Shintani correspondence for Maass forms of level N and prehomogeneous zeta functions

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(Communicated by Masaki KASHIWARA, M.J.A., June 13, 2022)

Abstract: A Shintani-Katok-Sarnak type correspondence for Maass cusp forms of level N is shown to be derived from analytic properties of prehomogeneous zeta functions whose coefficients involve periods of Maass forms.

Key words: Shintani correspondence; Maass forms; prehomogeneous zeta functions.

In [10], Shimura constructed a lifting from holomorphic cusp forms of half-integral weight to cusp forms of integral weight. Shimura's original proof depends on the Rankin-Selberg method and Weil's converse theorem [13]. In [11], Shintani constructed a lifting from holomorphic cusp forms of integral weight to cusp forms of half-integral weight by using theta functions. In the case of non-holomorphic modular forms, a prototype of the lifting had already appeared in the work of Maaß [6]. Katok and Sarnak [5] developed the method of [6] to prove the Shintani correspondence for Maass cusp forms of weight 0 for $SL_2(\mathbf{Z})$. The Katok-Sarnak formula reveals a relation between the periods of Maass forms of weight 0 and the Fourier coefficients of the corresponding form of weight $\frac{1}{2}$, and now plays an important role in number theory. The Katok-Sarnak formula has been extended in many directions; we refer to Baruch-Mao [1], Biró [2], Duke-Imamoḡlu-Tóth [3], Imamoḡlu-Lägeler-Tóth [4]. On the other hand, F. Sato [9] constructed a theory of prehomogeneous zeta functions whose coefficients involve periods of automorphic forms. In this note, we show that a Shintani-Katok-Sarnak type correspondence is derived from analytic properties of a certain zeta function investigated in [9]. The proof relies on a Weil type converse theorem for Maass forms [7].

This is an announcement whose details will appear elsewhere.

1. Statement of the result. The group $G = SL_2(\mathbf{R})$ acts on the Poincaré upper half plane $\mathcal{H} = \{z = x + iy \in \mathbf{C} \mid y > 0\}$ via the linear fractional transformation. Let N be a positive integer and take a congruence subgroup $\Gamma_0(N)$ of level N defined by

$$\Gamma_0(N) = \{\gamma \in SL_2(\mathbf{Z}) \mid \gamma_{21} \equiv 0 \pmod N\},$$

where γ_{21} the $(2,1)$-entry of γ. Let χ be a Dirichlet character of mod N satisfying $\chi(-1) = 1$. We use the same symbol χ to denote the induced character of $\Gamma_0(N)$ defined by $\chi(\gamma) = \chi(\gamma_{22})$ for $\gamma = (\gamma_{ij}) \in \Gamma_0(N)$. A C^∞-function $\Phi : \mathcal{H} \to \mathbf{C}$ is called a *Maass cusp form* of weight 0 for $\Gamma_0(N)$ with character χ if

(1) $\Delta_0\Phi = \lambda(1-\lambda)\Phi$ for a $\lambda \in \mathbf{C}$, where

$$\Delta_0 = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is the hyperbolic Laplacian on $\mathcal{H}$,

(2) $\Phi(\gamma z) = \chi(\gamma)\Phi(z)$ for $\gamma \in \Gamma_0(N)$, and

(3) Φ has exponential decay at all cusps of $\Gamma_0(N)$.

Let $\mathfrak{S}_0(N, \lambda, \chi)$ be the space of all such functions. For $g \in G$, we put $\phi(g) = \Phi(g^{-1} \cdot \sqrt{-1})$. Let

$$g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

be the Iwasawa decomposition of $g \in G$. Then we have $\phi(g) = \Phi(x + y\sqrt{-1})$. Let $V = \mathrm{Sym}_2(\mathbf{R})$ be the space of real symmetric matrices of degree 2. Then $\widetilde{G} = \mathbf{R}^\times \times G$ acts on V by $v \mapsto t \cdot gv\,{}^tg$ for $v \in V$ and $(t, g) \in \widetilde{G}$. Let $V_+ = \{v \in V \mid \det v > 0\}$ and $V_- = \{v \in V \mid \det v < 0\}$. We have $V_+ = \widetilde{G} \cdot I_2$ and $V_- = \widetilde{G} \cdot J_2$, where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2020 Mathematics Subject Classification. Primary 11S90; Secondary 11F37.

doi: 10.3792/pjaa.98.008

Further, we put $H_+ = SO(I_2)$ and $H_- = SO(J_2)$ so that

$$H_+ = SO(2) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, 0 \le \theta < 2\pi \right\},$$

$$H_- = SO(1,1) = \left\{ a_y = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \,\middle|\, y \in \mathbf{R}^\times \right\}.$$

