $(\Phi 8)$, and if $0 \in \sigma(L)$ assume further that Φ satisfies the additional condition $(\Phi 8')$. Then Φ fulfills the $(WPS)^*_c$ condition for every $c \in \mathbf{R}$.*

Let us also recall the following result concerning the verification of (ii) in $(\Phi 3)$.

LEMMA 17 ([2, Remark 3.3]). *Suppose that Φ satisfies $(\Phi 7)$ and there exist no critical points in $B_{2r}E$ other than $0 \in E$. In addition, let*

$$\Psi(v) \geq a\|v\|^\alpha_H \quad (\forall v \in H)$$

hold for some constants $a > 0, \alpha > 0$. Then every $(PS)^*_0$ sequence in $B_{2r}E$ has a strongly convergent subsequence.

The following proposition deals with a sufficient condition for $(WPS)^*$ condition, which is not covered by Lemma 16.

PROPOSITION 18. *Suppose that Φ satisfies $(\Phi 8)$ and there exist some constants α, β satisfying $0 < \alpha \leq \beta < 1$ and $\beta - \alpha/2 < 1/2$, for which the following conditions* (a) *and* (b) *hold.*

(a) *For some constant $c_0 > 0$, $\|\nabla_E \Psi_1(u)\| \leq c_0(\|u\|^\beta + 1) \quad (\forall u \in E)$. (Here we consider $\nabla\Psi_1(u) \in H^* \subset E^*$.)*

(b) *For some constants c_1 and d, one of the following conditions hold where $X^0 :=$* ker *L:*
   (i) $\forall u^0 \in X^0 \quad \Psi_1(u^0) \geq c_1\|u^0\|^{\alpha+1} - d\|u^0\|$,
   (ii) $\forall u^0 \in X^0 \quad \Psi_1(u^0) \leq -c_1\|u^0\|^{\alpha+1} + d\|u^0\|$.

Then Φ satisfies the $(WPS)^*_c$ condition for every $c \in \mathbf{R}$ (w.r.t. $\{E_n\}_n$ in $(\Phi 8)$).

PROOF. We first treat the case where (i) of the condition (b) holds. By Lemma 15, it remains to show that any $(PS)^*_c$ sequence is bounded. We assume that $\{u_j\}_j$ is a $(PS)^*_c$ sequence. According to the spectral decomposition of $\sigma(L)$ we have $E = X^+ \oplus X^0 \oplus X^-$ where $X^\pm$ and X^0 correspond to $\sigma(L) \cap \mathbf{R}^\pm$ and ker L, respectively. We write $u = u^+ + u^0 + u^-$ for $u \in E$ corresponding to this decomposition. By the condition (b) in $(\Phi 8)$, there exists $\gamma > 0$ such that $\pm\langle Lu^\pm, u^\pm\rangle_E \geq \gamma\|u^\pm\|^2_E$ for $\forall u^\pm \in X^\pm$.

If j is sufficiently large, the first inequality of the following formula holds, and the assumption (a) yields

$$\begin{aligned}\|u^+_j\|_E &\geq \langle u^+_j, P_{n_j}\nabla\Phi(u_j)\rangle_E = \langle u^+_j, Lu^+_j\rangle_E - \langle u^+_j, \nabla\Psi_1(u_j)\rangle_E \\ &\geq \gamma\|u^+_j\|^2_E - \|u^+_j\|_E\|\nabla\Psi_1(u_j)\|_E \geq \gamma\|u^+_j\|^2_E - c_0\|u^+_j\|_E(\|u_j\|^\beta_E + 1)\,.\end{aligned}$$

Hence $c_0 + 1 \geq \gamma \|u_j^+\|_E - c_0\|u_j\|_E^\beta$ for j large enough. Similarly $c_0 + 1 \geq \gamma\|u_j^-\|_E - c_0\|u_j\|_E^\beta$ for j large enough. Therefore $2(c_0 + 1) \geq \gamma(\|u_j^+\|_E + \|u_j^-\|_E) - 2\|u_j\|_E^\beta$ for j large enough. Let $y_j := \|u_j^+\|_E + \|u_j^-\|_E$, if $\{y_j\}$ is not bounded, then we may assume, going if necessary to a subsequence, $y_j \to \infty$ (as $j \to \infty$). Since $2(c_0+1) \geq \gamma\, y_j - 2c_0\|u_j\|_E^\beta$ for j large enough, there exists a $C > 0$ such that $C\|u_j^0\|_E^\beta \geq y_j$ for j large enough. Hence by using the condition (a) and (i) of (b), we obtain for sufficiently large j

$$\begin{aligned}
\Phi(u_j) &= \frac{1}{2}\langle Lu_j, u_j\rangle_E - \Psi_1(u_j)\\
&= \frac{1}{2}\langle Lu_j^+, u_j^+\rangle_E + \frac{1}{2}\langle Lu_j^-, u_j^-\rangle_E - \Psi_1(u_j^0)\\
&\quad - \int_0^1 \langle \nabla\Psi_1(u_j^0 + s(u_j^+ + u_j^-)), \quad u_j^+ + u_j^-\rangle_E ds\\
&\leq \frac{1}{2}\|L\|_E\|u_j^+\|_E^2 - \frac{\gamma}{2}\|u_j^-\|_E^2 - c_1\|u_j^0\|_E^{\alpha+1} + d\|u_j^0\|_E\\
&\quad + (\|u_j^+\|_E + \|u_j^-\|_E)\int_0^1 \|\nabla\Psi_1(u_j^0 + s(u_j^+ + u_j^-))\|_E ds\,.\\
&\leq \frac{C^2}{2}\|L\|_E\|u_j^0\|_E^{2\beta} - c_1\|u_j^0\|_E^{\alpha+1} + d\|u_j^0\|_E + c_2\|u_j^0\|_E^{2\beta} + c_3\|u_j^0\|_E^{\beta(\beta+1)}
\end{aligned}$$

