A quantitative study of orbit counting and discrete spectrum for anti-de Sitter 3-manifolds

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Abstract: Let Γ be a discontinuous group for the 3-dimensional *anti-de Sitter space* $\mathrm{AdS}^3 := \mathrm{SO}_0(2,2)/\mathrm{SO}_0(2,1)$. In this article, we discuss a growth rate of the counting of Γ-orbits at infinity and the discrete spectrum of the hyperbolic Laplacian of the complete anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$.

Key words: Anti-de Sitter manifold; anti-de Sitter space; discontinuous group; counting problem; hyperbolic Laplacian.

1. Introduction. The 3-dimensional anti-de Sitter space $\mathrm{AdS}^3 := \mathrm{SO}_0(2,2)/\mathrm{SO}_0(2,1)$ is a Lorentzian manifold with constant sectional curvature -1 of which the identity component of the isometry group is the Lie group $\mathrm{SO}_0(2,2)$. Discontinuous groups for AdS^3 and their deformation theory have been developed by renowned mathematicians, William Goldman, Toshiyuki Kobayashi, and Fanny Kassel, among others.

In this article, we discuss a growth rate of the counting of orbits of a discontinuous group Γ for AdS^3 at infinity and the discrete spectrum of the hyperbolic Laplacian of the complete anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$. Detailed proofs of the results will appear elsewhere.

2. Relationship between the sharpness of the Γ-action and a growth rate of the counting at infinity. In old days, the terminology "discontinuous groups" was used to denote the same meaning of discrete subgroups. Indeed, the action of a discrete group of isometries is automatically properly discontinuous in the Riemannian setting. In his study of the action of discrete groups beyond the Riemannian setting, Kobayashi [13] advocated to make a difference of two terminologies: discontinuous groups for the property of *actions*, and discrete subgroups for the property of *groups*. Following this principle, we call a discrete subgroup Γ of a Lie group G a *discontinuous group for* a homogeneous manifold G/H if the natural Γ-action on G/H from the left is properly discontinuous and free [13, Def. 1.3]. Then any Γ-orbit meets a compact subset of G/H in at most finitely many points, and thus we may consider the number of the intersection points. Kassel-Kobayashi [6] introduced a compact subset $B(R)$ called a pseudo-ball of radius $R > 0$ in any semisimple symmetric space G/H, in particular, in AdS^3, of which the volume is of exponential growth as $R \to \infty$. Moreover, they studied a growth rate of the *counting*

$$N_\Gamma(x, R) := \#(\Gamma x \cap B(R))$$

of the Γ-orbit through $x \in G/H$ as $R \to \infty$.

When the metric tensor is indefinite as in the anti-de Sitter space AdS^3, an isotropy subgroup of the isometry group is not necessarily compact and an orbit of a discrete subgroup Γ of isometries may have accumulation points. In particular, Γ may not act on G/H properly discontinuously. Generalizing a pioneering work of Kobayashi [10] on the properness criterion by means of the Cartan projection for homogeneous manifolds of reductive type, Kobayashi [11] and Benoist [1] established a criterion for a general discrete subgroup Γ of a reductive Lie group G to act properly discontinuously on G/H. As a slightly stronger condition than this criterion, Kassel-Kobayashi [6] introduced the notion of (c, C)-*sharpness* ($c > 0$, $C \geq 0$) of a discontinuous group which quantifies proper discontinuity. Loosely speaking, the parameter $c > 0$ indicates that the "degree of proper discontinuity" of the Γ-action is weaker if c approaches to 0. Then they gave an upper estimate of the counting for

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(c,C)-sharp discontinuous groups for any semisimple symmetric space G/H, in particular, for AdS^3 by means of the two constants c and C, and proved that the counting $N_\Gamma(x,R)$ is of exponential growth uniformly with respect to $x \in G/H$ as $R \to \infty$:

Fact 1 (Kassel-Kobayashi [6, Lem. 4.6 (4)])**.** *There exists $A > 0$ such that for any $c > 0$, $C \geq 0$, and torsion-free (c,C)-sharp discontinuous group Γ for AdS^3, one has*

$$\forall x \in \mathrm{AdS}^3,\ \forall R > 0,\ N_\Gamma(x,R) \leq A \exp\left(\frac{4(R+C)}{c}\right).$$

On the other hand, there has been no existing literature about the counting for a non-sharp discontinuous group (the case $c = 0$) to the best knowledge of the author. We find non-sharp discontinuous groups Γ with various behaviors of the counting of Γ-orbits:

Theorem 2. *There exists a non-sharp discontinuous group Γ for AdS^3 such that*

$$\forall x \in \mathrm{AdS}^3,\ \forall R > 0,\ N_\Gamma(x,R) \leq 4^R.$$

In particular, $N_\Gamma(x,R)$ is of exponential growth uniformly with respect to $x \in \mathrm{AdS}^3$ as $R \to \infty$.