We normalize the Haar measures $d\mu_\pm$ on $H_\pm$ by

$$d\mu_+(k_\theta) = \frac{d\theta}{2}, \quad d\mu_-(a_y) = \frac{dy}{4|y|}.$$

Let V_+^p (resp. V_+^n) be the set of positive (resp. negative) definite symmetric matrices in V_+. For $v \in V$ with $\det v \neq 0$, we take $t_v > 0$ and $g_v \in G$ such that

$$v = \begin{cases} t_v(g_v I_2 \, {}^t g_v) & \text{if } v \in V_+^p, \\ -t_v(g_v I_2 \, {}^t g_v) & \text{if } v \in V_+^n, \\ t_v(g_v J_2 \, {}^t g_v) & \text{if } v \in V_-. \end{cases}$$

For $v \in V_{\mathbf{Q}} \cap V_\pm$, we define the *period* $\mathcal{M}\phi(v)$ of ϕ by

$$\mathcal{M}\phi(v) = \int_{H_\pm / g_v^{-1}\Gamma_{0,v} g_v} \phi(h g_v^{-1}) d\mu_\pm(h),$$

where $\Gamma_{0,v} = \{\gamma \in \Gamma_0(N) \mid \gamma v \, {}^t\gamma = v\}$. Then $\mathcal{M}\phi(v)$ is absolutely convergent and does not depend on the choice of g_v. By [9, Lemma 6.3], for $v \in V_+$, we have

$$\mathcal{M}\phi(v) = \frac{\pi}{\varepsilon(v)} \cdot \Phi(z_v),$$

where $\varepsilon(v) = \sharp(\Gamma_{0,v})$ and $z_v = g_v \cdot \sqrt{-1}$. Note that z_v coincides with the so-called Heegner point associated with v. If $v \in V_-$, then $\{g_v a_y \cdot \sqrt{-1} \mid y > 0\}$ is the Heegner cycle associated with v, and thus $\mathcal{M}\phi(v)$ coincides (up to constant) with a certain cycle integral of Φ. Following the formulation of Shintani, we take a lattice $\mathcal{L}_N$ defined by

$$\mathcal{L}_N = \left\{ v = \begin{pmatrix} v_1 & N v_2 \\ N v_2 & N v_3 \end{pmatrix} \,\middle|\, v_1, v_2, v_3 \in \mathbf{Z} \right\}.$$

(see [11, p. 109]). We note that $\Gamma_0(N)$ acts on $\mathcal{L}_N$ via the restriction of the representation $(\widetilde{G}, V)$. Namely, $v \mapsto \gamma v \, {}^t\gamma$ for $\gamma \in \Gamma_0(N)$ and $v \in \mathcal{L}_N$. Further, for $v \in \mathcal{L}_N$, we put

$$d_N(v) := N(v_2)^2 - v_1 v_3 \quad \left(= -\frac{1}{N} \det v \right).$$

Let $V_{\mathbf{Z}}$ be the set of half-integral symmetric matrices of degree 2:

$$V_{\mathbf{Z}} = \left\{ w^* = \begin{pmatrix} w_1^* & \frac{w_2^*}{2} \\ \frac{w_2^*}{2} & w_3^* \end{pmatrix} \,\middle|\, w_1^*, w_2^*, w_3^* \in \mathbf{Z} \right\},$$

and for $w^* \in V_{\mathbf{Z}}$, we put

$$\mathrm{disc}(w^*) := (w_2^*)^2 - 4 w_1^* w_3^* \quad (= -4 \det w^*).$$

Note that $\Gamma_0(N)$ also acts on $V_{\mathbf{Z}}$ in the same way as above. We take an automorphic factor $J(\gamma, z)$ of weight $\frac{1}{2}$ defined by

$$J(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}, \quad \text{with} \quad \theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi i n^2 z).$$

Let

$$\Delta_{\frac{1}{2}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{iy}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

be the hyperbolic Laplacian of weight $\frac{1}{2}$ on $\mathcal{H}$, and ψ a Dirichlet character of mod $4N$. A C^∞-function $F : \mathcal{H} \to \mathbf{C}$ is called a Maass cusp form of weight $\frac{1}{2}$ for $\Gamma_0(4N)$ with character ψ if

(1) $\Delta_{\frac{1}{2}} F = (\mu - \frac{1}{4})(\frac{3}{4} - \mu) F$ for a $\mu \in \mathbf{C}$,
(2) $F(\gamma z) = \psi(\gamma) J(\gamma, z) F(z)$ for $\gamma \in \Gamma_0(4N)$, and
(3) $F(z)$ has exponential decay at all cusps of $\Gamma_0(4N)$.