where $c_2 > 0$ and $c_3 > 0$ are constants independent of j. Since $\beta(\beta+1) < 2\beta < \alpha+1$, $\Phi(u_j) \to -\infty$ as $j \to \infty$. Thus we obtain the contradiction. Next if $\|u_j^0\|_E$ is not bounded, then similarly we may assume, going if necessary to a subsequence, $\|u_j^0\|_E \to \infty$ (as $j \to \infty$) and $\Phi(u_j) \leq c_4 - c_1\|u_j^0\|_E^{\alpha+1} + d\|u_j^0\|_E + c_5\|u_j^0\|_E^\beta$ for j large enough. Therefore $\Phi(u_j) \to -\infty$ as $j \to \infty$, and this is a contradiction.

Similar argument also proves the assertion of the theorem in the case (ii) in (b).

The following technical Lemma is for the verification of $(\widetilde{\Phi}4)$.

LEMMA 19. *Suppose that Φ satisfies $(\Phi 8)$ and let α, β, λ, p and q be constant numbers satisfying*

$$0 < \alpha \leq \beta < 1\,, \quad \beta < \frac{\alpha+1}{2}, \tag{13}$$

$$\max\left\{\frac{1}{2}, \beta\right\} < \lambda < \frac{\alpha+1}{2}\,, \tag{14}$$

$$\max\left\{2, \frac{1}{\beta}\right\} < p < \frac{2\lambda}{\beta}\,, \quad \frac{1}{p} + \frac{1}{q} = 1\,. \tag{15}$$

Moreover, let $X^0 := \ker L$ and let $X^\pm$ be the spectral subspace for L w.r.t. $\sigma(L) \cap \mathbf{R}^\pm$. Then if there exist positive constants C_1, C_2 and C_3 for which one of the following conditions $(b1)$ or

$(b2)$ holds, Φ or $-\Phi$ satisfies $(\widetilde{\Phi}4)$ with $\delta = 1$, λ as above, and $(W_\infty, V_\infty) = (X^0 \oplus X^-, X^+)$ or $(W_\infty, V_\infty) = (X^0 \oplus X^+, X^-)$, respectively. Moreover for every $r > 0$, the set U defined by (3) *can be made to satisfy $U \supset B_{2r}$.*

(In the sequel, $u^0 \in X^0$, $u^+ \in X^+$ and $u^- \in X^-$ denote arbitrary elements in respective subspaces.)

(b1) (i) $\Psi_1(u^0) \geq C_1\|u^0\|_E^{\alpha+1} - C_2\|u^0\|_E$.

(ii) $|\Psi_1(u^0 + u^- + u^+) - \Psi_1(u^0)|$
$\leq C_3(\|u^- + u^+\|_E^{\beta+1} + \|u^- + u^+\|_E + \|u^0\|_E^{p\beta} + \|u^- + u^+\|_E^q)$.

(iii) $|\langle \nabla_H \Psi_1(u^0 + u^- + u^+), u^+\rangle_H|$
$\leq C_3(\|u^+\|_E^{\beta+1} + \|u^+\|_E + \|u^0 + u^-\|_E^{p\beta} + \|u^+\|_E^q)$.

(iv) $|\langle \nabla_H \Psi_1(u^0 + u^- + u^+), u^0 + u^-\rangle_H| \leq C_3(\|u^+\|_E^{\beta+1} + \|u^0 + u^-\|_E^{\beta+1})$.

(b2) (i) $\Psi_1(u^0) \leq -C_1\|u^0\|_E^{\alpha+1} + C_2\|u^0\|_E$.

(ii) $|\Psi_1(u^0 + u^- + u^+) - \Psi_1(u^0)|$
$\leq C_3(\|u^- + u^+\|_E^{\beta+1} + \|u^- + u^+\|_E + \|u^0\|_E^{p\beta} + \|u^- + u^+\|_E^q)$.

(iii) $|\langle \nabla_H \Psi_1(u^0 + u^- + u^+), u^-\rangle_H|$
$\leq C_3(\|u^-\|_E^{\beta+1} + \|u^-\|_E + \|u^0 + u^+\|_E^{p\beta} + \|u^-\|_E^q)$.

(iv) $|\langle \nabla_H \Psi_1(u^0 + u^- + u^+), u^0 + u^+\rangle_H| \leq C_3(\|u^-\|_E^{\beta+1} + \|u^0 + u^+\|_E^{\beta+1})$.

PROOF. We treat only the case of (b1), since we can similarly show that $-\Phi$ satisfies $(\widetilde{\Phi}4)$ in the case of (b2). So we assume that the condition (b1) holds and shall show that Φ satisfies $(\widetilde{\Phi}4)$ with $\delta = 1$. First note that there exists $\gamma > 0$ such that $\pm\langle Lu^\pm, u^\pm\rangle_E \geq \gamma\|u^\pm\|_E^2$ for $\forall u^\pm \in X^\pm$, since $0 \in \sigma(L)$ is an isolated point of the spectrum. Utilizing these inequalities, we shall show (iii) of $(\widetilde{\Phi}4)$ for Φ. From (i) and (ii) of (b1), for u with $\|u^+\|_E \leq \|u^0 + u^-\|_E^\lambda$, we obtain