Theorem 3. *For any monotone increasing function $f\colon \mathbf{R} \to \mathbf{R}_{>0}$ and any $x \in \mathrm{AdS}^3$, there exists a discontinuous group $\Gamma \equiv \Gamma_{f,x}$ for AdS^3 satisfying*

$$\lim_{R\to\infty} \frac{N_\Gamma(x,R)}{f(R)} = \infty.$$

For example, applying Theorem 3 to $f(R) = \exp(e^R)$, we can construct a discontinuous group Γ satisfying

$$\lim_{R\to\infty} \frac{\#(\Gamma x \cap B(R))}{\mathrm{vol}(B(R))} = \infty.$$

It should be noted that Eskin-Mcmullen [2] also considered the counting of a Γ-orbit Γx for a general semisimple symmetric space G/H. They dealt with the case where Γ is a lattice of G and x is a special point in G/H, and thus their setting is completely different from [6] and also from ours.

3. Construction of non-sharp discontinuous groups. In this section, we describe how to construct non-sharp discontinuous groups for AdS^3 used in the proofs of Theorems 2 and 3. We note that the product group $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ acts isometrically on $\mathrm{AdS}^3 = SO_0(2,2)/SO_0(2,1)$ via the double covering $SL(2,\mathbf{R}) \times SL(2,\mathbf{R}) \to SO_0(2,2)$.

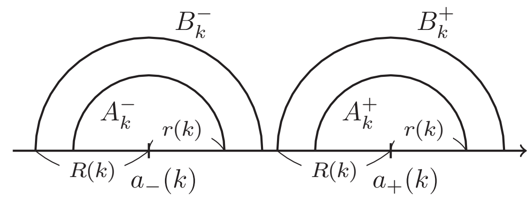

Fig. 1. $A_k^\pm$ and $B_k^\pm$ in $\mathbf{H}^2$.

Generalizing a non-sharp example of Guéritaud-Kassel [3, Sect. 10.1], we construct a family of infinitely generated subgroups of $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$. Our subgroup has four sequences $(a_-(k), a_+(k), r(k), R(k))_{k\in\mathbf{N}}$ as parameters. We find a properness criterion and a sharpness criterion for the actions of our subgroups on AdS^3 using the asymptotic behaviors of these sequences.

For a quadruple of real-valued sequences (a_-, a_+, r, R), we define $\alpha_k, \beta_k \in SL(2,\mathbf{R})$ by

$$\alpha_k = \frac{1}{r(k)}\begin{pmatrix} a_+(k) & -(a_-(k)a_+(k) + r(k)^2) \\ 1 & -a_-(k) \end{pmatrix},$$
$$\beta_k = \frac{1}{R(k)}\begin{pmatrix} a_+(k) & -(a_-(k)a_+(k) + R(k)^2) \\ 1 & -a_-(k) \end{pmatrix},$$

and denote by $\Gamma_\nu(a_-, a_+, r, R)$ for sufficiently large $\nu \in \mathbf{N}$ the subgroup generated by $(\alpha_k, \beta_k) \in SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ for all $k = \nu, \nu+1, \ldots$.

Let A_k^ϵ and B_k^ϵ for $\epsilon \in \{+,-\}$ be respectively the half-disks in the upper half plane $\mathbf{H}^2 = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ defined by

$$A_k^\epsilon := \{z \in \mathbf{H}^2 \mid |z - a_\epsilon(k)| \leq r(k)\},$$
$$B_k^\epsilon := \{z \in \mathbf{H}^2 \mid |z - a_\epsilon(k)| \leq R(k)\},$$

see Fig. 1. Then we note

$$\alpha_k(A_k^-) \subset \mathbf{H}^2 \setminus A_k^+,\ \beta_k(B_k^-) \subset \mathbf{H}^2 \setminus B_k^+,$$

where $SL(2,\mathbf{R})$ acts on $\mathbf{H}^2$ as linear fractional transformations. One can see by an elementary argument of general topology called the ping-pong argument that the subgroup $\Gamma_\nu(a_-, a_+, r, R)$ is discrete and free if the half-disks $A_\nu^\pm, A_{\nu+1}^\pm, \ldots$ (resp. $B_\nu^\pm, B_{\nu+1}^\pm, \ldots$) are disjoint.