We denote by $\mathfrak{S}_{\frac{1}{2}}(4N, \mu, \psi)$ the space of all such functions. Any $F \in \mathfrak{S}_{\frac{1}{2}}(4N, \mu, \psi)$ has a Fourier expansion of the form

$$F(z) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c(n) \cdot W_{1,\mu}(n, y) \boldsymbol{e}[nx], \tag{1.1}$$

where $\boldsymbol{e}[x] = \exp(2\pi\sqrt{-1}x)$ and for $\ell \in \mathbf{Z}$,

$$W_{\ell,\mu}(n, y) = y^{-\frac{\ell}{4}} W_{\frac{\mathrm{sgn}(n)\ell}{4}, \mu - \frac{1}{2}}(4\pi |n| y). \tag{1.2}$$

Here $W_{\kappa,\nu}(z)$ denotes the Whittaker function. For a Dirichlet character χ of mod N, let

$$\tau_\chi(n) = \sum_{\substack{m \bmod N \\ (m,N)=1}} \chi(m) \boldsymbol{e}\left[\frac{mn}{N} \right]$$

be the Gauss sum associated with χ. Now we state our main theorem.

Theorem 1. *Let* $\lambda \neq \frac{1}{2}$ *and assume that* $\Phi(z) \in \mathfrak{S}_0(N, \lambda, \chi^2)$. *We put*

$$\mu = \frac{2\lambda + 1}{4}, \quad \chi_N(r) = \chi(r) \left(\frac{N}{r} \right).$$

Then there exists an $F(z) \in \mathfrak{S}_{\frac{1}{2}}(4N, \mu, \chi_{4N})$ *such that the Fourier coefficients* $c(n)$ *in* (1.1) *are given by*

$$c(n) = 2\pi^{-\frac{1}{2}} \cdot n^{-\frac{3}{4}} \sum_{\substack{v \in \Gamma_0(N)\backslash \mathcal{L}_N \\ d_N(v)=n}} \chi(v_1)\mathcal{M}\phi(v),$$

$$c(-n) = n^{-\frac{3}{4}} \sum_{\substack{v \in \Gamma_0(N)\backslash \mathcal{L}_N \\ d_N(v)=-n}} \frac{\chi(v_1)\Phi(z_v)}{\epsilon(v)},$$

for $n = 1, 2, 3, \ldots$. *Furthermore, if we put*

$$c^*(n) = 2^{\lambda}\pi^{-\frac{1}{2}} \cdot n^{-\frac{3}{4}} \sum_{\substack{w^* \in \Gamma_0(N)\backslash V_{\mathbf{Z}} \\ \operatorname{disc} w^*=n}} \tau_\chi(w_3^*)\mathcal{M}\phi(w^*),$$

$$c^*(-n) = 2^{\lambda-1} \cdot n^{-\frac{3}{4}} \sum_{\substack{w^* \in \Gamma_0(N)\backslash V_{\mathbf{Z}} \\ \operatorname{disc} w^*=-n}} \frac{\tau_\chi(w_3^*)\Phi(z_{w^*})}{\epsilon(w^*)},$$

for $n = 1, 2, 3, \ldots,$ *and define a function* $G(z)$ *on* $\mathcal{H}$ *by*

$$G(z) = N^{-\frac{3}{4}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c^*(n) \cdot W_{1,\mu}(n, y)\boldsymbol{e}[nx],$$

then we have $G(z) \in \mathfrak{S}_{\frac{1}{2}}(4N, \mu, \overline{\chi})$ *and*

$$F\left(-\frac{1}{4Nz}\right)(\sqrt{N}z)^{-\frac{1}{2}} = \boldsymbol{e}\left[-\frac{1}{8}\right] \cdot G(z).$$

2. A Weil type converse theorem for Maass forms. Our proof for Theorem 1 relies on a converse theorem given in [7]. Here let us recall briefly the result, with some modifications. For the convenience of readers, we give the statement for general weights. Fix an integer ℓ and a positive integer N. We assume that N is a multiple of 4 when ℓ is odd. Let $\alpha = \{\alpha(n)\}_{n \in \mathbf{Z}\backslash\{0\}}$ and $\beta = \{\beta(n)\}_{n \in \mathbf{Z}\backslash\{0\}}$ be complex sequences of polynomial growth. For α, β, we can define the L-functions $\xi_\pm(\alpha; s), \xi_\pm(\beta; s)$ by

$$\xi_\pm(\alpha; s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{n^s}, \quad \xi_\pm(\beta; s) = \sum_{n=1}^{\infty} \frac{\beta(\pm n)}{n^s},$$

and the completed L-functions $\Xi_\pm(\alpha; s)$ and $\Xi_\pm(\beta; s)$ by $\Xi_\pm(\alpha; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\alpha; s)$ and $\Xi_\pm(\beta; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\beta; s)$.

Now we assume the following two conditions:

[C1] The L-functions $\xi_\pm(\alpha; s), \xi_\pm(\beta; s)$ have analytic continuations to *entire* functions of s, and are of finite order in any vertical strip.