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\langle Lu, u\rangle_E - \Psi_1(u) \\
&\leq \frac{1}{2}\|L\|\|u^+\|_E^2 - \frac{\gamma}{2}\|u^-\|_E^2 - C_1\|u^0\|_E^{\alpha+1} + C_2\|u^0\|_E \\
&\quad + C_3\big(\|u^- + u^+\|_E^{\beta+1} + \|u^- + u^+\|_E + \|u^0\|_E^{p\beta} + \|u^- + u^+\|_E^q\big) \\
&\leq C_4(\|u^0\|_E^{2\lambda} + \|u^-\|_E^{2\lambda}) - \frac{\gamma}{2}\|u^-\|_E^2 - C_1\|u^0\|_E^{\alpha+1} + C_2\|u^0\|_E \\
&\quad + C_4(\|u^-\|_E^{\lambda(\beta+1)} + \|u^0\|_E^{\lambda(\beta+1)} + \|u^-\|_E + \|u^0\|_E^{p\beta} + \|u^0\|^\lambda + \|u^-\|^\lambda \\
&\quad + \|u^-\|_E^q + \|u^0\|_E^{\lambda q} + \|u^-\|_E^{\lambda q})
\end{aligned}$$

where $C_4 > 0$ is constant. Now since $\lambda(\beta + 1)$, λq and $p\beta < 2\lambda < \alpha + 1$ from (13) and (14), $\Phi(u) \to -\infty$ as $\|u^- + u^0\|_E$ go to ∞ with $\|u^+\|_E \leq \|u^0 + u^-\|_E^\lambda$. Thus we have proved that Φ satisfies (iii) of $(\widetilde{\Phi}4)$.

Next we prove that Φ satisfies (ii) of $(\widetilde{\Phi 4})$. By (iii) of (b1), we obtain for u with $\|u^0 + u^-\|_E^\lambda \le \|u^+\|_E$

$$\begin{aligned}\langle \nabla\Phi(u)\,, u^+\rangle_E &= \langle Lu^+, u^+\rangle_E - \langle \nabla_H \Psi_1(u)\,,\ u^+\rangle_H \\ &\ge \gamma\|u^+\|_E^2 - C_3(\|u^+\|_E^{\beta+1} + \|u^+\| + \|u^0 + u^-\|_E^{p\beta} + \|u^+\|^q) \\ &\ge \gamma\|u^+\|_E^2 - C_5(\|u^+\|_E^{\beta+1} + \|u^+\| + \|u^+\|_E^{\frac{p\beta}{\lambda}} + \|u^+\|_E^q)\,,\end{aligned}$$

where $C_5 > 0$ is a constant. Since $\beta + 1$, $\frac{p\beta}{\lambda}$ and $q < 2$ from (13) and (14), and so $\langle \nabla\Phi(u), u^+\rangle_E > 0$ for $\|u^+\|_E$ large enough with $\|u^0 + u^-\|_E^\lambda \le \|u^+\|_E$.

Finally we show that Φ satisfies (i) of $(\widetilde{\Phi 4})$. By (iii) and (iv) of (b1), the following inequality holds for u with $\|u^0 + u^-\|_E^\lambda = \|u^+\|_E$:

$$\begin{aligned}&\left\langle \nabla\Phi(u)\,,\ u^+ - \lambda\frac{u^0 + u^-}{\|u^0 + u^-\|_E^{2-2\lambda}}\right\rangle_E \\ &\ge \langle Lu^+, u^+\rangle_E - \langle \nabla_H\Psi_1(u), u^+\rangle_H + \frac{\lambda}{\|u^0 + u^-\|_E^{2-2\lambda}}\langle \nabla_H\Psi_1(u), u^0 + u^-\rangle_H \\ &\ge \gamma\|u^+\|_E^2 - C_3(\|u^+\|_E^{\beta+1} + \|u^+\|_E + \|u^0 + u^-\|_E^{p\beta} + \|u^+\|_E^q) \\ &- \frac{\lambda C_3}{\|u^0 + u^-\|_E^{2-2\lambda}}(\|u^+\|_E^{\beta+1} + \|u^0 + u^-\|_E^{\beta+1}) \\ &\ge \gamma\|u^0 + u^-\|_E^{2\lambda} - C_3(\|u^0 + u^-\|_E^{\lambda(\beta+1)} + \|u^0 + u^-\|_E^\lambda) \\ &- C_3(\|u^0 + u^-\|_E^{p\beta} + \|u^0 + u^-\|_E^{\lambda q}) \\ &- \lambda C_3(\|u^0 + u^-\|_E^{\lambda(\beta+1)-(2-2\lambda)} + \|u^0 + u^-\|_E^{2\lambda+\beta-1})\,.\end{aligned}$$

Since $p\beta$, $\lambda(\beta+1)$ and $\lambda q < 2\lambda$ from (13) and (14), we can show that $\langle \nabla\Phi(u), u^+ - \lambda(u^0 + u^-)/\|u^0 + u^-\|_E^{2-2\lambda}\rangle_E > 0$ as $\|u^+\|_E$ large enough with $\|u^0 + u^-\|_E^\lambda = \|u^+\|_E$.

The last statement of the Lemma is easily verified. □

REMARK 20. Under the assumptions (13) to (15), $1 < \beta + 1 < q$ holds, and so we can omit the term $\|u^- + u^+\|^{\beta+1}$ in (ii) of (b1), for example.

3. Statement and proof of the main theorem

3.1. Statement of the result. Let us return to the nonlinear wave equation (WE):

$$\text{(WE)}\qquad \begin{cases} \Box u(x,t) = h(x,t,u(x,t)) & (0 < x < \pi,\ t \in \mathbf{R})\,, \\ u(0,t) = u(\pi,t) = 0 & (t \in \mathbf{R})\,, \\ u(x, t+2\pi) = u(x,t) & (0 < x < \pi,\ t \in \mathbf{R})\,. \end{cases}$$

The nonlinear term $h : [0,\pi] \times \mathbf{R}^2 \to \mathbf{R}$ is assumed to satisfy the following conditions (h1) to (h3).