Let $p(x)$ be a real-valued monotone increasing C^2-function defined for sufficiently large $x \in \mathbf{R}$ such that $\lim_{x\to\infty} p(x) = \infty$ and that the second derivative $p''(x)$ is nowhere vanishing. In this article, for simplicity, we assume that the pair of sequences $(a_+(k), a_-(k))$ can be expressed as

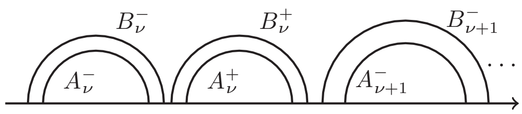

Fig. 2. $A_\nu^\pm$, $A_{\nu+1}^\pm$, ... and $B_\nu^\pm$, $B_{\nu+1}^\pm$, ...

Table I. Sharpness of the Γ_ν-action on AdS^3.

$p(x)$	$r(k)$	$R(k)$	the Γ_ν-action
e^x	$e^{-(k+k^2)}$	e^{-k^2}	non-sharp
e^x	1	e^k	sharp
$\log x$	$(k^2\log k)^{-1}$	k^{-2}	non-sharp
$\log x$	k^{-3}	k^{-2}	sharp

$$(3.1) \qquad a_-(k) = p(k),\ a_+(x) = p\left(k+\frac{1}{2}\right)$$

for sufficiently large $k \in \mathbf{N}$. Moreover, we suppose

$$(3.2) \qquad R(k) > r(k),$$

$$(3.3) \qquad \lim_{k\to\infty} \frac{R(k)}{\min\{p'(k-1), p'(k+1)\}} = 0.$$

Then $B_k^\epsilon \supset A_k^\epsilon$ holds and an easy calculation shows that the half-disks $B_\nu^\pm$, $B_{\nu+1}^\pm$, ... are disjoint for sufficiently large $\nu \in \mathbf{N}$, see Fig. 2.

The following are a properness criterion and a sharpness criterion for the action on AdS^3 of the discrete subgroup $\Gamma_\nu(a_-, a_+, r, R)$:

Proposition 4. *Let (a_-, a_+, r, R) be a quadruple of sequences satisfying* (3.1)–(3.3) *as above. The action on* AdS^3 *of the discrete subgroup* $\Gamma_\nu(a_-, a_+, r, R)$ *for sufficiently large* $\nu \in \mathbf{N}$ *is*:

(1) *properly discontinuous if and only if*

$$\lim_{k\to\infty} \frac{R(k)}{r(k)} = \infty;$$

(2) *sharp if and only if*

$$\liminf_{k\to\infty} \log\left(\frac{R(k)}{r(k)}\right)\left(\log\frac{a_-(k)a_+(k)}{r(k)}\right)^{-1} \neq 0.$$

Example 5. For the triples $(p(x), r(k), R(k))$ in Table I, we form the subgroups $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ with (3.1)–(3.3). Then Proposition 4 shows that Γ_ν are all discontinuous groups for AdS^3 for sufficiently large $\nu \in \mathbf{N}$ but not always sharp as summarized in Table I.

4. Discrete spectrum of non-sharp anti-de Sitter manifolds. Next we consider discrete spectrum of the Laplacian of the noncompact anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ for a non-sharp discontinuous group Γ.

Let us recall some basic notions. A *pseudo-Riemannian manifold* is a C^∞-manifold equipped with a smooth non-degenerate symmetric bilinear tensor of signature (p, q). It is called *Riemannian* if $q = 0$ and *Lorentzian* if $q = 1$. As in the Riemannian case, $\square = \operatorname{div}\circ\operatorname{grad}$ defines a second order differential operator (the *Laplacian*) on a pseudo-Riemannian manifold. In contrast to the Riemannian setting, the Laplacian on a Lorentzian manifold is not an elliptic differential operator but a hyperbolic differential operator, and its eigenfunction is not analytic in general.