[C2] The following functional equation holds:

$$\gamma(s)\begin{pmatrix} \Xi_+(\alpha; s) \\ \Xi_-(\alpha; s) \end{pmatrix} = N^{2-2\mu-s} \cdot \Sigma(\ell) \cdot \gamma(2-2\mu-s)\begin{pmatrix} \Xi_+(\beta; 2-2\mu-s) \\ \Xi_-(\beta; 2-2\mu-s) \end{pmatrix}, \tag{2.1}$$

where $\gamma(s)$ and $\Sigma(\ell)$ are given by

$$\gamma(s) = \begin{pmatrix} e^{\pi si/2} & e^{-\pi si/2} \\ e^{-\pi si/2} & e^{\pi si/2} \end{pmatrix}, \quad \Sigma(\ell) = \begin{pmatrix} 0 & i^\ell \\ 1 & 0 \end{pmatrix}.$$

For an odd prime number r with $(N, r) = 1$ and a Dirichlet character ψ mod r, the twisted L-functions $\xi_\pm(\alpha, \tau_\psi; s), \xi_\pm(\beta, \tau_\psi; s)$ are defined by

$$\xi_\pm(\alpha, \tau_\psi; s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)\tau_\psi(\pm n)}{n^s},$$

$$\xi_\pm(\beta, \tau_\psi; s) = \sum_{n=1}^{\infty} \frac{\beta(\pm n)\tau_\psi(\pm n)}{n^s},$$

where $\tau_\psi(n)$ is the Gauss sum associated with ψ. The complete L-functions $\Xi_\pm(\alpha, \tau_\psi; s)$ and $\Xi_\pm(\beta, \tau_\psi; s)$ are defined by $\Xi_\pm(\alpha, \tau_\psi; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\alpha, \tau_\psi; s)$ and $\Xi_\pm(\beta, \tau_\psi; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\beta, \tau_\psi; s)$, respectively.

Let $\mathbf{P}_N$ be a set of odd prime numbers not dividing N such that, for any positive integers a, b coprime to each other, $\mathbf{P}_N$ contains a prime number r of the form $r = am + b$ for some $m \in \mathbf{Z}_{>0}$. For an $r \in \mathbf{P}_N$, denote by X_r the set of all Dirichlet characters mod r (including the principal character). For $\psi \in X_r$, we define the Dirichlet character ψ^* by

$$\psi^*(k) = \overline{\psi(k)}\left(\frac{k}{r}\right)^\ell. \tag{2.2}$$

For an odd integer d, we put $\varepsilon_d = 1$ or $\sqrt{-1}$ according as $d \equiv 1$ or $3 \pmod 4$. Let

$$C_{\ell,r} = \begin{cases} 1 & (\ell \text{ is even}), \\ \varepsilon_r^\ell & (\ell \text{ is odd}). \end{cases}$$

In the following, we fix a Dirichlet character χ mod N that satisfies $\chi(-1) = (\sqrt{-1})^\ell$ (resp. $\chi(-1) = 1$) when ℓ is even (resp. odd).

For an $r \in \mathbf{P}_N$ and a $\psi \in X_r$, we consider the following conditions $[\mathrm{C1}]_{r,\psi} - [\mathrm{C2}]_{r,\psi}$ on $\xi_\pm(\alpha, \tau_\psi; s)$ and $\xi_\pm(\beta, \tau_{\psi^*}; s)$.

$[\mathrm{C1}]_{r,\psi}$ $\xi_\pm(\alpha, \tau_\psi; s), \xi_\pm(\beta, \tau_{\psi^*}; s)$ have analytic continuations to entire functions of s, and are of finite order in any vertical strip.

$[\mathrm{C2}]_{r,\psi}$ $\Xi_\pm(\alpha, \tau_\psi; s)$ and $\Xi_\pm(\beta, \tau_{\psi^*}; s)$ satisfy the following functional equation:

$$(2.3)\quad \gamma(s)\begin{pmatrix}\Xi_+(\alpha,\tau_\psi;s)\\ \Xi_-(\alpha,\tau_\psi;s)\end{pmatrix} = \chi(r)\cdot C_{\ell,r}\cdot\psi^*(-N)\cdot r^{2\mu-2}\cdot(Nr^2)^{2-2\mu-s}\cdot\Sigma(\ell)\cdot\gamma(2-2\mu-s)\begin{pmatrix}\Xi_+(\beta,\tau_{\psi^*};2-2\mu-s)\\ \Xi_-(\beta,\tau_{\psi^*};2-2\mu-s)\end{pmatrix}.$$

