A note on obstructions to weak approximation and Brauer and R-equivalence relations for homogeneous spaces over global fields

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Abstract: We give some new formulae relating an obstruction to the weak approximation on homogeneous spaces to the set of local and global Brauer and R-equivalence classes.

Key words: Brauer groups; Brauer equivalence; R-equivalence; Weak approximation; Galois cohomology; reductive group.

1. Introduction. This paper is the continuation of [Th21], where we give some applications of the results obtained there, to which we refer as the Part (I). We first recall briefly some notation and convention used in Part (I) and in the present paper.

Let k be a field, k_s a separable closure of k in an algebraic closure $\bar{k}$ of k, and let $\Gamma := \mathrm{Gal}(k_s/k)$ be the absolute Galois group of k. Denote by V the set of all places of k and let k_v be the completion of k at $v \in V$.

Let X be a smooth, geometrically integral k-variety and assume that $X(k) \neq \emptyset$. We say that X has the *weak approximation property with respect to a finite subset* $S \subset V$ if $X(k)$ is dense in the product $X_S := \prod_{v\in S} X(k_v)$ via the diagonal embedding and that X has the *weak approximation property over* k if the above holds for any finite set $S \subset V$. Denote $\overline{X(k)}^S$ (resp. $\overline{X(k)}$) the closure of $X(k)$ being taken in the product $X_S := \prod_{v\in S} X(k_v)$ (resp. in the product $X_V := \prod_v X(k_v)$).

If X is a smooth variety over k, let $\mathrm{Br}(X) := \mathrm{H}^2_{et}(X, \mathbf{G}_m)$ denote the cohomological Brauer group of X. For a field extension K/k we denote $X \times_k K$ the base change of X from k to K; if $K = k_s$ (resp. $K = k_v$), we denote $X_s = X \times_k k_s$ (resp. $X_v = X \times_k k_v$) for short. Then we have natural homomorphisms $\mathrm{Br}(k) \to \mathrm{Br}(X) \to \mathrm{Br}(X_s)$, where the image of the former lies in the kernel of the latter. Following [CTS87], [Sk], [Sa], we set $\mathrm{Br}_1(X) := \mathrm{Ker}\,(\mathrm{Br}(X) \to \mathrm{Br}(X_s))$, $\mathrm{Br}_0(X) := \mathrm{Im}\,(\mathrm{Br}(k) \to \mathrm{Br}(X))$, $\mathrm{Br}_a(X) := \mathrm{Br}_1(X)/\,\mathrm{Br}_0(X)$, (the "arithmetic" Brauer group of X), $Б_\omega(X) := \{x \in \mathrm{Br}_a(X) \mid x_v = 0 \text{ for almost all } v \in V\}$, and finally $Б(X) := \{x \in \mathrm{Br}_a(X) \mid x_v = 0 \text{ for all } v \in V\}$, where for $x \in \mathrm{Br}_a(X)$, we denote by x_v the image of x in $\mathrm{Br}_a(X_v)$. For a subset of places $S \subset V$, we denote $Б_S(X) := \mathrm{Ker}\,(\mathrm{Br}_a(X) \to \prod_{v\notin S} \mathrm{Br}_a(X_v))$, then $Б_\omega(X) := \varinjlim_S Б_S(X) = \bigcup_S Б_S(X)$, and notice that $Б(X) = Б_\emptyset(X)$. Denote by $Y^D = \mathrm{Hom}(Y, \mathbf{Q}/\mathbf{Z})$ the Pontrjagin dual of an abelian torsion group Y.

Now let X be a smooth, geometrically integral variety defined over a global field k and assume that $\prod_{v\in V} X(k_v) \neq \emptyset$.

Consider the following natural (Brauer–Manin) pairings $\prod_v X(k_v) \times Б_\omega(X) \to \mathbf{Q}/\mathbf{Z}$, and $\prod_{v\in S} X(k_v) \times Б_S(X) \to \mathbf{Q}/\mathbf{Z}$, $((x_v), b) \mapsto \langle (x_v), b\rangle := \sum_v \mathrm{inv}_v(b_v(x_v))$ (cf. [Sa, Lem. 6.2]), and inv_v denotes the local Hasse invariant of the local field k_v.