(h1) h is continuous and $h(x,t+2\pi,\xi) = h(x,t,\xi)$ $((x,t,\xi) \in [0,\pi] \times \mathbf{R}^2)$.
(h2) h is non-decreasing in ξ and $h(x,t,\xi) \neq 0$ $(\xi \neq 0)$.
(h3) There exist constants $b_0 \geq 0$, $b > 0$ that satisfy the following properties:
$g_0(x,t,\xi) := h(x,t,\xi) - b_0\xi = o(|\xi|)$ as $\xi \to 0$ uniformly in (x,t),
$g(x,t,\xi) := h(x,t,\xi) - b\xi = o(|\xi|)$ as $|\xi| \to \infty$ uniformly in (x,t).

Let $Q := (0,\pi) \times (0,2\pi)$ and define

$$b_0^+ := \min\{\lambda \mid \lambda \in \sigma(\Box),\ b_0 < \lambda\}, \quad b_0^- := \max\{\lambda \mid \lambda \in \sigma(\Box),\ \lambda < b_0\},$$

where $\Box$ (D'Alembertian) mean the self-adjoint operator in $L^2(Q)$ obtained as the closure of $\partial^2/\partial t^2 - \partial^2/\partial x^2$ with domain $\{u \in C^2([0,\pi] \times \mathbf{R}) \mid u(x,t+2\pi) = u(x,t),\ u(0,t) = u(\pi,t) = 0\}$.

Now we state and prove the main theorem of this paper, of which the part of cases (A3) and (A4) referring to $(C2)$ are new.

THEOREM 21. *As to the equation* (WE), *assume that the non-linear term h satisfies the conditions* (h1) $\sim$ (h3) *and let* b_0, g_0, b *and* g *be as in* (h3). *Set* $G(x,t,\xi) := \int_0^\xi g(x,t,s)ds$, $G_0(x,t,\xi) := \int_0^\xi g_0(x,t,s)ds$ *and consider the following conditions*:

(C1) *g is bounded, and* $G(x,t,\xi) \to +\infty$ $(as\ |\xi| \to \infty)$ *uniformly in* (x,t),
(C2) *the following condition* (a1) *or* (a2) *holds for some constants* $0 < \alpha \leq \beta < 1$, $\beta - \frac{\alpha}{2} < \frac{1}{2}$, $c_1, c_2 > 0$, *and* $d_1, d_2 \geq 0$:

(a1) $|g(x,t,\xi)| \leq c_1|\xi|^\beta + d_1$, $\quad G(x,t,\xi) \geq c_2|\xi|^{\alpha+1} - d_2|\xi|$,

(a2) $|g(x,t,\xi)| \leq c_1|\xi|^\beta + d_1$, $\quad G(x,t,\xi) \leq -c_2|\xi|^{\alpha+1} + d_2|\xi|$.

(C3) There exists a $\delta > 0$ such that $G_0(x,t,\xi) \geq 0$ if $|\xi| \leq \delta$,
(C4) There exists a $\delta > 0$ such that $G_0(x,t,\xi) \leq 0$ if $|\xi| \leq \delta$.

Then (WE) *has a non-trivial weak solution in each of the following cases*:

(A1) $b_0 \notin \sigma(\Box)$, $b \notin \sigma(\Box)$ *and* $b \notin [b_0^-, b_0^+)$;
(A2) $b_0 \in \sigma(\Box)$, $b \notin \sigma(\Box)$, *and one of the following conditions holds*:
 (1) $b \notin [b_0, b_0^+)$ *and* (C3);
 (2) $b \notin [b_0^-, b_0)$ *and* (C4);
(A3) $b_0 \notin \sigma(\Box)$, $b \in \sigma(\Box)$, *and one of the following conditions holds*:
 (1) $b_0 < b$ *and* (C1) *or* (a1) of (C2);
 (2) $b_0 > b$ *and* (a2) of (C2);
(A4) $b_0 \in \sigma(\Box)$, $b \in \sigma(\Box)$, *and one of the following conditions holds*:
 (1) (C3), $b_0 < b$ *and* (C1) *or* (a1) of (C2);
 (2) (C3), $b_0 \geq b$ *and* (a2) of (C2);
 (3) (C4), $b_0 \leq b$ *and* (C1) *or* (a1) of (C2);
 (4) (C4), $b_0 > b$ *and* (a2) of (C2);

3.2. Framework for variational method. For convenience sake, we state here the standard setting for the variational treatment of the equation (WE).

3.2.1. The function space. By the Fourier series expansion, every real-valued $u \in L^2(Q)$ can be written as

$$u(x,t) = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} u_{kj} \sin jx \, e^{ikt}$$

with $\overline{u_{kj}} = u_{-kj}$ for all j, k. Using this expansion, we set

$$\|u\|_E := \left(\pi^2 \sum_{j \neq |k|} |j^2 - k^2| |u_{kj}|^2 + \pi^2 \sum_{j=|k|} |u_{kj}|^2 \right)^{1/2}$$

and we define the space E by

$$E := \{u \in L^2(Q) \mid \|u\|_E < \infty\} .$$

Then E is a Hilbert space with the inner-product

$$\langle u, v \rangle_E := \pi^2 \sum_{j \neq |k|} |j^2 - k^2| u_{kj} \overline{v_{kj}} + \pi^2 \sum_{j=|k|} u_{kj} \overline{v_{kj}} ,$$

where u_{kj}, v_{kj} are Fourier coefficients of u, v respectively. E has an orthogonal decomposition $E = E^+ \oplus E^0 \oplus E^-$ where

$$E^+ := \left\{ u \in E \ : \ u(x,t) = \sum_{j^2 > k^2} u_{kj} \sin jx e^{ikt} \right\},$$

$$E^0 := \left\{ u \in E \ : \ u(x,t) = \sum_{j^2 = k^2} u_{kj} \sin jx e^{ikt} \right\},$$

and

$$E^- := \left\{ u \in E \ : \ u(x,t) = \sum_{j^2 < k^2} u_{kj} \sin jx e^{ikt} \right\}.$$

The orthogonal projections onto E^-, E^0 and E^+ are designated by P^-, P^0 and P^+, respectively.

