

Morphisms between Grassmannians

By Angelo NALDI and Gianluca OCCHETTA

Dipartimento di Matematica - Università di Trento, via Sommarive 14 I-38123 Trento (TN), Italy

(Communicated by Shigefumi MORI, M.J.A., Nov. 14, 2022)

Abstract: Denote by $\mathbf{G}(k, n)$ the Grassmannian of linear subspaces of dimension k in \mathbf{P}^n .

We show that if $n > m$ then every morphism $\varphi : \mathbf{G}(k, n) \rightarrow \mathbf{G}(l, m)$ is constant.

Key words: Grassmannians; Schubert calculus.

1. Introduction. In [11], H. Tango proved that there are no nonconstant morphisms from \mathbf{P}^n to the Grassmannian $\mathbf{G}(l, m)$ if $n > m$; this result was later used by E. Sato in [10] to study uniform vector bundles on \mathbf{P}^n . In fact, given a uniform vector bundle \mathcal{E} of rank r on the projective space, one can construct a morphism from the variety of lines passing through a point p – which is \mathbf{P}^{n-1} – to an appropriate Grassmannian of linear subspaces of $\mathbf{P}(\mathcal{E}_p)$, whose dimension depends on the splitting type of \mathcal{E} . If those maps are constant for every p , then one can prove that the bundle \mathcal{E} is decomposable as a sum of line bundles.

The study of uniform bundles on other Fano manifolds brought along the problem to prove the constancy of morphisms $M \rightarrow \mathbf{G}(l, m)$, where M is the Variety of Minimal Rational Tangents (VMRT) of the chosen Fano manifold. Tango’s proof made use of the particular structure of the cohomology ring of \mathbf{P}^n , so it was first extended to varieties whose cohomology ring is, up to some degree, isomorphic to the one of the projective space (see [7, Lemma 3.4]).

In [9, Theorem 4.2] X. Pan showed that, in order to obtain a Tango type result, a weaker property of the cohomology ring was needed, the so called *good divisibility* up to some degree. A further step – see [8, Section 4] – was to replace the notion of good divisibility with the weaker notion of *effective good divisibility* – see Section 2 for the definitions of good divisibility and effective good divisibility.

The notion of effective good divisibility was used in [8, Proposition 4.7] to prove the constancy of morphisms from a variety with “large” effective

good divisibility not only to Grassmannians, but also to other rational homogeneous varieties with Picard number one. The aim of [8] was to prove the diagonalizability of uniform flag bundles of small invariants on some rational homogeneous varieties. Therefore in that paper the effective good divisibility has been computed for projective spaces, quadrics and Grassmannians of lines, since those varieties appear as (factors of) the VMRT of the rational homogeneous varieties considered.

However, leaving aside the applications to uniform bundles, the problem of computing the effective good divisibility in order to obtain Tango type results seems worth to be considered also for other varieties.

In the present paper we address it for the Grassmannians $\mathbf{G}(k, n)$ of linear subspaces of dimension k of \mathbf{P}^n ; this choice seems a natural generalization of Tango’s setting, since the Grassmannians $\mathbf{G}(k, n)$ are rational homogeneous manifolds obtained as a quotient of the same group as \mathbf{P}^n ; moreover, the cohomology ring of Grassmannians has a clear description in terms of Schubert cycles, which are effective.

We will prove that $\mathbf{G}(k, n)$ has effective good divisibility up to degree n , and derive from this the main result of the paper.

Theorem 1.1. *If $n > m$ then every morphism $\varphi : \mathbf{G}(k, n) \rightarrow \mathbf{G}(l, m)$ is constant.*

A straightforward consequence of Theorem 1.1 is the following result.

Corollary 1.2. *If $n > m$ then every morphism from $\mathbf{G}(k, n)$ to a rational homogeneous variety of type A_m is constant.*

An anonymous referee suggested us that Proposition 4.1, that we prove using Schubert calculus, can be easily deduced using the fact that the product of two Schubert cycles is never zero in the

2010 Mathematics Subject Classification. Primary 14M15; Secondary 14J45.

(small) quantum cohomology ring of the Grassmannian. We will explain this in Remark 4.4.

2. Divisibility. In this section we will recall the notions of good divisibility and effective good divisibility and explain how to use them to prove constancy of morphisms to Grassmannians.