Lemma 1. *Let $\mu\notin\frac{1}{2}\mathbf{Z}$. We assume that $\xi_\pm(\alpha;s)$ and $\xi_\pm(\beta;s)$ satisfy the conditions* [C1] *and* [C2]. *We assume furthermore that, for any $r\in\mathbf{P}_N$ and $\psi\in X_r$, $\xi_\pm(\alpha,\tau_\psi;s)$ and $\xi_\pm(\beta,\tau_{\psi^*};s)$ satisfy the conditions $[\mathrm{C1}]_{r,\psi}$ and $[\mathrm{C2}]_{r,\psi}$. We define the function $\widetilde{W}_{\ell,\mu}(n,y)$ by*

$$\widetilde{W}_{\ell,\mu}(n,y)=\frac{|n|^{\mu-1}}{\Gamma\Big(\mu+\frac{\mathrm{sgn}(n)\ell}{4}\Big)}\cdot W_{\ell,\mu}(n,y),$$

where $W_{\ell,\mu}(n,y)$ is given as (1.2), *and the functions $F_\alpha(z)$ and $G_\beta(z)$ on $\mathcal{H}$ by*

$$F_\alpha(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\alpha(n)\cdot\widetilde{W}_{\ell,\mu}(n,y)\mathbf{e}[nx],$$

$$G_\beta(z)=N^{1-\mu}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\beta(n)\cdot\widetilde{W}_{\ell,\mu}(n,y)\mathbf{e}[nx].$$

Then $F_\alpha(z)$ (resp. $G_\beta(z)$) gives a Maass form for $\Gamma_0(N)$ of weight $\frac{\ell}{2}$ with character χ (resp. $\chi_{N,\ell}$), and eigenvalue $(\mu-\ell/4)(1-\mu-\ell/4)$, where

$$\chi_{N,\ell}(d)=\overline{\chi(d)}\left(\frac{N}{d}\right)^{\ell}.$$

Moreover, we have

$$F_\alpha\left(-\frac{1}{Nz}\right)(\sqrt{N}z)^{-\ell/2}=G_\beta(z).$$

Remark 1. Here we have assumed a stronger condition $\mu\notin\frac{1}{2}\mathbf{Z}$ than that given in the previous paper [7]. This enables us to remove conditions on zeros of L-functions (cf. [7, p. 33]).

3. Prehomogeneous zeta functions. As an example of the theory of [9], Sato investigated the zeta functions associated to the vector space of symmetric matrices of degree 2 whose coefficients involve the periods $\mathcal{M}\phi(v)$ of Maass cusp forms Φ. In this section, we introduce twisted versions of these zeta functions and give their analytic properties such as analytic continuations and functional equations.

Keep the notation as in the previous sections. We define zeta functions $\zeta_\pm(\phi,\chi;s)$ and $\zeta^*_\pm(\phi,\tau_\chi;s)$ by

$$\zeta_\pm(\phi,\chi;s)=\sum_{\substack{v\in\Gamma_0(N)\backslash\mathcal{L}_N\\ \mathrm{sgn}\,d_N(v)=\pm}}\frac{\chi(v_1)\mathcal{M}\phi(v)}{|d_N(v)|^s},$$

$$\zeta^*_\pm(\phi,\tau_\chi;s)=\sum_{\substack{w^*\in\Gamma_0(N)\backslash V_{\mathbf{Z}}\\ \mathrm{sgn}\,\mathrm{disc}(w^*)=\pm}}\frac{\tau_\chi(w^*_3)\mathcal{M}\phi(w^*)}{|\mathrm{disc}\,w^*|^s}.$$

Then we have the following lemma, whose proof is similar to that of [9, Theorem 6.7].

Lemma 2. *The zeta functions $\zeta_\pm(\phi,\chi;s)$ and $\zeta^*_\pm(\phi,\tau_\chi;s)$ have analytic continuations to entire functions of s and satisfy the following functional equation*:

$$(3.1)\quad \begin{pmatrix}\zeta_+(\phi,\chi;\frac{3}{2}-s)\\ \zeta_-(\phi,\chi;\frac{3}{2}-s)\end{pmatrix} = \pi^{\frac{1}{2}-2s}N^{s-\frac{3}{2}}\Gamma\left(s+\frac{\lambda-1}{2}\right)\Gamma\left(s-\frac{\lambda}{2}\right)\cdot\Psi_\lambda(s)\begin{pmatrix}\zeta^*_+(\phi,\tau_\chi;s)\\ \zeta^*_-(\phi,\tau_\chi;s)\end{pmatrix},$$

where $\Psi_\lambda(s)$ is a 2×2 matrix given by

$$\Psi_\lambda(s)=\begin{pmatrix}\sin\pi s & \dfrac{2^{\lambda-1}\cdot\pi\Gamma(1-\lambda)}{\Gamma(1-\frac{\lambda}{2})^2}\cos\dfrac{\pi\lambda}{2}\\ \dfrac{\Gamma(1-\frac{\lambda}{2})^2}{2^{\lambda-1}\cdot\pi\Gamma(1-\lambda)}\sin\dfrac{\pi\lambda}{2} & \cos\pi s\end{pmatrix}.$$

