Local universality of determinantal point processes on Riemannian manifolds

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Abstract: We consider the Laplace-Beltrami operator Δ_g on a smooth, compact Riemannian manifold (M, g) and the determinantal point process $\mathcal{X}_\lambda$ on M associated with the spectral projection of $-\Delta_g$ onto the subspace corresponding to the eigenvalues up to λ^2. We show that the pull-back of $\mathcal{X}_\lambda$ by the exponential map $\exp_p : T_p^*M \to M$ under a suitable scaling converges weakly to the universal determinantal point process on T_p^*M as $\lambda \to \infty$.

Key words: Determinantal point process on Riemannian manifolds; local universality; spectral projection; pointwise Weyl law; Euclidean motion group; Bessel functions.

1. Introduction. Let (M, g) be a smooth, compact, Riemannian manifold of dimension m with no boundary. We fix an orthonormal basis $\{\varphi_i\}_{i\geq 0}$ of eigenfunctions of the Laplace-Beltrami operator Δ_g acting on $L^2(M) := L^2(M, \mathrm{vol}_g)$:

$$-\Delta_g \varphi_i = \lambda_i^2 \varphi_i, \quad \langle \varphi_i, \varphi_j \rangle_{L^2(M)} = \delta_{ij},$$

with $0 = \lambda_0^2 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \nearrow +\infty$. Here $\{\lambda_i\}_{i=0}^\infty$ are the eigenvalues of $\sqrt{-\Delta_g}$. We denote the eigenspace corresponding to an eigenvalue λ_i by W_{λ_i}. The projection operator E_λ on $L^2(M, \mathrm{vol}_g)$ onto the closed subspace $W_{\leq\lambda} := \bigoplus_{\lambda_i \leq \lambda} W_{\lambda_i}$ admits the following integral kernel

$$E_\lambda(x, y) = \sum_{\lambda_i \leq \lambda} \varphi_i(x)\overline{\varphi_i(y)} \quad (x, y \in M). \tag{1.1}$$

The projection kernel $E_\lambda(x, y)$ is the reproducing kernel of $W_{\leq\lambda}$ and thus defines a determinantal point process (DPP) $\mathcal{X}_\lambda$ on M, which is a random simple point configuration on M whose n-point correlation function with respect to vol_g is given by

$$\rho_n(x_1, x_2, \dots, x_n) = \det(E_\lambda(x_i, x_j))_{i,j=1}^n.$$

In particular, the 1-point correlation function, the density of points, is

$$\rho_1(x) = E_\lambda(x, x).$$

See Section 2 for the definition of DPP.

The number of points in $\mathcal{X}_\lambda$ on M is equal to the eigenvalue counting function given by

$$N(\lambda) = \sum_{\lambda_i \leq \lambda} 1 = \mathrm{rank}\, E_\lambda = \int_M E_\lambda(x, x)\mathrm{vol}_g(dx).$$

Since $\{\lambda_i\}_{i=0}^\infty$ are the eigenvalues of $\sqrt{-\Delta_g}$, it is known as the classical Weyl law (cf. [11]) that

$$N(\lambda) \sim \frac{\lambda^m}{(2\pi)^m} |B_1^{(m)}| \,\mathrm{vol}_g(M) \quad (\lambda \to \infty), \tag{1.2}$$

where $|B_1^{(m)}|$ is the volume of a unit ball in $\mathbf{R}^m$, i.e., $|B_1^{(m)}| = \pi^{m/2}/\Gamma(m/2 + 1)$. This means that the points in $\mathcal{X}_\lambda$ on M become dense as $\lambda \to \infty$.

Example 1. When $M = \mathbf{S}^1$, for every $\lambda > 0$, the DPP associated with E_λ is the random eigenvalues of Circular Unitary Ensemble (CUE) of size $N(\lambda)$ (cf. [9]). More generally, when $M = \mathbf{S}^m$, the corresponding DPPs are called *harmonic ensembles* on $\mathbf{S}^m$ (cf. [13]). These point processes are homogeneous in the sense that they are invariant under the $\mathrm{O}(m)$-action.

