A proof of the second Rogers-Ramanujan identity via Kleshchev multipartitions

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Abstract: We give another proof of the second Rogers-Ramanujan identity by Kashiwara crystals.

Key words: Integer partitions; Rogers-Ramanujan identities; Kashiwara crystals; quantum groups; Hecke algebras.

1. Introduction. In [9], Lepowsky and Milne observed a similarity between the characters of the level 3 standard modules of the affine Lie algebra of type $A_1^{(1)}$

$$\text{(1)}\qquad \operatorname{ch} V(2\Lambda_0+\Lambda_1)=\frac{1}{(q;q^2)_\infty}\frac{1}{(q,q^4;q^5)_\infty},$$
$$\operatorname{ch} V(3\Lambda_0)=\frac{1}{(q;q^2)_\infty}\frac{1}{(q^2,q^3;q^5)_\infty},$$

and the infinite products of the Rogers-Ramanujan identities

$$\text{(2)}\qquad \sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty},$$
$$\sum_{n\geq 0}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}.$$

Here, the q-Pochhammer symbols are defined for $n\in\mathbf{Z}_{\geq 0}\sqcup\{\infty\}$ as follows:

$$(a;q)_n=\prod_{0\leq j<n}(1-aq^j),$$
$$(a_1,\dots,a_k;q)_n=(a_1;q)_n\cdots(a_k;q)_n.$$

Later, Lepowsky and Wilson promoted the observation to a vertex operator proof and gave a Lie theoretic interpretation of the infinite sums in the Rogers-Ramanujan identities [10]. The goal of this paper is to show that a result of Kashiwara crystals which is motivated by the representation theory of Hecke algebras [1, Corollary 9.6] promotes the equality (1) into a proof of the second Rogers-Ramanujan identity (2). Note that it is well-known that the Rogers-Ramanujan identities and the solvable lattice models from which the quantum groups originated are related (see [3, Chapter 8]). Several relationships between Rogers-Ramanujan type identities and Kashiwara crystals are also known (see [5] and the references therein). The author was inspired by a recent work of Corteel which gave a proof of (2) using the cylindric partitions and the Robinson-Schensted-Knuth correspondence [4].

2. The main result. A partition (resp. strict partition) is a weakly (resp. strictly) decreasing sequence $\lambda=(\lambda_1,\dots,\lambda_\ell)$ of positive integers, i.e., $\lambda_1\geq\cdots\geq\lambda_\ell\geq 1$ (resp. $\lambda_1>\cdots>\lambda_\ell\geq 1$). We denote the set of partitions (resp. strict partitions) by Par (resp. Str). We also denote the size $\lambda_1+\cdots+\lambda_\ell$ (resp. the length ℓ) of λ by $|\lambda|$ (resp. $\ell(\lambda)$). When λ is empty (i.e., $\ell(\lambda)=0$), we put $\lambda_1=0$.

Theorem 2.1 ([1, (The transposed version of) Proposition 9.7]). *Let $k\geq 1$. Under the $A_1^{(1)}$-crystal isomorphism $\mathsf{Str}\cong B(\Lambda_0)$ due to Misra-Miwa [11], the canonical image $B(k\Lambda_0)$ in the tensor product $B(\Lambda_0)^{\otimes k}$ coincides with*

$$S_k=\{\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(k)})\in\mathsf{Str}^k\mid \ell(\lambda^{(i)})\geq(\lambda^{(i+1)})_1 \text{ for } 1\leq i<k\}.$$

This result is credited to Mathas in [1, §9]. It is also a Corollary of [7, Theorem 3.8] and [8, Theorem 10.1]. An element of the connected component S_k is called a Kleshchev multipartition in the context of the representation theory of Hecke algebras. For a generalization to $A_p^{(1)}$-crystal, where $p\geq 2$, see [1, Corollary 9.6]. For a different charac-

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terization, see [6].