Let r be an odd prime number r with $(N,r)=1$, and ψ a Dirichlet character of mod r. We denote by ψ^* the Dirichlet character defined as (2.2) with $\ell=1$. We define $\zeta_\pm(\phi,\chi,\tau_\psi;s)$ and $\zeta^*_\pm(\phi,\tau_\chi,\tau_{\psi^*};s)$ by

$$\zeta_\pm(\phi,\chi,\tau_\psi;s)=\sum_{\substack{v\in\Gamma_0(N)\backslash\mathcal{L}_N\\ \mathrm{sgn}\,d_N(v)=\pm}}\frac{\chi(v_1)\mathcal{M}\phi(v)\tau_\psi(d_N(v))}{|d_N(v)|^s},$$

$$\zeta^*_\pm(\phi,\tau_\chi,\tau_{\psi^*};s) = \sum_{\substack{w^*\in\Gamma_0(N)\backslash V_{\mathbf{Z}}\\ \mathrm{sgn}\,\mathrm{disc}(w^*)=\pm}}\frac{\tau_\chi(w^*_3)\mathcal{M}\phi(w^*)\tau_{\psi^*}(\mathrm{disc}\,w^*)}{|\mathrm{disc}\,w^*|^s}.$$

Then we have the following lemma.

Lemma 3. *The zeta functions $\zeta_\pm(\phi,\chi,\tau_\psi;s)$ and $\zeta^*_\pm(\phi,\tau_\chi,\tau_{\psi^*};s)$ have analytic continuations to entire functions of s and satisfy the following functional equation*:

$$(3.2)\quad \begin{pmatrix} \zeta_+(\phi,\chi,\tau_\psi;\frac{3}{2}-s) \\ \zeta_-(\phi,\chi,\tau_\psi;\frac{3}{2}-s) \end{pmatrix} = \varepsilon_r \chi_N(r)\psi^*(-4N) \cdot r^{2s-\frac{3}{2}}\pi^{\frac{1}{2}-2s}N^{s-\frac{3}{2}}\Gamma\left(s+\frac{\lambda-1}{2}\right)\Gamma\left(s-\frac{\lambda}{2}\right) \cdot \Psi_\lambda(s)\begin{pmatrix} \zeta_+^*(\phi,\tau_\chi,\tau_{\psi^*};s) \\ \zeta_-^*(\phi,\tau_\chi,\tau_{\psi^*};s) \end{pmatrix}.$$

The proof of Lemma 3 goes along the same line as Sato [8], Ueno [12]. In this case, however, it is necessary to calculate a kind of Gauss sums that have not appeared in the previous works. The author has learned such calculation from unpublished notes of Sato. We quote his result, which is a key ingredient and of independent interest. Let $f_{\psi,\chi}(v)$ be a function on $V_{\mathbf{Q}}$ defined by

$$f_{\psi,\chi}(v) = \begin{cases} \tau_\psi(d_N(v))\cdot\chi(v_1) & (v\in\mathcal{L}_N) \\ 0 & (v\notin\mathcal{L}_N) \end{cases}.$$

Let $\langle v, v^*\rangle$ be the inner product on V defined by $\langle v,v^*\rangle = \operatorname{tr}(vwv^*w^{-1})$ with $w=\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. For $v^*\in V_{\mathbf{Q}}$, we define the Fourier transform $\widehat{f_{\psi,\chi}}(v^*)$ by

$$(3.3)\quad \widehat{f_{\psi,\chi}}(v^*) = \frac{1}{[V_{\mathbf{Z}}:L]}\sum_{v\in V_{\mathbf{Q}}/L} f_{\psi,\chi}(v)\boldsymbol{e}[\langle v,v^*\rangle],$$

where L is a sufficiently small lattice so that $L\subset V_{\mathbf{Z}}$ and the value $f_{\psi,\chi}(v)\boldsymbol{e}[\langle v,v^*\rangle]$ depends only on the residue class $v+L$.

Lemma 4 (F. Sato). *If $v^*\notin\frac{1}{Nr}V_{\mathbf{Z}}$, then we have $\widehat{f_{\psi,\chi}}(v^*)=0$. If $v^*\in\frac{1}{Nr}V_{\mathbf{Z}}$, we have*

$$\widehat{f_{\psi,\chi}}(v^*) = \frac{\varepsilon_r}{2r^{\frac{3}{2}}N^3}\chi_N(r)\cdot\psi^*(-4N)\tau_\chi(w_3^*)\tau_{\psi^*}(\operatorname{disc}(w^*)).$$