Let M be a smooth complex projective manifold. Set $H^j(M) := H^j(M, \mathbf{R})$.

Definition 2.1 ([9, Cf. Definition 4.1]). A variety M has *good divisibility up to degree s* if, given $x_i \in H^{2i}(M)$, $x_j \in H^{2j}(M)$ with $i + j \leq s$ and $x_i x_j = 0$, we have $x_i = 0$ or $x_j = 0$. The *good divisibility* of M , denoted by $\text{g.d.}(M)$, is the maximum integer s such that M has good divisibility up to degree s .

The concept of good divisibility can be refined, by considering only effective classes, i.e., classes that can be written as a real linear combination of classes of subvarieties with non negative coefficients.

Definition 2.2 ([8, Definition 4.2]). A variety M has *effective good divisibility up to degree s* if, given effective $x_i \in H^{2i}(M)$, $x_j \in H^{2j}(M)$ with $i + j \leq s$ and $x_i x_j = 0$, we have $x_i = 0$ or $x_j = 0$. The *effective good divisibility* of M , denoted by $\text{e.d.}(M)$ is the maximum integer s such that M has effective good divisibility up to degree s .

It is clear from the definition that $\text{g.d.}(M) \leq \text{e.d.}(M) \leq \dim(M)$.

Example 2.3. In [8, Section 4] the divisibilities are computed in the following cases:

| M | $\text{g.d.}(M)$ | $\text{e.d.}(M)$ |
|---------------------|------------------|------------------|
| \mathbf{P}^n | n | n |
| \mathbf{Q}^{2n-1} | $2n - 1$ | $2n - 1$ |
| \mathbf{Q}^{2n} | n | $2n - 1$ |
| $\mathbf{G}(1, n)$ | $n - 1$ | n |

The last two examples show that, in general $\text{g.d.}(M) < \text{e.d.}(M)$.

The notion of effective good divisibility has been used in [8, Section 4] to study morphisms to rational homogeneous varieties of classical type of Picard number one. The following is a special case of [8, Proposition 4.7], that we report here for the reader's convenience, since it is a key step in the proof of Theorem 1.1.

Proposition 2.4. *Let M be a smooth complex projective variety. If $\text{e.d.}(M) > m$ then any morphism $\varphi : M \rightarrow \mathbf{G}(l, m)$ is constant.*

Proof. Consider the exact sequence of vector

bundles on $\mathbf{G}(l, m)$:

$$(1) \quad 0 \rightarrow \mathcal{S}^\vee \rightarrow \mathcal{O}^{\oplus m+1} \rightarrow \mathcal{Q} \rightarrow 0$$

where \mathcal{S} , of rank $m - l$, and \mathcal{Q} , of rank $l + 1$, are the universal subbundle and the universal quotient bundle. Set

$$\begin{aligned} \lambda_i &= \varphi^* c_i(\mathcal{Q}) & \text{for } i = 0, \dots, \text{rk } \mathcal{Q}, \\ \mu_j &= \varphi^* c_j(\mathcal{S}^\vee) & \text{for } j = 0, \dots, \text{rk } \mathcal{S}^\vee. \end{aligned}$$

The Chern classes of the nef bundles \mathcal{Q} and \mathcal{S} are effective and non zero (see [2, Section 5.6.2]), hence $\lambda_i \neq 0$ is effective for every $i = 0, \dots, \text{rk } \mathcal{Q}$ and $\mu_j \neq 0$ is effective or antieffective for every $j = 0, \dots, \text{rk } \mathcal{S}^\vee$. Set

$$\lambda(t) = \sum_i \lambda_i t^i \quad \mu(t) = \sum_i \mu_i t^i$$

by the exact sequence (1) we have

$$\lambda(t)\mu(t) = 1.$$

Let i_0 and j_0 be the maximum indexes for which $\lambda_{i_0} \neq 0$ and $\mu_{j_0} \neq 0$. If $i_0 + j_0 \neq 0$, then

$$\lambda_{i_0} \mu_{j_0} = 0,$$

but the assumption implies that $i_0 + j_0 \leq \text{e.d.}(M)$, so we must have $i_0 = j_0 = 0$. In particular

$$\lambda_1 = \varphi^* \det \mathcal{Q} = \varphi^* \mathcal{O}_{\mathbf{G}(l,m)}(1) = 0$$

hence φ is constant. □

3. Schubert calculus. We will now recall some basic facts about Schubert calculus in $\mathbf{G}(k, n)$. We refer to [2] and [3] for a complete account on the subject.