The quantum ergodicity theorem originated by Šnirel'man [19,20] and also studied in [6,22] states that if the geodesic flow on M is ergodic then $N(\lambda)^{-1}E_\lambda(x, x)\,\mathrm{vol}_g(dx)$ converges weakly to $\mathrm{vol}_g(dx)$ as $\lambda \to \infty$, in other words, so does the normalized first correlation measure of the DPP $\mathcal{X}_\lambda$. This theorem describes the global behavior of random points of the DPP on M.

In this paper, we focus on the local statistics of points in the DPP by taking a scaling as in (1.4) below so that we define a DPP $\Xi_{\lambda,p}$ on the cotangent space T_p^*M by taking the pull-back of the DPP $\mathcal{X}_\lambda$ on M by the exponential map.

We denote the Riemannian metric on T_p^*M by $\langle \cdot, \cdot \rangle_{g_p} : T_p^*M \times T_p^*M \to \mathbf{R}$ and the corresponding norm by $|\cdot|_{g_p}$. Here $|\xi|_{g_p}$ is the same as the principal symbol of $\sqrt{-\Delta_g}$ locally given by

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$$|\xi|_{g_p} = \left(\sum_{i,j=1}^{m} g^{ij}(p)\xi_i\xi_j \right)^{1/2},$$

and $(g^{ij}(p))_{i,j=1}^m$ is the inverse matrix g_p^{-1} of $g_p = (g_{ij}(p))_{i,j=1}^m$. The so-called pointwise Weyl law can be expressed as follows: as $\lambda \to \infty$,

$$E_\lambda(x,x) = \frac{1}{(2\pi)^m} \int_{|\xi|_{g_x} < \lambda} \frac{d\xi}{\sqrt{\det g_x}} + R_\lambda(x) = \frac{|B_1^{(m)}|}{(2\pi)^m} \lambda^m + R_\lambda(x) \tag{1.3}$$

with the uniform bound $\sup_{x \in M} |R_\lambda(x)| \le C\lambda^{m-1}$ [11], which leads to the classical Weyl law (1.2).

Since M is compact, the injectivity radius $\mathrm{inj}^*(M)$ is positive, i.e., the exponential map $\exp_p : T_p^*M \to M$ is injective on the subset $\{\xi \in T_p^*M : |\xi|_{g_p} < \mathrm{inj}^*(M)\}$ for any $p \in M$. We fix a point $p \in M$ and positive $\epsilon < \mathrm{inj}^*(M)$. Let B_ϵ be the open ball of radius ϵ in T_p^*M centered at the origin and denote the image $\exp_p(B_\epsilon)$ by $\mathcal{B}_{p,\epsilon}$. For $\lambda > 0$, we define a point process $\Xi_{\lambda,p,\epsilon}$ on the cotangent space T_p^*M by

$$\Xi_{\lambda,p,\epsilon} := \sum_{x \in \mathcal{X}_\lambda \cap \mathcal{B}_{p,\epsilon}} \delta_{\lambda \exp_p^{-1}(x)}, \tag{1.4}$$

which defines the pull-back of $\mathcal{X}_\lambda$ restricted on $\mathcal{B}_{p,\epsilon}$ by the exponential map and is scaled by λ. Here, we identified $\mathcal{X}_\lambda$ with a subset in M (see Section 2.1). It turns out again to be a DPP on T_p^*M (see Lemma 3).