4. An outline of the proof of Theorem 1. We construct L-functions satisfying two conditions [C1] and [C2]. In the functional equation (2.1), we let $\ell=1$ and $\mu=\frac{2\lambda+1}{4}$, and replace N by $4N$. Then it follows from an elementary calculation that (2.1) is transformed as

$$(4.1)\quad \begin{pmatrix} \xi_+(\alpha;s) \\ \xi_-(\alpha;s) \end{pmatrix} = (4N)^{\frac{3}{2}-\lambda-s}\cdot 2^{2s+\lambda-\frac{3}{2}} \cdot \pi^{2s+\lambda-\frac{5}{2}}\cdot\boldsymbol{e}\left[\frac{1}{8}\right]\Gamma(1-s)\Gamma\left(\frac{3}{2}-\lambda-s\right) \cdot \begin{pmatrix} -\cos\pi(s+\frac{\lambda}{2}) & \sin\frac{\pi\lambda}{2} \\ \cos\frac{\pi\lambda}{2} & -\sin\pi(s+\frac{\lambda}{2}) \end{pmatrix} \cdot \begin{pmatrix} \xi_+(\beta;\frac{3}{2}-\lambda-s) \\ \xi_-(\beta;\frac{3}{2}-\lambda-s) \end{pmatrix}.$$

We put

$$\widetilde{\zeta}_+(\phi,\chi;s) := 2^{2-\lambda}\cdot\frac{\Gamma(\lambda)}{\Gamma(\frac{\lambda}{2})^2}\cdot\zeta_+\left(\phi,\chi;s+\frac{\lambda}{2}\right),$$

$$\widetilde{\zeta}_-(\phi,\chi;s) := \zeta_-\left(\phi,\chi;s+\frac{\lambda}{2}\right),$$

$$\widetilde{\zeta}_+^*(\phi,\tau_\chi;s) := 2^{\frac{1}{2}}\cdot N^{-\frac{3}{2}+\frac{\lambda}{2}}\cdot\frac{\Gamma(\lambda)}{\Gamma(\frac{\lambda}{2})^2} \cdot\zeta_+^*\left(\phi,\tau_\chi;s+\frac{\lambda}{2}\right),$$

$$\widetilde{\zeta}_-^*(\phi,\tau_\chi;s) := 2^{\lambda-\frac{3}{2}}\cdot N^{-\frac{3}{2}+\frac{\lambda}{2}}\cdot\zeta_-^*\left(\phi,\tau_\chi;s+\frac{\lambda}{2}\right).$$

Then (3.1) can be rewritten as

$$(4.2)\quad \begin{pmatrix} \widetilde{\zeta}_+(\phi,\chi;s) \\ \widetilde{\zeta}_+(\phi,\chi;s) \end{pmatrix} = (4N)^{\frac{3}{2}-\lambda-s}\cdot 2^{2s+\lambda-\frac{3}{2}} \cdot \pi^{2s+\lambda-\frac{5}{2}}\cdot\boldsymbol{e}\left[\frac{1}{8}\right]\Gamma(1-s)\Gamma\left(\frac{3}{2}-\lambda-s\right) \cdot \begin{pmatrix} -\cos\pi(s+\frac{\lambda}{2}) & \sin\frac{\pi\lambda}{2} \\ \cos\frac{\pi\lambda}{2} & -\sin\pi(s+\frac{\lambda}{2}) \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\zeta}_+^*(\phi,\tau_\chi;\frac{3}{2}-\lambda-s) \\ \widetilde{\zeta}_-^*(\phi,\tau_\chi;\frac{3}{2}-\lambda-s) \end{pmatrix},$$

which agrees with (4.1). Similarly, the functional equation (3.2) of the twisted zeta functions can be compared to (2.3) in the condition $[\mathrm{C2}]_{r,\psi}$. Now the converse theorem (Lemma 1) applies, and we obtain a Maass form $F(z)$. To prove the cuspidality of $F(z)$, we compare an integral representation of the zeta functions $\zeta_\pm(\phi,\chi;s)$ with an integral of some theta series. Let $\mathcal{S}(V)$ be the space of rapidly decreasing functions on $V=\operatorname{Sym}_2(\mathbf{R})$, and $S=\{v\in V;\det v=0\}$. For $f_\infty(v)\in\mathcal{S}(V)$, we set

$$\Theta(\phi,\mathcal{L}_N,f_\infty) = \int_{SL_2(\mathbf{R})/\Gamma_0(N)}\phi(g)\sum_{v\in\mathcal{L}_N\setminus S}\chi(v_1)f_\infty(gv\,{}^tg)dg,$$

and

$$Z_\phi(\mathcal{L}_N,f_\infty;s) = \int_0^\infty t^{2s-1}\Theta(\phi,\mathcal{L}_N,f_\infty^t)dt,$$

where $f_\infty^t(v)=f_\infty(tv)$. Then, as shown in [9, Proposition 6.4], the zeta integral can be decomposed as

$$Z_\phi(\mathcal{L}_N, f_\infty; s) = \sum_{\varepsilon=\pm} \zeta_\varepsilon(\phi, \chi; s)\Gamma_\varepsilon(f_\infty; \lambda, s),$$

where $\Gamma_\varepsilon(f_\infty; \lambda, s)$ is a certain local zeta integral. On the other hand, let $\sigma \mapsto r(\sigma)$ be the Weil representation of $G = SL_2(\mathbf{R})$ on $\mathcal{S}(V)$ given as in [11, p. 91]. For $z = x + iy \in \mathcal{H}$, we set

$$\sigma_z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \in G,$$

and $f_{\infty,z}(v) = \{r(\sigma_z) f_\infty\}(v)$. We define a function $H_\Theta(z)$ on $\mathcal{H}$ by

$$H_\Theta(z) = \Theta(\phi, \mathcal{L}_N, f_{\infty,z}) = \Theta(\phi, \mathcal{L}_N, r(\sigma_z) f_\infty).$$