We write $L^2(M)$ for the Hilbert space of square integrable functions with respect to the volume form induced by the pseudo-Riemannian structure of M, and denote by $L^2_\lambda(M)$ for $\lambda \in \mathbf{C}$ the space of square integrable eigenfunctions

$$\{f \in L^2(M) \mid \square_M f = \lambda f \text{ in the weak sense}\}.$$

Then the set of L^2-eigenvalues

$$\mathrm{Spec}_d(\square_M) := \{\lambda \in \mathbf{C} \mid L^2_\lambda(M) \neq 0\}$$

is called the *discrete spectrum* of the Laplacian of M.

We recall the theory of Kassel-Kobayashi [6] on the discrete spectrum of "intrinsic" differential operators on locally semisimple symmetric spaces by limiting ourselves to the case AdS^3. Let Γ be a discontinuous group for AdS^3. Then the quotient space $\Gamma\backslash\mathrm{AdS}^3$ is a C^∞-manifold and the quotient map $\mathrm{AdS}^3 \to \Gamma\backslash\mathrm{AdS}^3$ is a covering map of C^∞-class. The quotient manifold $\Gamma\backslash\mathrm{AdS}^3$ admits a Lorentzian structure with constant sectional curvature -1 via this covering map. Kassel-Kobayashi [6] and Kobayashi [14] initiated the study of spectral analysis on locally symmetric spaces, in particular, that of the discrete spectrum $\mathrm{Spec}_d(\square)$ of the hyperbolic Laplacian $\square$ on the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$.

They introduced "the Γ-averages of non-periodic eigenfunctions" as a generalization of Poincaré series to construct L^2-eigenvalues. If an eigenfunction φ of the Laplacian on AdS^3 is integrable, then the *generalized Poincaré series*

$$\varphi^\Gamma(\Gamma x) := \sum_{\gamma\in\Gamma} \varphi(\gamma^{-1}x)$$

defines an integrable function on the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$, and is an eigenfunction of the Laplacian with same eigenvalue. It is known that the Laplacian on AdS^3 has the following L^2-eigenvalues:

$$\lambda_m := 4m(m-1) \quad (m \in \mathbf{Z} \text{ and } m \geq 2).$$

As an application of an upper estimate of the counting as in Fact 1, they proved L^2-convergence and non-vanishing of the generalized Poincaré series of eigenfunctions for sufficiently large eigenvalue λ_m, and obtained the following theorem:

Fact 6 [6]. *For any sharp discontinuous group Γ for AdS^3, there exists a constant $m_0(\Gamma) > 0$ such that*

$$\mathrm{Spec}_d(\Box_{\Gamma\backslash\mathrm{AdS}^3}) \supset \{\lambda_m \mid m \in \mathbf{Z},\ m > m_0(\Gamma)\}.$$

A natural question would be whether the Laplacian on an anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ still has an L^2-eigenvalue if the discontinuous group Γ is non-sharp. As an application of an upper estimate of the counting as in Theorem 2, we see that there exist countably many L^2-eigenvalues for some non-sharp Γ by applying the machinery developed in [6]:

Theorem 7. *There exist a non-sharp discontinuous group Γ for AdS^3 and a constant $m_0'(\Gamma) > 0$ such that*

$$\mathrm{Spec}_d(\Box_{\Gamma\backslash\mathrm{AdS}^3}) \supset \{\lambda_m \mid m \in \mathbf{Z},\ m > m_0'(\Gamma)\}.$$

5. Multiplicity of the discrete spectrum. In the final section we discuss the multiplicity of the L^2-eigenvalue λ_m of the Laplacian of an anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ constructed by the generalized Poincaré series. Here, for a pseudo-Riemannian manifold M,

$$\mathcal{N}_M(\lambda) := \dim_{\mathbf{C}} L^2_\lambda(M) \in \mathbf{N} \cup \{\infty\}$$

is called the multiplicity of an L^2-eigenvalue λ. The Laplacian on a Riemannian manifold is an elliptic differential operator and the multiplicity of an L^2-eigenvalue is always finite if M is compact. However, in the Lorentzian setting, the multiplicity may be finite or may not even if M is compact (e.g., [8,14]).