Let us identify $\mathbf{G}(k, n)$ with the Grassmannian $G(k + 1, V)$ of vector subspaces of dimension $k + 1$ in a vector space V of dimension $n + 1$; consider a complete flag \mathcal{V} of vectors subspaces of V :

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n \subsetneq V.$$

Given a sequence of integers $a = (a_1, \dots, a_{k+1})$ such that

$$n - k \geq a_1 \geq a_2 \geq \dots \geq a_{k+1} \geq 0$$

the Schubert variety $\Sigma_a(\mathcal{V})$ is defined as

$$\begin{aligned} \Sigma_a(\mathcal{V}) &= \{W \in G(k + 1, n + 1) \mid \\ &\dim(V_{n-k+i-a_i} \cap W) \geq i \text{ for all } i\}. \end{aligned}$$

The codimension of $\Sigma_a(\mathcal{V})$ is $|a| := \sum a_i$, the class $[\Sigma_a(\mathcal{V})] \in H^{2|a|}(\mathbf{G}(k, n))$ does not depend on the choice of \mathcal{V} , will be denoted by σ_a and called a

Schubert cycle. The Schubert cycles form a basis for the cohomology of $\mathbf{G}(k, n)$.

The cycle σ_a may be represented by the corresponding Young diagram, which is a collection of left-justified rows of boxes in which the i -th row has length a_i . The product of two Schubert cycles can be computed using the the Littlewood–Richardson rule, as follows:

$$(2) \quad \sigma_a \cdot \sigma_b = \sum_c \delta_{a,b}^c \sigma_c$$

where $|c| = |a| + |b|$, the Young diagram of c contains the Young diagram of a and $\delta_{a,b}^c$ is equal to the number of Littlewood–Richardson tableaux of skew shape c/a and of weight b , i.e., to the number of ways in which on can fill the boxes of c/a with integers $1, \dots, k + 1$ in such a way that:

- the integer i appears b_i times;
- the integers are non decreasing along the rows (from left to right);
- the integers are strictly increasing along the columns (from up to down);
- the word obtained concatenating the reversed rows is a lattice word, that is, in every initial part of the sequence any number i occurs at least as often as the number $i + 1$.

4. Proofs. In this section we will prove Theorem 1.1 and Corollary 1.2. In view of Proposition 2.4, to prove Theorem 1.1 it is enough to show that the effective good divisibility of $\mathbf{G}(k, n)$ is n . We start with a special case.

Proposition 4.1. *Let σ_a and σ_b be two Schubert cycles in $\mathbf{G}(k, n)$, such that $|a| + |b| \leq n$. Then $\sigma_a \cdot \sigma_b > 0$.*

Proof. Via the duality of \mathbf{P}^n we have isomorphisms $\mathbf{G}(k, n) \simeq \mathbf{G}(n - k - 1, n)$, so we may assume that $2k \leq n - 1$. In particular

$$(3) \quad n - k > n/2,$$

hence

$$(4) \quad |a| + |b| \leq n < 2(n - k).$$

By formula (2), since all the $\delta_{a,b}^c$ are nonnegative and the Schubert cycles are effective, it is enough to prove that $\delta_{a,b}^c > 0$ for some c .

We may assume that $a_1 + b_1 > n - k$, otherwise taking $c = a + b$ and numbering the boxes in c/a with the number of the row to which they belong, we find a Littlewood–Richardson tableaux of skew shape c/a and of weight b . By formula (4) we then

have

$$a_i + b_j < n - k \text{ for every } i, j \geq 2.$$

In particular

$$(5) \quad b_j < n - k - a_2 \text{ for every } j \geq 2$$

and

$$\sum_{i=2}^{k+1} (a_i + b_i) < k.$$

Thus $a_{k+1} = b_{k+1} = 0$.

We can also assume, up to swap the cycles, that $|b| < |a|$; in particular $b_1 < n - k$.