It is well known that the inclusions $E^\pm \hookrightarrow L^2(Q)$ are compact and E^0 is a closed subspace of $L^2(Q)$. Moreover, $E^\pm \hookrightarrow L^p$ $(1 \leq p < 4)$ are continuous by the interpolation theory [15, Proposition 2.2].

For each $n \in \mathbf{N}$, we set

$$E_n := \operatorname{span} \{\sin jx \sin kt, \sin jx \cos kt \ : \ 0 < j \leq n, |k| \leq n\} .$$

Then $\{E_n\}_n$ is an increasing sequence of finite dimensional subspace of E with $\cup_{n=1}^{\infty} E_n$ being dense in E. Let us note that this sequence is compatible with the decomposition $E = E^- \oplus E^0 \oplus E^+$, i.e., the orthogonal projection onto E_n commutes with P^-, P^0 and P^+ for every n.

3.2.2. The functional and two decompositions of the space. Consider the functional Φ defined on E by

$$\Phi(u) := \frac{1}{2}\int_Q (u_x^2 - u_t^2)dxdt - \int_Q H(x,t,u)dxdt \tag{16}$$

$$= \frac{1}{2}(\|P^+u\|^2 - \|P^-u\|^2) - \Psi(u)\,, \tag{17}$$

where $H(x,t,\xi) := \int_0^\xi h(x,t,s)ds$, $\Psi(u) := \int_Q H(x,t,u)dxdt$. Under the conditions (h1) to (h3), it is clear that $\Phi(u)$ is a C^1 class functional on E with $\langle \nabla\Phi(u), v\rangle_E = \langle (P^+ - P^-)u, v\rangle_E - \int_Q h(x,t,u(x,t))v(x,t)dxdt$.

From now on, let G, G_0, b_0 and b be as defined in the statement of Theorem 21. Then define $\Psi_1 \in C^1(L^2, \mathbf{R})$ by $\Psi_1(u) := \int_Q G(x,t,u)dxdt$, and let $S_\infty\colon L^2 \to L^2$ be the bounded self-adjoint operator defined by $S_\infty u := bu$, let $S\colon E \to E$ be the bounded self-adjoint operator defined by $\langle Su,\ v\rangle_E = \langle S_\infty u,\ v\rangle_{L^2}$ for all $u, v \in E$ and define

$$L := P^+ - P^- - S\,. \tag{18}$$

We observe that 0 is at most an isolated point in $\sigma(L)$ and $0 \in \sigma(L)$ if and only if $b \in \sigma(\Box)$. The operator $L = P^+ - P^- - S$ yields an orthogonal decomposition $E = X^+ \oplus X^0 \oplus X^-$ with $X^0 = \ker L$ and $X^\pm$ corresponding to $\sigma(L) \cap \mathbf{R}^\pm$ respectively. Here we have

$$X^+ := \left\{ u \in E \ :\ u(x,t) = \sum_{j^2-k^2>b} u_{kj} \sin jx e^{ikt} \right\},$$

$$X^0 := \left\{ u \in E \ :\ u(x,t) = \sum_{j^2-k^2=b} u_{kj} \sin jx e^{ikt} \right\},$$

$$X^- := \left\{ u \in E \ :\ u(x,t) = \sum_{j^2-k^2<b} u_{kj} \sin jx e^{ikt} \right\}.$$

We have $\dim X^0 < \infty$ (note that $b > 0$). Similarly we define the following operators:

$$L_0 := P^+ - P^- - S_0\,,$$

$$S_0\colon E \to E\,, \quad \langle S_0 u, v\rangle_E := \int_Q b_0\, u(x,t)v(x,t)dxdt \quad (u, v \in E)\,.$$

We also consider the decomposition $E = X_0^+ \oplus X_0^0 \oplus X_0^-$ determined by the spectral projections related to L_0 ($X^0 = \ker L_0$ and $X_0^\pm$ corresponds to $\sigma(L) \cap \mathbf{R}^\pm$):

$$X_0^+ := \left\{ w \in E \ : \ w(x,t) = \sum_{j^2-k^2>b_0} u_{kj} \sin jx e^{ikt} \right\},$$

$$X_0^0 := \left\{ w \in E \ : \ w(x,t) = \sum_{j^2-k^2=b_0} u_{kj} \sin jx e^{ikt} \right\},$$

$$X_0^- := \left\{ w \in E \ : \ w(x,t) = \sum_{j^2-k^2<b_0} u_{kj} \sin jx e^{ikt} \right\}.$$

3.3. Proof of the main theorem. Since a non-zero critical point of the function Φ defined by (16) is a weak solution to (WE), our main theorem will be obtained by applying the abstract existence theorem (Theorem 12) to Φ. To do so, we shall verify step by step that the assumptions in Theorem 12 are satisfied by our variational framework described in the preceding subsection.