Assume that $X(k) \neq \emptyset$. Then for two elements $x, y \in X(k)$, we say that x, y are *Brauer* (*Br-*) *equivalent* in $X(k)$ (and denote it simply by $x \sim_{Br} y$, if X, k are clearly indicated), if via the natural pairing $X(k) \times \mathrm{Br}(X) \to \mathrm{Br}(k)$, $(x, b) \mapsto b(x)$, where $b(x)$ denotes the evaluation of b at x, we have $b(x) = b(y)$ for all elements $b \in \mathrm{Br}(X)$. (If we consider the group $\mathrm{Br}_1(X)$ instead of $\mathrm{Br}(X)$, then we get Br_1-equivalence relation.)

Further, for a set T of places of k, we define Brauer equivalence relation on the product $\prod_{v\in T} X(k_v)$ just as the product of the Brauer equivalence relation, i.e., if $(x_v), (y_v) \in \prod_v X(k_v)$, then $(x_v) \sim_{Br} (y_v)$ if and only if $x_v \sim_{Br} y_v$ (with respect to $\mathrm{Br}(X_v)$) for all $v \in T$. Thus the quotient set $\prod_{v\in T} X(k_v)/Br$ is by definition, the product $\prod_{v\in T}(X(k_v)/Br)$.

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Two k-points x, y are *R-equivalent* if there is a chain of k-points $x = x_1, x_2, \ldots, x_n = y$ in $X(k)$ and a chain of k-rational maps $f_i : \mathbf{P}^1 \to X$, $(i = 1, \ldots, n-1)$ such that $f_1(0) = x_1, f_1(1) = x_2, \ldots, f_{n-1}(0) = x_{n-1}, f_{n-1}(1) = x_n$. In general, the R-equivalence relation is finer than the Brauer equivalence relations (cf. [CTS77]).

If G is a connected reductive group defined over a field k and $1 \to F \to G_1 \to G \to 1$ is a flasque resolution of G (cf. [CT08, Sec. 2]), then there is an exact sequence of abelian groups

$$\mathrm{Pic}(G) \to \mathrm{Pic}(G_1) \to \mathrm{Pic}(F) \xrightarrow{\theta} \mathrm{Br}_1(G) \to \mathrm{Br}_1(G_1),$$

and the equivalence relation on the group $G(k)$ which is coming from the pairing (induced from that of $G(k) \times \mathrm{Br}(G) \to \mathrm{Br}(k)$)

$$G(k) \times \theta(\mathrm{Pic}(F)) \to \mathrm{Br}(k)$$

is called *weak Brauer equivalence* (denoted by Br_f-equivalence for short) (cf. [Th20, Sec. 3.1.3]). Let denote $B_f G(k) := \{x \in G(k) \mid x \sim_{\mathrm{Br}_f} 1\}$, $BG(k) := \{x \in G(k) \mid x \sim_{Br} 1\}$, $RG(k) := \{x \in G(k) \mid x \sim_R 1\}$. These are subgroups of $G(k)$ (cf. [Th20, Sec. 3]).

Further, for a linear algebraic k-group G let denote by $\mathrm{H}^i(k, G) := \mathrm{H}^i(\Gamma, G(k_s))$ the Galois cohomology in degree i of G. Then we denote $Ш^i(G) := \mathrm{Ker}\,(\mathrm{H}^i(k, G) \to \prod_{v \in V} \mathrm{H}^i(k_v, G)), i \geq 0,$ the *Tate–Shafarevich kernel* in degree i of G. One denotes by

$$\mathrm{A}(G) := \prod_{v \in V} G(k_v)/\overline{G(k)},\ \mathrm{A}(S, G) := \prod_{v \in S} G(k_v)/\overline{G(k)}^S,$$

the *obstruction* (or *defect*) *to the weak approximation property of G over k* and *obstruction to weak approximation of G at S*, respectively. It is well-known that if G is a connected reductive group defined over a global field k, then we have the following exact sequence of *groups*

(1.1) $\quad 1 \to \overline{G(k)}^S \to G_S \to \mathrm{A}(S, G) \to 1,$

(1.2) $\quad 1 \to \overline{G(k)} \to G_V \to \mathrm{A}(G) \to 1,$

where $\mathrm{A}(S, G)$, $\mathrm{A}(G)$ are finite abelian groups (cf. [Sa, Thm. 3.3] for number fields and [Th13, Thm. 2.3] for global function fields).