Set $\bar{a} = (n - k - a_1, a_1 - a_2, \dots, a_k - a_{k+1})$, and consider the skew shape \bar{a}/a , which contains $n - k$ boxes, no two of which on the same row.

Mark b_1 boxes of \bar{a}/a with the integer 1, starting from the first row and moving from left to right. Denote by β_j^1 the number of boxes in the j -th row that have been marked with 1's and by β^1 the skew Tableaux consisting of the marked boxes in \bar{a}/a .

Set $a_1 = (a_1 + \beta_1^1, \dots, a_{k+1} + \beta_{k+1}^1)$, and consider the sequence

$$\bar{a}^1 = a^1 + (0, \beta_1^1, \beta_2^1, 0, \dots, 0)$$

and the skew shape \bar{a}^1/a^1 , which contains $\beta_1^1 + \beta_2^1$ boxes, no two of which are on the same row, since $a_3 + \beta_3^1 \leq a_2$. Mark b_2 boxes of \bar{a}^1/a^1 with the integer 2, starting from the first non empty row and moving from left to right. This is possible since $\beta_1^1 + \beta_2^1 = \min\{b_1, n - k - a_2\}$, so, by (5), we have $b_2 \leq \beta_1^1 + \beta_2^1$. Denote by β_j^2 the number of boxes in the j -th row that have been marked with 2's and by β^2 the skew Tableaux consisting of the marked boxes in \bar{a}^1/a^1 .

Repeat the procedure, starting at the step i from the sequence

$$\bar{a}^{i-1} = a^{i-1} + (0, \dots, 0, \beta_1^i, \beta_2^i, 0, \dots, 0)$$

where the two nonzero entries are the ones in positions $i, i + 1$. Consider the skew shape \bar{a}^{i-1}/a^{i-1} , which contains $\beta_1^{i-1} + \beta_2^{i-1}$ boxes, no two of which are on the same row, since $a_{i+1} + \beta_{i+1}^i \leq a_i$.

Mark b_i boxes of this skew shape with the integer i , starting from the first non empty row and moving from left to right. This is possible since $\beta_1^{i-1} + \beta_2^{i-1} = b_{i-1}$, so, $b_i \leq \beta_1^{i-1} + \beta_2^{i-1}$. Denote by β_j^i the number of boxes in the j -th row that have been marked with i 's and by β^i the skew tableaux

consisting of the marked boxes in \bar{a}^{i-1}/a^{i-1} . Set $a^i = a^{i-1} + \beta^i$.

Set $c = a^k$; we claim that $\delta_{a,b}^c > 0$. The entries of the sequence c are

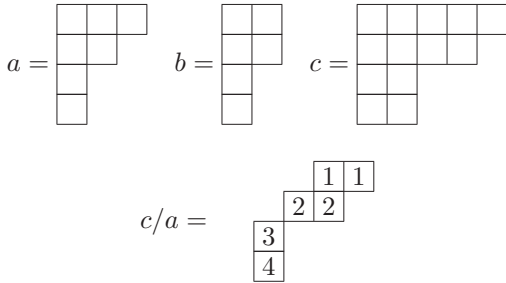
$$\begin{aligned} c_1 &= a_1 + \beta_1^1 = n - k, \\ c_2 &= a_2 + \beta_2^1 + \beta_1^2, \\ &\dots\dots \\ c_j &= a_j + \beta_j^1 + \beta_2^{j-1} + \beta_1^j, \\ &\dots\dots \\ c_{k+1} &= \beta_{k+1}^1 + \beta_2^k. \end{aligned}$$

By construction $\beta_j^1 + a_j \leq a_{j-1}$ and $\beta_j^i \leq \beta_j^{i-1}$, hence the sequence is not increasing.

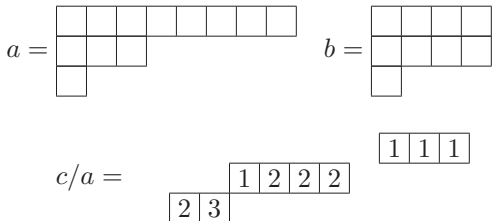
The Young diagram of the skew shape c/a is the diagram of the tableaux $\beta^1 + \dots + \beta^k$; in this tableaux by construction the rows are non decreasing and the columns are strictly increasing.