3.3.1. Preliminaries. For our purpose, we prepare the following technical, but straightforward, lemmas.

LEMMA 22. *Suppose that* (h1) *to* (h3) *hold. Then, Φ satisfies* (Φ8) (*hence* (Φ5)), *with $\Phi_1 = -\Psi_1$, $H := L^2(Q)$ and L in* (18)*. Moreover, if* (C1) *holds in addition, then Φ satisfies* ($\Phi 8'$) *(hence* (Φ6)).

PROOF. The conditions (a) and (c) in (Φ7) are satisfied by our framework described in Section 3.2. As to (b), our functional Φ defined in Section 3.2 is indeed of the form described in (c) (*cf.* (17)). The convexity of Ψ follows from $\nabla_H \Psi(u) = h(x,t,u)$ and the condition (h2). Other conditions in (b) are also readily seen to be satisfied.

Next we verify the extra condition in (Φ8). Since $\Psi_1(u) = \int_Q G(x,t,u)dxdt$, we have $\nabla_H \Psi_1(u) = g(x,t,u)$. On the other hand, the condition (h3) implies that for every $\varepsilon > 0$ there exists some constant $C_\varepsilon > 0$ for which $|g(x,t,\xi)| \le \varepsilon|\xi| + C_\varepsilon$ holds for every $\xi \in \mathbf{R}$ and $(x,t) \in Q$. Therefore we obtain $\|\nabla_H \Psi_1(u)\|_H \le \varepsilon C \|u\|_H + C'_\varepsilon$ and so $\nabla_H \Psi_1(u) = o(\|u\|_H)$ (as $\|u\|_H \to \infty$). Hence we have $\|\nabla \Phi_1(u)\|_E = o(\|u\|_E)$ (as $\|u\|_E \to \infty$).

Hereafter suppose that (C1) holds. Then $\nabla\Phi_1$ and $\nabla_H \Psi_1$ are clearly bounded. Finally we shall give a proof of $|\Psi_1(u)| \to \infty$ ($u \in \ker L$, $\|u\|_E \to \infty$), since we could not find a proof in the literature. Since $\dim\ker L < \infty$, we set $m := \dim\ker L$ and take a complete orthonormal system $\{e_j\}_{j=1}^m$ in $\ker L$. Let $\{u_n\}_n \subset \ker L$ such that $\|u_n\|_E \to \infty$ (as $n \to \infty$). Then there exists $\{\lambda_n\}_n \subset \mathbf{R}^m$ such that $u_n(x,t) = \lambda_n \cdot e(x,t)$ for $(x,t) \in Q$ and $|\lambda_n| = \|u_n\|_E \to \infty$ $(n \to \infty)$ where $e(x,t) := {}^t(e_1(x,t), \cdots, e_m(x,t))$. We let $\{u_k\}_k$ and $\{\lambda_k\}_k$ are arbitrary subsequences of $\{u_n\}_n$ and $\{\lambda_n\}_n$, respectively. Since $\lambda_k/|\lambda_k|$ is bounded in $\mathbf{R}^m$, there exist a subsequence $\{\lambda_{k_l}\}_l \subset \{\lambda_k\}_k$ and $0 \neq \lambda \in \mathbf{R}^m$ such that $\lambda_k/|\lambda_k| \to \lambda$ as $l \to \infty$.

We define $Q^+ := \{(x,t) \in Q \ : \ \lambda \cdot e(x,t) > 0\}$, $Q^- := \{(x,t) \in Q \ : \ \lambda \cdot e(x,t) < 0\}$ and $Q^0 := \{(x,t) \in Q \ : \ \lambda \cdot e(x,t) = 0\}$. We can show $\mathrm{meas}(Q \setminus Q^0) > 0$. Indeed if $\mathrm{meas}(Q \setminus Q^0) = 0$, then we obtain by Lebesgue's theorem

$$\lim_{l\to\infty} \frac{\|u_{k_l}\|_{L^1}}{|\lambda_{k_l}|} = \int_{Q^0} \lim_{l\to\infty} \left| \frac{\lambda_{k_l}}{|\lambda_{k_l}|} \cdot e(x,t) \right| dxdt = 0\,.$$

On the other hand, $\inf_l \|u_{k_l}\|_{L^1}/|\lambda_{k_l}| > 0$ because of $\dim\ker L < \infty$ and $\|u_{k_l}/|\lambda_{k_l}|\|_E = 1$, this yields a contradiction.

It is easy to see that G is uniformly bounded below by the assumption (C1). Therefore, by Fatou's lemma, we obtain

$$\liminf_{l\to\infty} \int_{Q^+\cup Q^-} G(x,t,u_{k_l})dxdt \ge \int_{Q^+\cup Q^-} \liminf_{l\to\infty} G(x,t,u_{k_l})dxdt = +\infty$$

since $|u_{k_l}(x,t)| = ||\lambda_{k_l}|[(\lambda_{k_l}/|\lambda_{k_l}|) \cdot e(x,t)]| \to \infty$ as $l \to \infty$ for every $(x,t) \in Q^+ \cup Q^-$ and $G(x,t,\xi) \to +\infty$ as $|\xi| \to \infty$. On the other hand we have

$$\begin{aligned} &\liminf_{l\to\infty} \int_Q G(x,t,u_{k_l})dxdt \\ &\quad \ge \liminf_{l\to\infty} \int_{Q^0} G(x,t,u_{k_l})dxdt + \liminf_{l\to\infty} \int_{Q^+\cup Q^-} G(x,t,u_{k_l})dxdt\,, \end{aligned}$$

and so we obtain $\lim_{l\to\infty} \int_Q G(x,t,u_{k_l})dxdt = +\infty$. Thus we have shown that every subsequence of $\{u_n\}_n$ has a subsequence for which Ψ_1 diverges to ∞. Hence we are done. □