We observe that $H_\Theta(z)$ closely resembles the theta integral (1.22) of [11, p. 97]. Since $f_{\infty,iy}(v) = y^{3/4} f_\infty(\sqrt{y}\, v)$, the Mellin transform of $H_\Theta(z)$ along the imaginary axis coincides with our zeta integral $Z_\phi(\mathcal{L}_N, f_\infty; s)$, and by the Mellin inversion, $H_\Theta(iy)$ equals to $F(iy)$ up to constant. Now let γ be an arbitrary element of $SL_2(\mathbf{Z})$. Then, by [11, p. 98],

$$r(\gamma) r(\sigma_z) = r(\sigma_{\gamma z}) r(k_\theta),$$

where $e^{i\theta} = J(\gamma, z)/|J(\gamma, z)|$. If f_∞ satisfies the condition (1.19) of [11], then $H_\Theta(\gamma z)$ coincides with

$$\Theta(\phi, \mathcal{L}_N, r(\gamma) r(\sigma_z) f_\infty)$$

up to constant, and the cuspidality of ϕ implies that $Z_\phi(\mathcal{L}_N, r(\gamma) r(\sigma_z) f_\infty; s)$ is an entire function of s. We therefore observe that $H_\Theta(\gamma z)$ has rapid decay as $y \to \infty$, and this proves that $F(\gamma z)$ has rapid decay at every cusp of $\Gamma_0(4N)$. Further details will be discussed elsewhere.

Remark 2.

(1) The argument above shows that the prehomogeneous zeta functions $\zeta_\pm(\phi, \chi; s)$ can be interpreted as the Mellin transform of some kind of theta lift of ϕ. In this sense, our lifting construction is not new. However, our proof does not rely on the theta transformation formula, and we expect that our method can be applied to various other cases.

(2) In a paper [4] that appeared very recently, the Katok-Sarnak formula is generalized for Maass forms of even weight and odd level with trivial characters. It is an interesting problem to combine their technique, such as use of differential operators, with our method.

Acknowledgements. The author wishes to thank Prof. Fumihiro Sato for his kind guidance and for giving the author permission to use his unpublished results. The author also thanks the referee for careful reading, and for pointing out that the proof for cuspidality in the original manuscript was incorrect.

References

[1] E. M. Baruch and Z. Mao, A generalized Kohnen-Zagier formula for Maass forms, J. Lond. Math. Soc. (2) **82** (2010), no. 1, 1–16.
[2] A. Biró, Cycle integrals of Maass forms of weight 0 and Fourier coefficients of Maass forms of weight 1/2, Acta Arith. **94** (2000), no. 2, 103–152.
[3] W. Duke, Ö. Imamoḡlu and Á. Tóth, Geometric invariants for real quadratic fields, Ann. of Math. (2) **184** (2016), no. 3, 949–990.
[4] Ö. Imamoğlu, A. Lägeler and Á. Tóth, The Katok-Sarnak formula for higher weights, J. Number Theory **235** (2022), 242–274.
[5] S. Katok and P. Sarnak, Heegner points, cycles and Maass forms, Israel J. Math. **84** (1993), 193–227.
[6] H. Maaß, Über die räumliche Verteilung der Punkte in Gittern mit indefiniter Metrik, Math. Ann. **138** (1959), 287–315.
[7] T. Miyazaki, F. Sato, K. Sugiyama and T. Ueno, Converse theorems for automorphic distributions and Maass forms of level N, Res. Number Theory **6** (2020), no. 1, Paper No. 6.
[8] F. Sato, On functional equations of zeta distributions, in *Automorphic forms and geometry of arithmetic varieties*, Adv. Stud. Pure Math., 15, Academic Press, Boston, 1989, pp. 465–508.
[9] F. Sato, Zeta functions of prehomogeneous vector spaces with coefficients related to periods of automorphic forms, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), no. 1, 99–135.
[10] G. Shimura, On modular forms of half integral weight, Ann. of Math. (2) **97** (1973), 440–481.
[11] T. Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J. **58** (1975), 83–126.
[12] T. Ueno, Modular forms arising from zeta functions in two variables attached to prehomogeneous vector spaces related to quadratic forms, Nagoya Math. J. **175** (2004), 1–37.
[13] A. Weil, Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Math. Ann. **168** (1967), 149–156.