Theorem 2.2. *For $k \geq 1$, we have*

$$\sum_{\boldsymbol{\lambda}\in S_k} x^{\ell(\boldsymbol{\lambda})} q^{|\boldsymbol{\lambda}|} = \sum_{i_1,\dots,i_k\geq 0} \frac{q^{\sum_{a=1}^k a\binom{1-i_a}{2}+\sum_{1\leq a<b\leq k} a i_a i_b}}{(q;q)_{i_1}\cdots(q;q)_{i_k}} x^{\sum_{a=1}^k a i_a}.$$

Here, for a k-tuple of strict partitions $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_k) \in \mathsf{Str}^k$, the size $|\boldsymbol{\lambda}|$ and the length $\ell(\boldsymbol{\lambda})$ are defined as follows:

$$|\boldsymbol{\lambda}| = |\lambda_1| + \cdots + |\lambda_k|, \quad \ell(\boldsymbol{\lambda}) = \ell(\lambda_1) + \cdots + \ell(\lambda_k).$$

3. A proof of Theorem 2.2. As usual (see [2, Definition 3.1]), we define the q-binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}$$

for $n \geq m \geq 0$. It is well-known (see [2, Theorem 3.1]) that we have

(3) $$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{\substack{\lambda\in\mathsf{Par}\\ \ell(\lambda)\leq m\\ \lambda_1\leq n-m}} q^{|\lambda|}.$$

For $i, j \geq 0$, considering the staircase $\Delta_j = (j, j-1, \dots, 1) \in \mathsf{Str}$, we see

(4) $$\sum_{\substack{\mu\in\mathsf{Str}\\ \ell(\mu)=j\\ \mu_1\leq i+j}} q^{|\mu|} = q^{|\Delta_j|} \sum_{\substack{\lambda\in\mathsf{Par}\\ \ell(\lambda)\leq j\\ \lambda_1\leq i}} q^{|\lambda|}.$$

Proposition 3.1. *For $k \geq 1$ and $j_1,\dots,j_k \geq 0$, there is a size preserving bijection*

$$f_{j_1,\dots,j_k} : V_{j_1,\dots,j_k} \to W_{j_1,\dots,j_k},$$

where

$$\begin{aligned} A_{j_1,\dots,j_k} &= \{\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(k)}) \in \mathsf{Str}^k \mid \ell(\lambda^{(i)}) \\ &\qquad = j_i + \cdots + j_k \text{ for } 1 \leq i \leq k\}, \\ V_{j_1,\dots,j_k} &= S_k \cap A_{j_1,\dots,j_k}, \\ W_{j_1,\dots,j_k} &= \{\boldsymbol{\lambda} \in A_{j_1,\dots,j_k} \mid ((\lambda^{(i)})_{j_i+1},\dots,(\lambda^{(i)})_{\ell(\lambda^{(i)})}) \\ &\qquad = \Delta_{\ell(\lambda^{(i+1)})} \text{ for } 1 \leq i < k\}. \end{aligned}$$

Proof. We prove the claim by induction on k. The case $k = 1$ is trivial.

Similarly to (4), for $i, j \geq 0$ we see

(5) $$\sum_{(\lambda,\mu)\in V_{i,j}} q^{|\lambda|+|\mu|} = \sum_{\substack{\mu\in\mathsf{Str}\\ \ell(\mu)=j\\ \mu_1\leq i+j}} q^{|\mu|} \frac{q^{|\Delta_{i+j}|}}{(q;q)_{i+j}},$$

$$\sum_{(\lambda,\mu)\in W_{i,j}} q^{|\lambda|+|\mu|} = \frac{q^{|\Delta_{i+j}|}}{(q;q)_i} \frac{q^{|\Delta_j|}}{(q;q)_j},$$

which are equal to each other thanks to (3) and (4). This settled the case $k = 2$.