LEMMA 23. *If* (a1) *or* (a2) *of* (C2) *holds and h satisfies the conditions* (h1) $\sim$ (h3), *then Φ satisfies* (a) *in Proposition* 18 *and* (b1) *or* (b2) *in Lemma* 19, *respectively.*

PROOF. We only prove that Φ satisfies (a) and (b1) of Lemma 19 when (a1) of (C2) holds, since we can similarly prove the other case. By Lemma 22, Φ satisfies (Φ8) with $\Psi_1(u) = \int_Q G(x,t,u)dxdt$. Since $\dim X^0 < \infty$ and $G(x,t,\xi) \ge c_2\,|\xi|^{\alpha+1} - d_2\,|\xi|$, there exists C_1, $C_2 > 0$ such that $\Psi_1(u^0) \ge C_1\,\|u^0\|_E^{\alpha+1} - C_2\,\|u^0\|_E$. By the assumption α and β satisfies (13), hence we can choose λ, p and q satisfying (14) and (15) in Lemma 19. Since the inclusions $E^\pm \hookrightarrow L^p$ $(1 \le p < 4)$ are continuous and $\dim X^0 < \infty$, with the aid of Hölder's inequality and Young's inequality, we have for every u, $v \in E$

$$\int_Q |u|^\beta \cdot |v| dxdt \le \|u\|_{L^{p\beta}}^\beta\,\|v\|_{L^q} \le C\|u\|_E^\beta\,\|v\|_E \tag{19}$$

$$\le C'(\|u\|_E^{p\beta} + \|v\|_E^q) \tag{20}$$

where C and C' are positive constants. (19) proves that the condition (a) in Proposition 18 is satisfied. Next we set $u = u^0 + u^- + u^+$ where $u^0 \in X^0$, $u^\pm \in X^\pm$. By the inequality (20) and the condition $|g(x,t,\xi)| \le c_1|\xi|^\beta + d_1$, we obtain

$$|\Psi_1(u^0 + u^- + u^+) - \Psi_1(u^0)|$$

$$
\begin{aligned}
&\leq \int_Q \int_0^1 |g(x,t,u^0 + s(u^+ + u^-))| \cdot |u^+ + u^-| ds dx dt \\
&\leq C(\|u^- + u^+\|_{L^{\beta+1}}^{\beta+1} + \|u^- + u^+\|_{L^1}) + C \int_Q |u^0|^\beta \cdot |u^- + u^+| dx dt \\
&\leq C(\|u^- + u^+\|_E^{\beta+1} + \|u^- + u^+\|_E + \|u^0\|_E^{p\beta} + \|u^- + u^+\|_E^q)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\langle \nabla_H \Psi_1(u^0 + u^- + u^+), u^+ \rangle_H| \\
&\qquad \leq \int_Q |g(x,t,u^0 + u^- + u^+)| \cdot |u^+| dx dt \\
&\qquad \leq C(\|u^+\|_{L^{\beta+1}}^{\beta+1} + \|u^+\|_{L^1}) + C \int_Q |u^0 + u^-|^\beta \cdot |u^+| dx dt \\
&\qquad \leq C(\|u^+\|_E^{\beta+1} + \|u^+\|_E + \|u^0 + u^-\|_E^{p\beta} + \|u^+\|_E^q)
\end{aligned}
$$

where C is a positive constant. Finally we similarly obtain

$$
\begin{aligned}
&|\langle \nabla_H \Psi_1(u^0 + u^- + u^+), u^0 + u^- \rangle_H| \\
&\quad \leq \int_Q |g(x,t,u^0 + u^- + u^+)| \cdot |u^0 + u^-| dx dt \\
&\quad \leq c_1 \|u^0 + u^-\|_{L^{\beta+1}}^{\beta+1} + d_1 \|u^0 + u^-\|_{L^1} + C \int_Q |u^+|^\beta \cdot |u^0 + u^-| dx dt \\
&\quad \leq C(\|u^0 + u^-\|_E^{\beta+1} + \|u^+\|_E^{\beta+1}).
\end{aligned}
$$

□

3.3.2. Proof of the main theorem. Now we begin with the proof of our main theorem, Theorem 21. We concentrate on the part of cases of assumption (A3) and (A4) where the condition (C2) is concerned. To this end, we verify that the functional Φ or $-\Phi$ satisfies the condition ($\Phi 1$) to ($\widetilde{\Phi} 4$) and the dimension condition (10). Once these conditions are verified, then we can apply Theorem 12 to Φ or $-\Phi$ and obtain a non-zero critical point of Φ, which yields a non-trivial weak solution to (WE).

In what follows, when we speak of the condition (A3) *or* (A4), *we understand that either* (a1) *or* (a2) *of* (C2) *holds.*

1. Condition ($\Phi 1$). Under the conditions (h1), (h2), (h3) and (C2), Lemma 22 implies Φ satisfies ($\Phi 8$) and ($\Phi 5$), while Lemma 23 shows that Φ satisfies the condition (a) in Proposition 18 and (b1) or (b2) in Lemma 19, according as (a1) or (a2) of (C2) holds. Therefore, noting that (b1) or (b2) in Lemma 19 implies the condition (b) in Proposition 18, Φ satisfies $(WPS)_c^*$ for every $c \in \mathbf{R}$ by Proposition 18 and so $-\Phi$ satisfies $(WPS)_c^*$ for every $c \in \mathbf{R}$ provided (A3) or (A4) holds.