Our main assertion in this paper is the following

Theorem 1. *As $\lambda \to \infty$, the point process $\Xi_{\lambda,p,\epsilon}$ converges weakly to the DPP Ξ_p on T_p^*M associated with the kernel*

$$K_{g_p}^{(m)}(u,v) = \frac{1}{(2\pi|u-v|_{g_p})^{m/2}} J_{m/2}(|u-v|_{g_p}) \tag{1.5}$$

and the reference measure $\mathrm{vol}_{G^{(p)}}$, where $J_\alpha(x)$ is the Bessel function of the first kind defined by

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha}$$

and $\mathrm{vol}_{G^{(p)}}$ is the Riemannian measure on T_p^*M with respect to the constant Riemannian metric $G^{(p)} = (G_u^{(p)})_{u \in T_p^*M}$ such that $G_u^{(p)} = (\exp_p^* g)_0$ for every $u \in T_p^*M$.

We remark that the limiting DPP Ξ_p does not depend on $\epsilon > 0$.

We consider the following correlation kernel on $\mathbf{R}^m$,

$$K^{(m)}(u,v) := \frac{1}{(2\pi|u-v|)^{m/2}} J_{m/2}(|u-v|) = \frac{1}{(2\pi)^m} \int_{|\xi|<1} e^{\sqrt{-1}(u-v,\xi)} d\xi,$$

where $(\cdot,\cdot)$ (resp. $|\cdot|$) is the standard inner product (resp. norm) on $\mathbf{R}^m$. The DPP on $\mathbf{R}^m$ associated with $K^{(m)}(u,v)$ is invariant under the action of the Euclidean motion group. When $m = 1$, $K^{(1)}(u,v)$ coincides with the sinc kernel

$$K^{(1)}(u,v) = \frac{\sin(u-v)}{\pi(u-v)},$$

which is the reproducing kernel of the classical Paley-Wiener space (see also Example 2 for $K^{(m)}$ given in Section 2.2). It is well known that the point process of eigenvalues of CUE (also GUE) under suitable scaling converges to the DPP associated with the sinc kernel. This DPP is also one of the most important examples of the class of DPPs associated with de Branges spaces discussed in [3]. In [13], we proved a special case of Theorem 1 when $M = \mathbf{S}^m$ by using spherical harmonics. Theorem 1 can be regarded as a generalization of these results to compact Riemannian manifolds. For the proof of Theorem 1, the pointwise Weyl law (1.3) plays a central role.

Theorem 1 shows the local universality of DPPs on Riemannian manifolds. This type of universality has been discussed as the asymptotic local structure of Szegő kernels, which is used to analyze random spherical harmonics and random section of holomorphic line bundles over a compact Kähler manifold. The former corresponds to the Euclidean class (real case) while the latter does the Heisenberg class (complex case) (cf. [2,23,24]). The terms "Euclidean" and "Heisenberg" are related to representations of the Euclidean and Heisenberg motion groups. The result in this paper falls in the Euclidean class in this terminology. It is shown in [8] that DPPs associated with certain Schrodinger operators on $\mathbf{R}^n$ fall in Euclidean class in the semiclassical limit. Theorem 1 could also be generalized to the case where the spectral projections of Laplace-Beltrami operators are replaced by those of general elliptic operators.

2. Determinantal point processes.

For the necessary background for determinantal point processes, see e.g. [12,14–18,21].

2.1. Definition.

Let S be a locally compact

Hausdorff space with countable base. A configuration Ξ on S is a non-negative integer-valued Radon measure and it can be expressed as $\Xi = \sum_i \delta_{x_i}$ $(x_i \in S)$. We denote by $\mathrm{Conf}(S)$ the totality of configurations on S, which we call a configuration space over S. An element Ξ of $\mathrm{Conf}(S)$ is sometimes regarded as an at most countable subset in S without accumulation, possibly with multiple points. Thus, $\Xi(A)$ is equal to the number of points in $A \in \mathscr{B}(S)$ with counted multiplicity, where $\mathscr{B}(S)$ is the totality of all bounded (i.e., relatively compact) sets in S. The configuration space $\mathrm{Conf}(S)$ equipped with vague topology turns out to be a Polish space, i.e., a complete, separable metrizable space. We equip the configuration space $\mathrm{Conf}(S)$ with the Borel structure with respect to this topology, which coincides with the Borel structure generated by the mapping $\mathrm{Conf}(S) \ni \Xi \mapsto \Xi(A) \in \mathbf{Z}_{\geq 0} := \{0, 1, 2, \dots\}$ for all bounded $A \in \mathscr{B}(S)$. A point process on S is a $\mathrm{Conf}(S)$-valued random variable $\Xi = \Xi_\omega$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If $\Xi(\{x\}) \leq 1$ for every $x \in S$ a.s., then Ξ is called a simple point process. In this case, by identifying Ξ with its support, we use the notation $x \in \Xi$ meaning that $\Xi(\{x\}) = 1$.