In [CTS77, Prop. 19] it has been shown that if T is a torus over a number field k, S the corresponding Néron-Severi torus, then there is an exact sequence of finite abelian groups

(CTS)

$$1 \to Ш^1(S) \to T(k)/R \to \prod_v T(k_v)/R \to \mathrm{A}(T) \to 1.$$

This sequence inspires another ones in this direction as follows:

In [Th97, Thm. 2.7, 2.8], it was shown that, if G is a connected reductive group defined over a global field k, S a finite set of places, then there are the following exact sequences of finite abelian groups

$$G(k)/R \to \prod_{v \in S} G(k_v)/R \to \mathrm{A}(S, G) \to 1,$$
$$G(k)/R \to \prod_v G(k_v)/R \to \mathrm{A}(G) \to 1.$$

Then [CTGP, Thm. 4.13] extends this to the case of fields of type (ll) and (gl) (we refer the interested readers to [CTGP] for the definition of fileds of type (ll) and (gl)). In [Th00, Prop. 2.4, Thm. 3.4] and [Th20, Th. 5.3, 5.10] we extend these sequences to the case of groups of Brauer equivalence classes, to get the following exact sequences of finite abelian groups

$$G(k)/Br \to \prod_{v \in S} G(k_v)/Br \to \mathrm{A}(S, G) \to 1,$$
$$1 \to G(k)/Br \to \prod_v G(k_v)/Br \to \mathrm{A}(G) \to 1,$$

where the same exact sequence also holds with respect to other Brauer equivalence relations: Br_1 or Br_f. Recall that by [Th21, Thm. 2.1], we also have canonical isomorphisms

$$\mathrm{A}(S, G) \simeq (Б_S(G)/Б(G))^D,$$
$$\mathrm{A}(G) \simeq (Б_\omega(G)/Б(G))^D,$$

so above exact sequences can be written as

(1.1a) $\quad 1 \to \overline{G(k)}^S \to G_S \to (Б_S(G)/Б(G))^D \to 1,$

(1.2a) $\quad 1 \to \overline{G(k)} \to G_V \to (Б_\omega(G)/Б(G))^D \to 1,$

$$G(k)/Br \to \prod_{v \in S} G(k_v)/Br \to (Б_S(G)/Б(G))^D \to 1,$$
$$1 \to G(k)/Br \to \prod_v G(k_v)/Br \to (Б_\omega(G)/Б(G))^D \to 1.$$

and (almost) similarly for R-equivalence relations:

$$G(k)/R \to \prod_{v \in S} G(k_v)/R \to (Б_S(G)/Б(G))^D \to 1,$$
$$G(k)/R \to \prod_v G(k_v)/R \to (Б_\omega(G)/Б(G))^D \to 1.$$

If X is a homogeneous space then the exact sequences (1.1a), (1.2a) have been extended to homogeneous spaces under connected reductive k-groups G with connected reductive k-stabilizers H in [Bo99, Thms. 1.3, 1.11] (number field case) and [Th21, Thm. 4.1] (global function field case). Namely, if we select the image of the trivial coset $[H]$ as the pointed element in $X(k)$ (resp. in X_S, X_V, via the diagonal map), then we have the following exact sequence of *pointed sets*

$$(1.1') \quad 1 \to \overline{X(k)}^S \to X_S \to (Б_S(X)/Б(X))^D \to 1,$$

and one derives from this the following exact sequence of *pointed sets*

$$(1.2') \quad 1 \to \overline{X(k)} \to X_V \to (Б_\omega(X)/Б(X))^D \to 1,$$

which generalize (1.1a) and (1.2a) to an important class of homogeneous spaces. (Here and in the sequel, for all considered exact pointed sets $1 \to A \xrightarrow{f} B \to C$, the exact on the left means the injectivity of f.) So X has weak approximation in S (resp. over k) if and only if the dual group $(Б_S(X)/Б(X))^D$ (resp. $(Б_\omega(X)/Б(X))^D$) is trivial. In other words, the abelian group $(Б_S(X)/Б(X))^D$ (resp. $(Б_\omega(X)/Б(X))^D$) is an obstruction to the weak approximation in S (resp. over k) for X.