2. Condition ($\Phi 2$). This condition is clearly satisfied under the condition (h3).

3. Condition ($\Phi 3$). Since the behavior of $h(x, t, \xi)$ as $\xi \to 0$ is solely concerned with the local linking property of Φ, the condition ($\Phi 3$) is already proved in [2]. For convenience sake, we quote here the relevant results.

LEMMA 24 ([2, Lemma 5.4]). *If h satisfies the conditions* (h1) $\sim$ (h3) *and $b_0 = 0$, then Φ has a strong local linking at* 0 *w.r.t.* (V_0, W_0) *with* $V_0 := E^+$, $W_0 := E^0 \oplus E^-$.

LEMMA 25 ([2, Lemma 5.6]). *If h satisfies the conditions* (h1) $\sim$ (h3), $b_0 > 0$ *and* $b_0 \notin \sigma(\Box)$ *or* (C3) *or* (C4) *holds, then Φ has a local linking at* 0 *w.r.t. the following* (V_0, W_0) *in each of the cases below:*

$$\begin{cases} V_0 := X_0^+,\ W_0 := X_0^- & \text{if } b_0 \notin \sigma(\Box) \\ V_0 := X_0^+,\ W_0 := X_0^0 \oplus X_0^- & \text{if (C3)} \\ V_0 := X_0^+ \oplus X_0^0,\ W_0 := X_0^- & \text{if (C4)} \end{cases}$$

Moreover, Φ satisfies the condition (ii) *of* ($\Phi 3$).

Note that the last assertion of Lemma 25 follows from Lemma 17 and the fact that there exists a constant $a > 0$ such that $H(x, t, \xi) \geq a|\xi|^2$ for all (x, t, ξ).

4. Condition ($\widetilde{\Phi 4}$). By Lemma 22 and Lemma 23, the assumptions in Lemma 19 are satisfied and hence Φ or $-\Phi$ satisfies ($\widetilde{\Phi 4}$) with $(W_\infty, V_\infty) = (X^0 \oplus X^-, X^+)$ or $(W_\infty, V_\infty) = (X^0 \oplus X^+, X^-)$, respectively according as (a1) or (a2) of (C2) holds.

5. Dimension condition. In each of the cases in (A3) and (A4), we can verify that the condition (10) holds by using Lemma 5.7 in [2], which seems too lengthy to be quoted. Here we shall deal with only two cases since the other cases would be similarly handled. First assume that case (1) of (A3) holds. Namely, suppose that $b_0 < b$ and (a1) of (C2) hold. Then Φ has a local linking at 0 w.r.t. $(W_0, V_0) = (X_0^-, X_0^+)$ and satisfies ($\widetilde{\Phi 4}$) w.r.t. $(W_\infty, V_\infty) = (X^0 \oplus X^-, X^+)$ (see above). Since we are assuming $b \in \sigma(\Box)$, the eigenspace $E(b)$ of $\Box$ pertaining to b is contained in X^0. On the other hand, the definition of X_0^0, X_0^- and the assumption $b_0 < b$ imply $X_0^0 \oplus X_0^- \subset X^-$. Moreover, $E(b) \subset E_n$ for large $n \in \mathbf{N}$. Therefore, if n is large enough, we obtain

$$E_n \cap (X_0^0 \oplus X_0^-) \subset E_n \cap X^- \subsetneq (E_n \cap X^-) \oplus E(b) = E_n \cap (X^- \oplus X^0)\,.$$

Hence

$$\liminf_{n\to\infty} \big[\dim E_n \cap W_\infty - \dim E_n \cap W_0\big] \geq \dim E(b) > 0\,,$$

and hence (10) holds in this case.

As another case, consider the case (3) of (A4). Namely assume that (C4), $b_0 \leq b$ and (a1) of (C2) hold. Then Φ has a local linking at 0 w.r.t. $(W_0, V_0) = (X_0^-, X_0^0 \oplus X_0^+)$ and satisfies ($\widetilde{\Phi 4}$) w.r.t. $(W_\infty, V_\infty) = (X^0 \oplus X^-, X^+)$. Since we are assuming $b \in \sigma(\Box)$, the eigenspace $E(b)$ of $\Box$ pertaining to b is contained in X^0. On the other hand, the definition of X_0^0, X_0^- and the assumption $b_0 \leq b$ imply $X_0^- \subset X^-$. Moreover, $E(b) \subset E_n$ for large $n \in \mathbf{N}$.

Therefore, if n is large enough, we obtain

$$E_n \cap X_0^- \subset E_n \cap X^- \subsetneq (E_n \cap X^-) \oplus E(b) = E_n \cap (X^- \oplus X^0)\,.$$

Hence

$$\liminf_{n\to\infty} \big[\dim E_n \cap W_\infty - \dim E_n \cap W_0\big] \geq \dim E(b) > 0\,,$$

and hence (10) holds in this case.

In other cases, the dimension condition can be similarly verified by considering $-\Phi$ instead of Φ if (a2) of (C2) is assumed.

3.3.3. Supplement. Here we sketch a proof for the case of (A1) and (A2), since the condition ($\Phi 4$) in [2] seems insufficient for the proof of Proposition 2.3 in [2]. In the case of (A1) and (A2), it is already proved in [2] that Φ or $-\Phi$ satisfies ($\Phi 1$), ($\Phi 2$), ($\Phi 3$) and the dimension condition. Moreover, Lemma 13 yields ($\widetilde{\Phi 4}$) (with $\lambda = 1$) because $b \notin \sigma(\Box)$ and so Φ satisfies ($\Phi 5$). Hence we can apply our abstract theorem to Φ or $-\Phi$ to obtain a non-trivial weak solution to (WE).

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