We fix a Radon measure ν on S as a reference measure. A symmetric measure ν_n on S^n is called the n-th *correlation measure* if it satisfies

$$\mathbf{E}\left[\prod_{i=1}^{p} \frac{\Xi(A_i)!}{(\Xi(A_i) - k_i)!}\right] = \nu_n(A_1^{k_1} \times \cdots \times A_p^{k_p})$$

for any disjoint bounded sets $A_1, \dots, A_p \in \mathscr{B}(S)$ and any $k_1, \dots, k_p \in \mathbf{Z}_{\geq 0}$ with $\sum_{i=1}^{p} k_i = n$. If ν_n is absolutely continuous with respect to the product measure $\nu^{\otimes n}$, the Radon-Nikodym derivative $\rho_n := d\nu_n / d\nu^{\otimes n}$ is called the *n-point correlation function* with respect to the reference measure ν;

$$\nu_n(dx_1 \dots dx_n) = \rho^n(x_1, \dots, x_n)\nu^{\otimes n}(dx_1 \dots dx_n).$$

Let $\mathscr{I}_1(S, \nu)$ be the ideal of trace class operators $K\colon L_2(S, \nu) \to L_2(S, \nu)$; we denote the $\mathscr{I}_1$-norm of the operator K by $||K||_{\mathscr{I}_1}$. Let $\mathscr{I}_{1,\mathrm{loc}}(S, \nu)$ be the space of operators $K\colon L_2(S, \nu) \to L_2(S, \nu)$ such that $\mathbf{1}_A K \mathbf{1}_A \in \mathscr{I}_1(S, \nu)$ for any bounded Borel subset $A \subset S$, where $\mathbf{1}_A$ is the indicator function of a set A. Such an operator K is called a locally trace class operator. We endow the space $\mathscr{I}_{1,\mathrm{loc}}(S, \nu)$ with a countable family of semi-norms $\|\mathbf{1}_A K \mathbf{1}_A\|_{\mathscr{I}_1}$ where A runs through an exhausting family A_n of bounded sets, i.e., A_n is increasing and $\bigcup_{n=1}^{\infty} A_n = S$. A locally trace class operator K admits a kernel (cf. [10,14]), for which, slightly abusing notation, we use the same symbol K.

A point process is called a *determinantal point process associated with* K *and* ν if there exists an operator $K \in \mathscr{I}_{1,\mathrm{loc}}(S, \nu)$ such that for any bounded measurable function h, for which $h - 1$ is supported in a bounded set A, we have

$$\mathbf{E}\Psi_h = \det(1 + (h - 1)K\mathbf{1}_A), \tag{2.1}$$

where $\Psi_h(\Xi) = \prod_{x \in \Xi} h(x)$ for $\Xi \in \mathrm{Conf}(S)$. The Fredholm determinant in (2.1) is well-defined since $K \in \mathscr{I}_{1,\mathrm{loc}}(S, \nu)$. For example, if K is a positive contraction operator $K \in \mathscr{I}_{1,\mathrm{loc}}(S, \nu)$, then there exists a DPP associated with K and ν. Equation (2.1) determines the law of the DPP uniquely ([16,17,21]). For the DPP associated with K, the n-th correlation function with respect to ν is given by

$$\rho_n(x_1, \dots, x_n) = \det(K(x_i, x_j))_{i,j=1}^{n}.$$

$K(x, y)$ is often called the correlation kernel and ν the reference measure. When $S = \mathbf{R}^m$, if ν is the Lebesgue measure and $K(x, y) = k(x - y)$ for some k, then the law of the DPP associated with K and ν is invariant under the action of the Euclidean motion group.