The aim of this note is to show that if X is a *homogeneous space* under a connected reductive k-group G with stabilizer a connected reductive k-subgroup H of G, then some of the above exact sequences also hold for Br, Br_1, Br_f and R-equivalence relations. Namely we have the following generalization of the above sequences to the case of homogeneous spaces.

2. Theorem. *Let k be a global field, X a homogeneous space under a connected reductive k-group G with stabilizer a connected reductive k-subgroup H of G. Then we have the following exact sequences of pointed sets, which connect the various (local and global) sets of Brauer (Br, Br_1, or Br_f), R-equivalence classes, the arithmetic Brauer groups and obstruction to weak approximation with each other*:

$$(2.1) \quad X(k)/B \to \prod_{v\in S} X(k_v)/B \to (Б_S(X)/Б(X))^D \to 1,$$

$$1 \to X(k)/B \xrightarrow{j_B} \prod_{v} X(k_v)/B \to (Б_\omega(X)/Б(X))^D \to 1$$

where B stands for Br, Br_1 or Br_f-equivalence relation;

$$(2.2) \quad X(k)/R \to \prod_{v\in S} X(k_v)/R \to (Б_S(X)/Б(X))^D \to 1,$$

$$X(k)/R \xrightarrow{j_R} \prod_{v} X(k_v)/R \to (Б_\omega(X)/Б(X))^D \to 1.$$

(Here, for any field extension L/k, the set $X(L)$ is a pointed set, with distinguished element $e := [H]$ given by the coset H and the set $X(L)/Br$ (resp. $X(L)/R$) is pointed with the equivalence class of e.)

Before proving the theorem, we need the following lemmas. Recall that a connected reductive group G defined over a field k is called *quasi-trivial* ([CT08, Sec. 2]) if the semisimple part G^{ss} of G is simply connected and $G^{tor} := G/G^{ss}$ is an induced k-torus.

3. Lemma. (1) (cf. [Th20, Corol. 4.4]). *If k is a local field, G is a quasi-trivial reductive k-group, then we have $G(k)/R = \{1\}, G(k)/Br = \{1\}, G(k)/Br_1 = \{1\}, G(k)/Br_f = \{1\}$.*
(2) (cf. [Bo99, Thms. 1.3, 1.11] (number field case) and [Th21, Thm. 4.1] (function field case)). *If k is a global field, with notation as in Theorem 2, there are the following exact sequences of pointed sets*

$$1 \to \overline{X(k)}^S \xrightarrow{f_S} X_S \xrightarrow{\zeta_S} (Б_S(X)/Б(X))^D \to 1,$$

$$1 \to \overline{X(k)} \xrightarrow{f} X_V \xrightarrow{\zeta} (Б_\omega(X)/Б(X))^D \to 1.$$

□

The following lemma complements [Th97, Lem. 2.5] (cf. [Ko, Corol. 1.5, Corol. 1.7] for smooth *proper* varieties over local fields).

4. Lemma. *With notation as in Theorem 2, let v be a place of k. Then*
(1) *Each Brauer (Br, Br_1 and Br_f) and R-equivalence class in $G(k_v)$ is open and closed in $G(k_v)$.*
(2) *Each Brauer-equivalence class in $X(k_v)$ is a union of finitely many $RG(k_v)$-orbits, so each Brauer and R-equivalence class in $X(k_v)$ is open and closed in $X(k_v)$.*
(3) *The set of Brauer (resp. R)-equivalence classes $X(k_v)/B$, where B stands for Br, Br_1 or Br_f-equivalence relation, (resp. $X(k_v)/R$) is finite.*