If a discontinuous group Γ for AdS^3 is standard [6, Def. 1.4] and torsion-free, $\mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) = \infty$ for sufficiently large $m \in \mathbf{N}$, which is derived from the results in Kassel-Kobayashi [7,8]. On the other hand, there exists a non-standard discontinuous group Γ, for example a finitely generated discontinuous group Γ which is Zariski-dense in the Lie group $SO(2,2)$ [9,12]. However, it is not known whether the multiplicities of the Laplacian are finite in this case. We see that the multiplicities of the Laplacian on the anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ for such Γ are unbounded as follows:

Theorem 8. *For any finitely generated discontinuous group Γ for AdS^3, there exists a constant $c_\Gamma > 0$ such that*

$$\mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) \geq \log_3 m - c_\Gamma. \tag{5.1}$$

In particular,

$$\lim_{m\to\infty} \mathcal{N}_{\Gamma\backslash\mathrm{AdS}^3}(\lambda_m) = \infty.$$

To prove this theorem, we use $SO(2) \times SO(2)$-finite L^2-eigenfunctions of the Laplacian on AdS^3 with eigenvalue λ_m vanishing at the origin. We note that such eigenfunctions decay more rapidly at infinity than at the origin with respect to geodesic parameters. We choose an L^2-eigenfunction with eigenvalue λ_m for each $j = 0, 1, \ldots, k-1$ which decays at the origin as rapidly as R^{3^j} when a "pseudo-distance" R from the origin tends to zero, and show the linear independence of their generalized Poincaré series when $m > 3^{k+c_\Gamma}$, which proves (5.1).

Finally we discuss a lower bound of the multiplicities of L^2-eigenvalues under a small deformation of a discrete subgroup. The general study of local rigidity and stability of discontinuous groups for non-Riemannian homogeneous manifolds was initiated by Kobayashi [12] and Kobayashi-Nasrin [15], and has been further developed by Kassel [5] and others in specific settings. In our AdS^3 setting, any cocompact discontinuous group is not locally rigid and its proper discontinuity is stable under any small deformation [9,12]. Moreover, Kassel-Kobayashi [6] constructed infinitely many *stable L^2-eigenvalues* of the Laplacian of any compact anti-de Sitter manifold $\Gamma\backslash\mathrm{AdS}^3$ under any small deformation of Γ. More specifically, for sufficiently large $m \in \mathbf{N}$, one has

$$\lambda_m \in \bigcap_{\Gamma'} \mathrm{Spec}_d(\Box_{\Gamma'\backslash\mathrm{AdS}^3}),$$

where Γ' runs over a sufficiently small neighborhood of Γ in the compact-open topology [6, Cor. 9.10], see [6, Def. 1.6] for the definition of stable eigenvalues in a much more general setting. We introduce a function $\widetilde{\mathcal{N}}_{\Gamma\backslash \mathrm{AdS}^3}\colon \mathbf{C} \to \mathbf{N} \cup \{\infty\}$ satisfying the following for the multiplicities of stable eigenvalues:

- $\widetilde{\mathcal{N}}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda) \neq 0$ if and only if λ is a stable L^2-eigenvalue of $\Box_{\Gamma\backslash \mathrm{AdS}^3}$;
- $\mathcal{N}_{\Gamma'\backslash \mathrm{AdS}^3}(\lambda) \geq \widetilde{\mathcal{N}}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda)$ for any Γ' sufficiently close to Γ.

Theorem 9. *For any cocompact discontinuous group* Γ *for* AdS^3,

$$\lim_{m\to\infty} \widetilde{\mathcal{N}}_{\Gamma\backslash \mathrm{AdS}^3}(\lambda_m) = \infty.$$

The constant c_Γ also plays a crucial role in the proof of Theorem 9. Here recall (5.1). The geometric constant c_Γ is defined by using

- a growth rate of the counting $N_\Gamma(x, R)$ as $R \to \infty$;
- the "injective radius" of the anti-de Sitter manifold $\Gamma\backslash \mathrm{AdS}^3$.

We control these two quantities simultaneously using Lipschitz constants associated to Γ introduced in Kassel [4] and Kassel-Kobayashi [6], and further investigated by Guéritaud-Kassel [3], and show that c_Γ depends "continuously" on a small deformation of Γ. We prove that the larger $m \in \mathbf{N}$ is, the more linearly independent L^2-eigenfunctions of the Laplacian of the compact anti-de Sitter manifold $\Gamma\backslash \mathrm{AdS}^3$ can be constructed and that their construction is stable under any small deformation of Γ.

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