Weak convergence for DPPs is characterized by the convergence of operators (cf. Proposition 3.10 in [17]) as follows:

Lemma 2. *Let* Ξ_n *(resp.* Ξ*) be a DPP on* S *associated with* K_n *(resp.* K*) and* ν*. Suppose* K_n *converges to* K *in* $\mathscr{I}_{1,\mathrm{loc}}(S, \nu)$ *as* $n \to \infty$*. Then* Ξ_n *converges weakly to* Ξ *as* $n \to \infty$*. In particular, if the kernel* $K_n(x, y)$ *converges to* $K(x, y)$ *uniformly on any compact set in* $S \times S$*, then the convergence above takes place.*

2.2. DPPs associated with reproducing kernel Hilbert spaces. Let S be a non-empty set and $\mathcal{F}(S)$ be the linear space of all complex-valued functions on S, i.e., $\mathcal{F}(S) := \{f : S \to \mathbf{C}\}$. A subspace H of $\mathcal{F}(S)$ is called a reproducing kernel Hilbert space (RKHS) if H is endowed with an inner product $\langle \cdot, \cdot \rangle_H$ which makes H a Hilbert space and the evaluation functional $E_s : H \to \mathbf{C}$ defined by $E_s(f) := f(s)$ is bounded for every $s \in S$. By the Riesz representation theorem, for each $s \in S$, there exists a unique element $k_s \in H$ such that $E_s(f) = \langle f, k_s \rangle_H = f(s)$. We define a kernel $K : S \times S \to \mathbf{C}$

by

$$K(s,t) := k_t(s) = \langle k_t, k_s\rangle_H,$$

which is called the reproducing kernel for H (see [1] for more details about RKHS). The integral operator K with kernel $K(s,t)$ defines an orthogonal projection onto H. Therefore, the DPP is associated with reproducing kernel $K(s,t)$, or equivalently, RKHS H.

Example 2. (1) For a given $a > 0$,

$$\mathrm{PW}_a := \{f \in C(\mathbf{R}) : \operatorname{supp}\widehat{f} \subset [-a,a]\}$$

is called the Paley-Wiener space or the space of band-limited functions. Here and in (2) below $\widehat{f}$ is the Fourier transform of f defined as

$$\widehat{f}(\xi) := \int_{\mathbf{R}^m} f(x)e^{-\sqrt{-1}(x,\xi)}dx.$$

The corresponding reproducing kernel K_a is given by

$$K_a(x,y) = \frac{\sin a(x-y)}{\pi(x-y)}$$

and the corresponding DPP is the limiting DPP obtained from CUE (also GUE) eigenvalues.
(2) A generalized Paley-Wiener space is similarly defined as follows: for a bounded Borel set $\Omega \subset \mathbf{R}^m$,

$$\mathrm{PW}_\Omega := \{f \in C(\mathbf{R}^m) : \operatorname{supp}\widehat{f} \subset \overline{\Omega}\}.$$

When $\Omega = B_1^{(m)} \subset \mathbf{R}^m$, the corresponding reproducing kernel is $K^{(m)}(x,y)$ which appeared in Theorem 1.
(3) Let (M,g) be a compact, smooth, Riemannian manifold and Δ_g be the Laplace-Beltrami operator on $L^2(M, \mathrm{vol}_g)$. We denote the resolution of the identity for Δ_g by $\{E(A) : A \in \mathscr{B}(\mathbf{R})\}$. Then the integral operator E_λ with kernel $E_\lambda(x,y)$ given in (1.1) coincides with the projection $E([0,\lambda^2])$ and $W_{\leq\lambda}$ turns out to be the RKHS admitting the reproducing kernel $E_\lambda(x,y)$.