Proof. (1) If $k_v \simeq \mathbf{C}$ then it is trivial that $RG(k_v) = G(k_v)$. If $k_v \simeq \mathbf{R}$, let $x \in G(\mathbf{R})$, $x = x_s x_u$ be the Jordan decomposition of x, $x_s, x_u \in G(\mathbf{R})$. Since the unipotent $\mathbf{R}$-groups are $\mathbf{R}$-rational, we have $x_u \in RG(\mathbf{R})$. There is an $\mathbf{R}$-subtorus T of G such that $x_s \in T(\mathbf{R})$ and by [CTS77, Corol. 3, p. 200], $T(\mathbf{R})/R = 1$. Thus $x_s \in RG(\mathbf{R})$. Hence $RG(\mathbf{R}) = G(\mathbf{R})$, thus $BG(\mathbf{R}) = G(\mathbf{R})$ and the assertion holds for archimedean v. Next we assume that v is *non-archimedean*. We know that the weak Brauer, Brauer and R-equivalence relations on the k_v-points $G(k_v)$ of any connected reductive k-group G coincide (cf. [Th20, Lem. 4.9]), so it suffices to show the assertion for the weak Brauer equivalence relation. Also, it suffices to show that the subgroup $B_f G(k_v) := \{x \in G(k_v) \mid x \sim_{\mathrm{Br}_f} 1\}$ is an open (and thus also closed) subgroup of $G(k_v)$. We know that

$B_fG(k_v) = f_v(G_1(k_v))$ (cf. [Th20, Lem. 4.9]) where $1 \to F \to G_1 \xrightarrow{f} G \to 1$ is a flasque resolution of G (cf. [CT08, Sec. 2]). Since F, being a k_v-torus, is smooth, so f is a separable morphism, which defines an open map $f_v : G_1(k_v) \to G(k_v)$ with respect to v-adic topologies on $G_1(k_v)$ and $G(k_v)$. In particular, $B_fG(k_v)$ is an open subgroup in $G(k_v)$ as asserted. Hence each Brauer and R-equivalence class in $G(k_v)$ is open and closed in $G(k_v)$.

(2) We have the following exact sequence

$$G(k_v) \xrightarrow{\pi_v} X(k_v) \xrightarrow{\delta_v} \mathrm{H}^1(k_v, H) \to \mathrm{H}^1(k_v, G).$$

It is well-known that for any connected reductive group K defined over a local field k_v, the cohomology set $\mathrm{H}^1(k_v, K)$ is finite (see [Se, Chap. III, Sec. 4], cf. also [Th19, Thm. 2.7]).

For every element $y \in X(k_v)$, let $[y]_{Br}$ be the Brauer equivalence class of y in $X(k_v)$. From what we have said, the image $\delta_v([y]_{Br})$ in $\mathrm{H}^1(k_v, H)$ is finite, say

$$\delta_v([y]_{Br}) = \{h_1, \ldots, h_r\}, h_i \neq h_j, i \neq j.$$

Let $x_i \in [y]_{Br}$ such that $\delta_v(x_i) = h_i$. If $x \in [y]_{Br}$ such that $\delta_v(x) = \delta_v(x_i)$, then we have $x = g_i x_i$, for some $g_i \in G(k_v)$, i.e.,

$$[y]_{Br} = \bigcup_i ([y]_{Br} \bigcap G(k_v)x_i).$$

Since $G(k_v)/Br = G(k_v)/R$ is finite (cf. [Th20, Lem. 4.9] and the finiteness of H^1 for connected reductive groups), let

$$G(k_v) = \bigcup_{1 \le s \le t} RG(k_v)z_s, z_s \in G(k_v)$$

be a disjoint union of cosets of $RG(k_v)$, so

$$[y]_{Br} = \bigcup_i \bigcup_s ([y]_{Br} \bigcap RG(k_v)z_s x_i).$$

Then we have either

$$[y]_{Br} \bigcap RG(k_v)z_s x_i = RG(k_v)z_s x_i$$

or

$$[y]_{Br} \bigcap RG(k_v)z_s x_i = \emptyset,$$

according to whether $z_s x_i \sim_{Br} x_i$ or not. Hence we have

$$[y]_{Br} = \bigcup_i \bigcup_{s, z_s x_i \sim_{Br} x_i} RG(k_v)z_s x_i.$$

Then by changing the notation appropriately, we have the following disjoint union