For $k \geq 3$, it is easily seen that the composite

$$\begin{aligned} \boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(k)}) &\mapsto \boldsymbol{\mu} = (\mu^{(1)},\dots,\mu^{(k)}) \\ &:= (f_{j_1,j_2+\cdots+j_k}(\lambda^{(1)},\lambda^{(2)}),\lambda^{(3)},\dots,\lambda^{(k)}) \\ &\mapsto (\mu^{(1)}, f_{j_2,\dots,j_k}(\mu^{(2)},\dots,\mu^{(k)})) \end{aligned}$$

is a size preserving bijection from $V_{j_1,\dots,j_k}$ to $W_{j_1,\dots,j_k}$. □

Theorem 2.2 is proved as follows: Clearly, we have

$$\sum_{\boldsymbol{\lambda}\in S_k} x^{\ell(\boldsymbol{\lambda})} q^{|\boldsymbol{\lambda}|} = \sum_{j_1,\dots,j_k\geq 0} x^{j_1+2j_2+\cdots+kj_k} \sum_{\boldsymbol{\lambda}\in V_{j_1,\dots,j_k}} q^{|\boldsymbol{\lambda}|}.$$

By Proposition 3.1, the right hand side is equal to

$$\sum_{j_1,\dots,j_k\geq 0} x^{j_1+2j_2+\cdots+kj_k} \sum_{\boldsymbol{\lambda}\in W_{j_1,\dots,j_k}} q^{|\boldsymbol{\lambda}|}.$$

Similarly to (5), we see

$$\sum_{\boldsymbol{\lambda}\in W_{j_1,\dots,j_k}} q^{|\boldsymbol{\lambda}|} = \prod_{a=1}^k \frac{q^{|\Delta_{j_a+\cdots+j_k}|}}{(q;q)_{j_a}}.$$

Using $|\Delta_{s+t}| = |\Delta_s| + |\Delta_t| + st$ for $s, t \geq 0$, we have

$$|\Delta_{j_a+\cdots+j_k}| = \sum_{b=a}^k |\Delta_{j_b}| + \sum_{a\leq b<b'\leq k} j_b j_{b'}$$

and thus we have

$$\sum_{a=1}^k |\Delta_{j_a+\cdots+j_k}| = \sum_{a=1}^k a|\Delta_{j_a}| + \sum_{1\leq b<b'\leq k} b j_b j_{b'}.$$

4. A proof of the second Rogers-Ramanujan identity. In the proof, let

$$F(x,q) = \sum_{s,t,u\geq 0} \frac{q^{\binom{s+1}{2}+2\binom{t+1}{2}+3\binom{u+1}{2}+st+su+2tu} x^{s+2t+3u}}{(q;q)_s (q;q)_t (q;q)_u},$$

$$G(x,q) = \sum_{s\geq 0} \frac{q^{s(s+1)} x^{2s}}{(q;q)_s}.$$

Proposition 4.1. *We have the following q-difference equation.*

$$G(x,q)=(1+x^2q^2+x^2q^3)G(xq,q)-x^4q^7G(xq^2,q).$$

Proof. It is easy to verify that for all $M\in\mathbf{Z}$ we have

$$(1-q^M)g_M-q^M(1+q)g_{M-2}+q^{2M-1}g_{M-4}=0,$$

where $g_{2s}=q^{s(s+1)}/(q;q)_s$ for $s\in\mathbf{Z}_{\geq 0}$ and $g_M=0$ for $M\in\mathbf{Z}\setminus 2\mathbf{Z}_{\geq 0}$. □

Proposition 4.2. *We have the following q-difference equation.*

$$\begin{aligned}F(x,q)&=(1+xq)(1+x^2q^2+x^2q^3)F(xq,q)\\&\quad-x^4q^7(1+xq)(1+xq^2)F(xq^2,q).\end{aligned}$$

Proof. Our proof is a typical application of a q-version of Wegschaider's improvement of Sister Celine's technique (see [12]).

Let $F(x)=\sum_{n\in\mathbf{Z}}f_n(q)x^n$ and put

$$f(n,t,u)=\frac{q^{\binom{n-2t-3u+1}{2}+2\binom{t+1}{2}+3\binom{u+1}{2}+(n-2t-3u)(t+u)+2tu}}{(q;q)_{n-2t-3u}(q;q)_t(q;q)_u}$$

for $n,t,u\in\mathbf{Z}$, where we regard $\frac{1}{(q;q)_v}=0$ if $v<0$. Because $f(n,t,u)$ is q-proper hypergeometric (see [12, §2.1]), one can automatically derive a q-holonomic recurrence for f_n thanks to $f_n=\sum_{t,u\in\mathbf{Z}^2}f(n,t,u)$.