3. Proof of the main theorem. We define $\phi_\lambda : T_p^*M \to M$ by $\phi_\lambda(u) = \exp_p(u/\lambda)$ for $u \in T_p^*M$. For $u,v \in T_p^*M$, we write $U_\lambda = \phi_\lambda(u)$ and $V_\lambda = \phi_\lambda(v)$. We consider the kernel

$$\begin{aligned}(3.1)\quad & K_{\lambda,p,\epsilon}(u,v) \\ &= \frac{1}{\lambda^m}E_\lambda(U_\lambda,V_\lambda)\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(U_\lambda)\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(V_\lambda).\end{aligned}$$

We have the following

Lemma 3. *The scaled point process $\Xi_{\lambda,p,\epsilon}$ defined by (1.4) is the DPP on T_p^*M associated with the kernel $K_{\lambda,p,\epsilon}(u,v)$ of (3.1) and $\lambda^m\phi_\lambda^*\mathrm{vol}_g$.*

Proof. We note that $\mathcal{X}_\lambda|_{\mathcal{B}_{p,\epsilon}}$ is the DPP associated with the kernel $E_\lambda(x,y)\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(x)\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(y)$ and the reference measure vol_g. Then the pull-back $\phi_\lambda^*\mathcal{X}_\lambda|_{\mathcal{B}_{p,\epsilon}}$ is the DPP associated with the kernel $E_\lambda(\phi_\lambda(u),\phi_\lambda(v))\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(\phi_\lambda(u))\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(\phi_\lambda(v))$ and $\phi_\lambda^*\mathrm{vol}_g$ since $\phi_\lambda|_{B_\epsilon} : B_\epsilon \to \mathcal{B}_{p,\epsilon}$ is a diffeomorphism. The law of this DPP is the same as that of the DPP associated with the kernel (3.1) and $\lambda^m\phi_\lambda^*\mathrm{vol}_g$ through the measure change by the factor λ^m (cf. [14, Section 2.3]). □

We remark that since $(d\phi_\lambda)_u = \lambda^{-1}(d\phi_1)_{u/\lambda}$, the pull-back of the Riemannian metric g on M is expressed as $\lambda^2(\phi_\lambda^*g)_u = (\phi_1^*g)_{u/\lambda}$ for $u \in T_p^*M$. Therefore, $\lambda^m\phi_\lambda^*\mathrm{vol}_g$ is equal to the Riemannian measure with respect to $(\phi_1^*g)_{\cdot/\lambda}$. For the proof of Theorem 1, we appeal to the pointwise Weyl law (1.3), which gives an off-diagonal asymptotics for the projection kernel $E_\lambda(x,y)$ as $\lambda \to \infty$ as follows: if x is close enough to y, i.e., $x \in \mathcal{B}_{y,\epsilon}$ with $\epsilon < \mathrm{inj}^*(M)$, then

$$\begin{aligned}(3.2)\quad E_\lambda(x,y) = \frac{\lambda^m}{(2\pi)^m}\int_{|\xi|_{g_y}<1} & e^{\sqrt{-1}\lambda\psi(x,y,\xi)}\frac{d\xi}{\sqrt{\det g_y}} \\ &+ R_\lambda(x,y),\end{aligned}$$

where $\psi(x,y,\xi)$ is a phase function which is adapted, in Hörmander's terminology [11], to the principal symbol $|\xi|_{g_y}$ of $\sqrt{-\Delta_g}$, vanishing on the diagonal $x = y$. This type of asymptotics for the spectral function was initiated by Hörmander [11] as an application of the theory of pseudo-differential operators and recovers the classical Weyl law (1.2). The choice of a phase function is not unique, and one can take

$$(3.3)\qquad \psi(x,y,\xi) = \langle \exp_y^{-1}(x), \xi\rangle_{g_y}$$

in a coordinate-independent way [4,25]. Indeed, the integral on the right-hand side of (3.2) with (3.3) is taken over the cotangent fiber T_y^*M and it is coordinate-independent since the measure $d\xi/\sqrt{\det g_y}$ is the quotient of the canonical symplectic form $d\xi \wedge dy$ on T^*M by the Riemannian volume form $\sqrt{\det g_y}dy$ on M. There are many papers estimating the remainder term $R_\lambda(x,y)$. From [4, Theorem 2], the remainder term is uniformly estimated as follows:

Theorem 4 ([4,5,11])**.** *We assume (3.3). Then, for any fixed $r > 0$, as $\lambda \to \infty$,*

$$\sup_{d_g(x,y)<r/\lambda} |R_\lambda(x,y)| = O(\lambda^{m-1}),$$

where $d_g(x,y)$ is the Riemannian distance.

Before giving a proof of the main theorem, we see a generalization of the following formula (cf. [14])

$$\text{(3.4)} \qquad \frac{1}{(2\pi)^{m/2}} \int_{|\omega|<1} e^{\sqrt{-1}(\eta,\omega)} d\omega = F_{m/2}(|\eta|),$$

where $F_\alpha(t) = J_\alpha(t)/t^\alpha$ for $\alpha > 0$.

Lemma 5. *Let $m = \dim M$. For $\eta \in T_p^*M$,*

$$\frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_p}<1} e^{\sqrt{-1}\langle\eta,\xi\rangle_{g_p}} \frac{d\xi}{\sqrt{\det g_p}} = F_{m/2}(|\eta|_{g_p}).$$

Proof. We note that

$$\langle\eta,\xi\rangle_{g_p} = (g_p^{-1/2}\eta, g_p^{-1/2}\xi),$$

where $g_p^{-1/2}$ is the positive definite square root of the inverse matrix g_p^{-1}. In particular, $|\eta|_{g_p} = |g_p^{-1/2}\eta|$. From (3.4), by change of variables $\omega = g_p^{-1/2}\xi$, we have

$$\begin{aligned} F_{m/2}(|\eta|_{g_p}) &= \frac{1}{(2\pi)^{m/2}} \int_{|\omega|<1} e^{\sqrt{-1}(g_p^{-1/2}\eta,\omega)} d\omega \\ &= \frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_p}<1} e^{\sqrt{-1}\langle\eta,\xi\rangle_{g_p}} \frac{d\xi}{\sqrt{\det g_p}}. \end{aligned}$$

We obtain the assertion. □

Remark 1. We have a similar formula

$$\frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_p}=1} e^{\sqrt{-1}\langle\eta,\xi\rangle_{g_p}} \frac{d\xi}{\sqrt{\det g_p}} = F_{(m-2)/2}(|\eta|_{g_p}).$$

We need one more fact for the local behavior of the Riemannian distance function.

Lemma 6. *For $u, v \in T_p^*M$, let c_1 and c_2 be C^1 curves in M such that $c_1(0) = c_2(0) = p$, $c_1'(0) = u$ and $c_2'(0) = v$. Then,*

$$\lim_{t\to 0+} \frac{d_g(c_1(t), c_2(t))}{t} = |u - v|_{g_p}.$$

Proof. See Corollary 3.1 in [7] for instance. □

Now we are in a position to give a proof of the main theorem.

Proof of Theorem 1. It suffices to show that the DPP associated with $K_{\lambda,p,\epsilon}(u,v)$ and $\lambda^m \phi_\lambda^* \mathrm{vol}_g$ converges as $\lambda \to \infty$. Suppose $d_g(x,y)$ is small enough. First we note that there exists $\zeta \in T_y^*M$ such that $|\zeta|_{g_y} = 1$ and $\exp_y^{-1}(x) = d_g(x,y)\zeta$. By using Lemma 5, we see that

$$\begin{aligned} &\frac{1}{(2\pi)^{m/2}} \int_{|\xi|_{g_y}<1} e^{\sqrt{-1}\langle \exp_y^{-1}(x),\xi\rangle_{g_y}} \frac{d\xi}{\sqrt{\det g_y}} \\ &\quad = F_{m/2}(d_g(x,y)). \end{aligned}$$