$$[y]_{Br} = RG(k_v).y_1 \bigcup \cdots \bigcup RG(k_v).y_t,$$

where $y_i \sim_{Br} y, i = 1, \ldots, t$. Since $RG(k_v)$ is open in $G(k_v)$ (cf. [Th97, Lem. 2.5]), and the map $G(k_v) \to X(k_v)$ is open, it follows that the image of $RG(k_v) \to RG(k_v).x_i$ is also open in $X(k_v)$, hence $[y]_{Br}$ is open in $X(k_v)$. Each $RG(k_v)$-orbit in $X(k_v)$ is the complement to the disjoint union of other $RG(k_v)$-orbits, hence is also closed, and hence so is each Brauer equivalence class $[y]_{Br}$.

Similarly, the same argument shows that each R-equivalence class $[y]_R$ is a disjoint union of $RG(k_v)$-orbits, thus is open and closed in $X(k_v)$.

(3) Since there is a natural surjective map $X(k_v)/R \to X(k_v)/Br$, so it suffices to show that $X(k_v)/R$ is finite. If $1 \to F \to G_1 \xrightarrow{f} G \to 1$ is a flasque resolution of G then we have the following exact sequence

$$1 \to F \to H_1 \to H \to 1,$$

where $H_1 := f^{-1}(H)$ is a connected reductive k_v-subgroup of G_1, and also that

$$X = G/H \simeq G_1/H_1.$$

Hence in the presentation of X as the quotient G/H, we may assume from the very beginning that G is quasi-trivial. In particular, we have $G(k_v) = RG(k_v)$ by Lemma 3 (1). Since $\mathrm{H}^1_{\mathrm{fppf}}(k_v, H)$ is finite, it follows from the exact sequence

$$G(k_v) \xrightarrow{\pi} X(k_v) \to \mathrm{H}^1_{\mathrm{fppf}}(k_v, H),$$

that the set of $G(k_v)$-orbits in $X(k_v)$ is finite. Thus we have

$$X(k_v) = \bigcup_{i \in I} G(k_v) \cdot x_i, \ x_i \in X(k_v),$$

where i runs over a finite set I of indices. Since all elements from $G(k_v)$ are R-equivalent to the identity element (Lemma 3 (1)), one can check that all the elements from $G(k_v) \cdot x_i$ are R-equivalent to x_i. In particular, the set $X(k_v)/R$ is finite as desired.

□

5. Lemma. *We have the following commutative diagrams with exact rows*

$$\begin{array}{ccccc}
1 \to \overline{X(k)}^S & \xrightarrow{f_S} & X_S & \xrightarrow{\zeta_S} & (\text{Б}_S(X)/\text{Б}(X))^D \to 1 \\
\alpha_S \downarrow & & \downarrow \beta_S & & \downarrow = \\
1 \to \overline{X(k)}^S/T & \xrightarrow{f'_S} & X_S/T & \xrightarrow{\zeta'_S} & (\text{Б}_S(X)/\text{Б}(X))^D \to 1 \\
1 \to \overline{X(k)} & \xrightarrow{f} & X_V & \xrightarrow{\zeta} & (\text{Б}_\omega(X)/\text{Б}(X))^D \to 1 \\
\alpha \downarrow & & \beta \downarrow & & \downarrow = \\
1 \to \overline{X(k)}/T & \xrightarrow{f'} & X_V/T & \xrightarrow{\zeta'} & (\text{Б}_\omega(X)/\text{Б}(X))^D \to 1,
\end{array}$$

where T stands for either Br, Br_1, Br_f *or R-equivalence relation and the Brauer (resp. R-) equiv-*

alence relation on $\overline{X(k)}^S$ *(resp.* $\overline{X(k)}$*) is the one induced by the relation on* X_S *(resp.* X_V*) and* $\overline{X(k)}^S/T$ *(resp.* $\overline{X(k)}/T$*) denotes the corresponding set of* T*-equivalence classes.*

Proof. We consider only the first diagram for the Br-equivalence (the other cases are similar). It is clear that f'_S is an injective map.