Let $(Ng)(n,t,u)=g(n-1,t,u)$, $(Tg)(n,t,u)=g(n,t-1,u)$, $(Ug)(n,t,u)=g(n,t,u-1)$ be the shift operators for $g:\mathbf{Z}^3\to\mathbf{Q}(q)$ and let

$$\begin{aligned}A&=(1-q^n)-q^nN-q^n(1+q)(N^2+N^3)\\&\quad+q^{2n-1}N^4+q^{2n-2}((1+q)N^5+N^6),\\B&=(q^n-q^{2t+u})+q^n(-1+q^t+q^u)N\\&\quad+q^{n+u}(1+q^{1+t})N^2+q^{n+2t+u}UN^3,\\C&=q^n(1-q^t)N+q^{1+n+u}(1-q^t)N^2\\&\quad+q^n(1+q^{1+u})N^3-q^{2n-1}N^4\\&\quad-q^{2n-2}((1+q)N^5+N^6).\end{aligned}$$

One can check that

$$(A+(1-T)B+(1-U)C)f(n,t,u)=0.$$

By this certificate recurrence operator (see [12, §3] and [13, §7.1]), we get

$$\begin{aligned}&(1-q^n)f_n-q^nf_{n-1}-q^n(1+q)(f_{n-2}+f_{n-3})\\&\quad+q^{2n-1}f_{n-4}+q^{2n-2}((1+q)f_{n-5}+f_{n-6})=0\end{aligned}$$

for $n\in\mathbf{Z}$. This is equivalent to the q-difference equation in the Proposition. □

Corollary 4.3. *We have* $F(x,q)=(-xq;q)_\infty G(x,q)$.

Proof. By Proposition 4.1 and Proposition 4.2, $F(x,q)$ and $(-xq;q)_\infty G(x,q)$ satisfy the same q-difference equation presented in Proposition 4.2. Then, the equality follows from the fact that both the coefficients of x^0 (resp. x^n for $n<0$) in $F(x,q)$ and $(-xq;q)_\infty G(x,q)$ are equal to 1 (resp. 0). □

Remark 4.4. After submission to arXiv of the first version of this paper, we learned from Ole Warnaar that Corollary 4.3 is easily deduced by a trick to use $f_n=\sum_{t,u\in\mathbf{Z}}f(n,t-u,u)$ instead of $f_n=\sum_{t,u\in\mathbf{Z}}f(n,t,u)$ noticing

$$f(n,t-u,u)=\frac{q^{\binom{n+1}{2}+t(t-n)}}{(q;q)_{n-2t}(q;q)_t}\frac{(q^{-t};q)_u(q^{-(n-2t)};q)_u}{(q;q)_u}.$$

Thanks to the q-Chu-Vandermonde identity ${}_2\phi_1(a,q^{-m};0;q,q)=a^m$ for a nonnegative integer m (see [3, (2.41)]), we have

$$f_n=\sum_{t=0}^{\lfloor n/2\rfloor}\frac{q^{\binom{n+1}{2}+t(t-n)-t(n-2t)}}{(q;q)_{n-2t}(q;q)_t}=\sum_{t=0}^{\lfloor n/2\rfloor}\frac{q^{\binom{n-2t+1}{2}+t(t+1)}}{(q;q)_{n-2t}(q;q)_t}.$$

This is equivalent to Corollary 4.3 by Euler's identity $(-xq;q)=\sum_{m\geq 0}\frac{q^{\binom{m+1}{2}}x^m}{(q;q)_m}$.

The second Rogers-Ramanujan identity (2) is proved as follows: By Theorem 2.2, Lepowsky-Milne's observation (1) is translated to

$$F(1,q)=\frac{1}{(q;q^2)_\infty}\frac{1}{(q^2,q^3;q^5)_\infty}.$$

By Corollary 4.3 and Euler's identity $(q;q^2)_\infty(-q;q)_\infty=1$, we have

$$G(1,q)=\frac{1}{(q^2,q^3;q^5)_\infty}.$$

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