From Lemma 3 with (3.1), (3.2) with (3.3), Theorem 4 and Lemma 5, as $\lambda \to \infty$, we have

$$\begin{aligned} &K_{\lambda,p,\epsilon}(u,v) \\ &\quad = \frac{1}{(2\pi)^m} \int_{|\xi|_{g_{V_\lambda}}<1} e^{\sqrt{-1}\lambda\langle \exp_{V_\lambda}^{-1}(U_\lambda),\xi\rangle_{g_{V_\lambda}}} \frac{d\xi}{\sqrt{\det g_{V_\lambda}}} \\ &\quad\quad \times \mathbf{1}_{\mathcal{B}_{p,\epsilon}}(U_\lambda)\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(V_\lambda) + O(\lambda^{-1}) \\ &\quad = \frac{1}{(2\pi)^{m/2}} F_{m/2}(\lambda d_g(U_\lambda, V_\lambda))\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(U_\lambda)\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(V_\lambda) \\ &\quad\quad + O(\lambda^{-1}). \end{aligned}$$

We note that $\lim_{t\to 0} F_\alpha(t) = 2^{-\alpha}\Gamma(\alpha+1)^{-1}$ and so $F_\alpha(t)$ is a bounded continuous function on $\mathbf{R}$. Since $\lambda d_g(U_\lambda, V_\lambda) \to |u-v|_{g_p}$ by Lemma 6 and $\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(U_\lambda)\mathbf{1}_{\mathcal{B}_{p,\epsilon}}(V_\lambda)$ is equal to 1 for any sufficiently large λ, we have

$$K_{\lambda,p,\epsilon}(u,v) \to \frac{1}{(2\pi)^{m/2}} F_{m/2}(|u-v|_{g_p})$$

uniformly on any compact set in T_p^*M. From the remark after Lemma 3, the reference measure is the Riemannian measure with respect to $(\phi_1^* g)_{\cdot/\lambda}$ and the Radon-Nikodym derivative relative to the Riemannian measure with respect to $(\phi_1^* g)_0$ is uniformly close to 1 on any compact set as $\lambda \to \infty$. Therefore, it follows from Lemma 2 that the scaled point process $\Xi_{\lambda,p,\epsilon}$ converges weakly to the DPP associated with the kernel $K_{g_p}^{(m)}(u,v)$ given by (1.5) and the reference measure given by the Riemannian measure with respect to the constant metric $(\phi_1^* g)_0$. The proof is completed. □

4. Concluding remarks. We have seen the local universality of DPPs on Riemannian manifolds. From this discussion, we came to several other questions.

(a) What is the universality when we consider the Heisenberg case in Zelditch's terminology? We only discussed the Euclidean case in this article. One can expect that the Bergman kernel is involved as in [2,13,23,24].

(b) We dealt with Laplace-Beltrami operators corresponding to the principal symbol $|\xi|_{g_y}$. What is the local universality result when we consider more general DPPs associated with the spectral projections of elliptic differential operators possibly with potentials?

(c) In this paper we have considered a point process $\Xi_{\lambda,p,\epsilon}$ on T_p^*M at each 'point' $p \in M$. We expect that the collection $\Xi_{\lambda,\epsilon} = \{\Xi_{\lambda,p,\epsilon}\}_{p\in M}$ will be regarded as a 'random field' on the cotangent bundle $T^*M = \cup_{p\in M} T_p^*M$. Theorem 1 determines the limit $\Xi_{\lambda,p,\epsilon} \to \Xi_p$ in $\lambda \to \infty$. How can we describe the limiting random field $\Xi_{\lambda,\epsilon} \to \Xi$ in $\lambda \to \infty$?

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