Next we define a map

$$\zeta'_S : \prod_{v\in S} X(k_v)/Br \to (\text{Б}_S(X)/\text{Б}(X))^D.$$

Recall that the map ζ_S is defined as follows. From the definition of $\text{Б}(X)$ it follows that the natural Brauer-Manin pairing $\prod_{v\in S} X(k_v) \times \text{Б}_S(X) \to \mathbf{Q}/\mathbf{Z}$, factors through the pairing $\prod_{v\in S} X(k_v) \times (\text{Б}_S(X)/\text{Б}(X)) \to \mathbf{Q}/\mathbf{Z}$, which in turn gives the map ζ_S. Notice that if $x = (x_v)_{v\in S} \sim_{Br} y = (y_v)_{v\in S}$, then their images via ζ_S are the same, thus ζ_S factors through $\prod_{v\in S} X(k_v)/Br$, and it defines a map $\zeta'_S : \prod_{v\in S} X(k_v)/Br \to (\text{Б}_S(X)/\text{Б}(X))^D$. It follows that the diagram stated above is commutative, where the first row is exact according to Lemma 3 (2).

It remains to show that the second row is exact. It is clear that we have $\zeta'_S \circ f'_S(\bar{x}) = 1$ for any $\bar{x} \in \overline{X(k)}^S/Br$. Conversely, if $\zeta'_S(\bar{p}) = 1$, $\bar{p} \in \prod_{v\in S} X(k_v)/Br$, then take $p \in \prod_{v\in S} X(k_v)$ with its image equal to $\bar{p}$. We have $1 = \zeta'_S(\beta_S(p)) = \zeta_S(p)$, so $p = f_S(x), x \in \overline{X(k)}^S$. Hence $\bar{p} = \beta_S(p) = \beta_S(f_S(x)) = f'_S(\alpha_S(x))$ as required.

To treat the case of R-equivalence, one uses the natural surjective maps $X(L)/R \to X(L)/Br$ for any field extension L/k (cf. [CTS77, p. 213], [Th20, 4.1(c), p. 1038]), and proceed as above. Consider the following diagrams

$$\begin{array}{ccccc}
\overline{X(k)}^S & \xrightarrow{f_S} & X_S & \xrightarrow{\zeta_S} & (\text{Б}_S(X)/\text{Б}(X))^D \\
\downarrow \alpha'_S & (1) & \beta'_S \downarrow & & \downarrow= \\
\overline{X(k)}^S/R & \xrightarrow{f'_S} & X_S/R & \xrightarrow{\zeta'_S} & (\text{Б}_S(X)/\text{Б}(X))^D \\
\downarrow \alpha''_S & (2) & \beta''_S \downarrow & & \downarrow= \\
\overline{X(k)}^S/Br & \xrightarrow{f''_S} & X_S/Br & \xrightarrow{\zeta''_S} & (\text{Б}_S(X)/\text{Б}(X))^D,
\end{array}$$

where the lower diagram (2) is commutative, and the following commutative diagram

$$\begin{array}{ccccc}
\overline{X(k)}^S & \xrightarrow{f_S} & X_S & \xrightarrow{\zeta_S} & (\text{Б}_S(X)/\text{Б}(X))^D \\
\downarrow \alpha_S & & \beta_S \downarrow & & \downarrow= \\
\overline{X(k)}^S/Br & \xrightarrow{f''_S} & X_S/Br & \xrightarrow{\zeta''_S} & (\text{Б}_S(X)/\text{Б}(X))^D,
\end{array}$$

where we have $\alpha_S = \alpha''_S \circ \alpha'_S$, $\beta_S = \beta''_S \circ \beta'_S$. Using this, we infer that the upper diagram (1) is also commutative. Further one proves the exactness of the middle row in a similar way as above. □