The fact that the word obtained concatenating the reversed rows is a lattice word follows from the inequalities $\beta_j^i \leq \beta_j^{i-1}$, which say that the number of $i - 1$ in row $j + i - 1$ is bigger than or equal to the number of i in row $j + i$. \square

Example 4.2. The following is an example with $n = 9, k = 4$, to illustrate the case $a_1 + b_1 \leq n - k$ of the proof of Proposition 4.1.



Example 4.3. The following is an example with $n = 21, k = 10$, to illustrate the case $a_1 + b_1 > n - k$ of the proof of Proposition 4.1.



Remark 4.4. As suggested by an anonymous referee, Proposition 4.1 can be proved using the fact that the product of two Schubert cycles in

the (small) quantum cohomology ring of the Grassmannian $\mathbf{G}(k, n)$ is never zero (see [1, Footnote 3, p. 828] for a proof of this statement). In fact, the quantum variable q has degree $n + 1$ (equal to the index of $\mathbf{G}(k, n)$, see [5, Section 6]), so in degree smaller than or equal to n the quantum product coincides with the cup product.

The nonvanishing of the product of Schubert cycles in the (small) quantum cohomology ring has been proved in [5, Theorem 9.1] for any rational homogeneous space, so with a similar argument one can obtain lower bounds on $\text{e.d.}(G/P)$ for every G/P . However, in general, these bounds are not optimal: for example, in the case of \mathbf{Q}^{2n-1} one gets $\text{e.d.}(\mathbf{Q}^{2n-1}) \geq 2n - 2$, while $\text{e.d.}(\mathbf{Q}^{2n-1}) = 2n - 1$.

We are now ready to compute the effective good divisibility of $\mathbf{G}(k, n)$.

Theorem 4.5. *The effective good divisibility of $\mathbf{G}(k, n)$ is n .*

Proof. By [4, Corollary of Theorem 1], as explained in [6, Section 3], the cones of effective classes of a fixed codimension i in $\mathbf{G}(k, n)$ are polyhedral cones generated by the Schubert classes of the same codimension. Therefore, if $\Gamma_i \in H^{2i}(\mathbf{G}(k, n))$, $\Delta_j \in H^{2j}(\mathbf{G}(k, n))$ are two effective nonzero classes we may write:

$$\Gamma_i = \sum_{|a|=i} \lambda_a \sigma_a, \quad \Delta_j = \sum_{|b|=j} \mu_b \sigma_b,$$

where $\lambda_a, \mu_b \geq 0$.

Assume that $i + j \leq n$; we want to show that $\Gamma_i \Delta_j \neq 0$. Since every intersection $\sigma_a \sigma_b$ is a combination of Schubert cycles, with nonnegative coefficients, due to the Littlewood–Richardson rule, it is enough to prove the statement for $\Gamma_i = \sigma_a$ and $\Delta_j = \sigma_b$, and this has been done in Proposition 4.1. Therefore $\text{e.d.}(\mathbf{G}(k, n)) \geq n$.

To show that $\text{e.d.}(\mathbf{G}(k, n)) = n$ it is enough to consider $a = (1, 1, \dots, 1)$ and $b = (n - k, 0, \dots, 0)$. For the corresponding cycles we have $\sigma_a \sigma_b = 0$, since there is no way of adding $n - k$ boxes marked with 1 to the diagram of a without having two boxes in the same column. \square

As observed at the beginning of the section, Theorem 1.1 follows combining Theorem 4.5 and Proposition 2.4.

Proof of Corollary 1.2. A rational homogeneous variety X of type A_m is a variety of partial flags of linear subspaces of dimensions (l_1, \dots, l_i) in the projective space \mathbf{P}^m .

The variety X has Picard number i , and morphisms $\pi_j: X \rightarrow \mathbf{G}(l_j, m)$ for every $j = 1, \dots, i$, which send a partial flag $\Lambda^1 \subset \Lambda^2 \subset \dots \subset \Lambda^i$ to the point of $\mathbf{G}(l_j, m)$ parametrizing Λ^j .

If $\varphi: \mathbf{G}(k, n) \rightarrow X$ is a morphism, then $\pi_j \circ \varphi$ is constant for every $j = 1, \dots, i$ by Theorem 1.1, hence φ is constant. \square

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