6. Lemma. *We have the following commutative diagrams with exact rows*

$$(6.1)\quad \begin{array}{ccccc}
X(k)/T & \xrightarrow{j_S} & X_S/T & \xrightarrow{\zeta_S} & (\text{Б}_S(X)/\text{Б}(X))^D \\
\gamma_S \downarrow & & =\downarrow & & \downarrow= \\
\overline{X(k)}^S/T & \xrightarrow{f'_S} & X_S/T & \xrightarrow{\zeta'_S} & (\text{Б}_S(X)/\text{Б}(X))^D,
\end{array}$$

$$(6.2)\quad \begin{array}{ccccc}
X(k)/T & \xrightarrow{j_T} & X_V/T & \xrightarrow{\zeta_T} & (\text{Б}_\omega(X)/\text{Б}(X))^D \\
\gamma_T \downarrow & & =\downarrow & & \downarrow= \\
\overline{X(k)}/T & \xrightarrow{f'_T} & X_V/T & \xrightarrow{\zeta_T} & (\text{Б}_\omega(X)/\text{Б}(X))^D,
\end{array}$$

where T *stands either for* Br, Br_1 *or* Br_f *or* R*-equivalence relation.*

Proof. We treat the diagram (6.1) for the case Br only; the other cases are treated in a similar way. Consider the following commutative diagram

$$\begin{array}{ccc}
X(k) & \xrightarrow{i_S} & \overline{X(k)}^S \\
p\downarrow & & \bar{p}\downarrow \\
X(k)/Br & \xrightarrow{\gamma_S} & \overline{X(k)}^S/Br.
\end{array}$$

Consider the quotient topology of the topology of the product $\prod_{v\in S} X(k_v)$ on the quotient set $\overline{X(k)}^S/Br$. Since i_S has dense image, it implies that $\gamma_S(p(X(k)))$ is also dense in $\overline{X(k)}^S/Br$.

As we have seen by Lemma 4, each Br-equivalence class in $X(k_v)$ is an open and closed subset, so the natural topology on $X(k_v)/Br$ is the discrete one, hence so is the topology on $\prod_{v\in S} X(k_v)/Br$, and thus also the one on $\overline{X(k)}^S/Br$. Therefore γ_S is surjective. Since the second row of diagram (6.1) is exact by Lemma 5, so from the surjectivity of γ_S it follows that so is the first row.

By passing to the limit for the first exact sequence of diagram (6.1), we get the following exact sequence

$$X(k)/Br \xrightarrow{j_B} \prod_v X(k_v)/Br \xrightarrow{\zeta} (\text{Б}_\omega(X)/\text{Б}(X))^D \to 1$$

and the lemma is proved. □

Proof of Theorem 2. By Lemma 5, it remains to show that the second sequence appearing in the diagram (2.1) is exact on the left, that is, the maps j_B are injective, for $B = \text{Br}$ or Br_1. But it is well-known (see [MaTs, Sec. 4.5]), that j_B is injective due to the Hasse principle for the Brauer group $\text{Br}(k)$, hence we have the following exact sequence

$$1 \to X(k)/Br \xrightarrow{j_B} \prod_v X(k_v)/Br$$
$$\xrightarrow{\zeta_B} (\text{Б}_\omega(X)/\text{Б}(X))^D \to 1.$$

The proof of Theorem 2, is therefore complete. □

Remark. Notice that in the case of R-equivalence, the last sequence may not be exact on the left side, i.e., $j_R : X(k)/R \to \prod_v X(k_v)/R$ *may not be injective*. In some special cases, we have a formula to compute the kernel Ker (j_R) for tori, see the exact sequence (CTS) ([CTS77, Prop. 19]), and for connected reductive groups, see [Th20, Thm. 5.11, Thm. 5.14]. It is still an open question to obtain such a formula for homogeneous spaces. Based on [Th20, Thm. 5.11, Thm. 5.14], one conjectures that the following holds. Assume k is a global field, such that for all simply connected semisimple k-group $\tilde{G}$, we have $\tilde{G}(k)/R = 1$. Then for any homogeneous k-space X with a smooth k-compactification $\mathscr{X}$, the following sequence of pointed sets is exact

$$1 \to \text{Ш}^1(S_{\mathscr{X}}) \to X(k)/R \xrightarrow{j_R} \prod_v X(k_v)/R \to$$
$$\to (\text{Б}_\omega(X)/\text{Б}(X))^D \to 1,$$

where $S_{\mathscr{X}}$ denotes the Néron-Severi k-torus of $\mathscr{X}$. According to Theorem 2, for this to hold, it is sufficient to show that there is a bijection

$$\text{Ker}\,(j_R) \simeq \text{Ш}^1(S_{\mathscr{X}}).$$

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