Rigidity of Euler products

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Abstract: We report simple rigidity theorems for Euler products under deformations of Euler factors. Certain products of the Riemann zeta function are rigid in the sense that there exist no deformations which preserve the meromorphy on **C**.

Key words: Zeta functions; Euler products; rigidity.

Introduction. For purely imaginary numbers $a,\ b,\ c \in i\mathbf{R}$ we study the meromorphy of the associated Euler product

$$Z^{abc}(s) = \prod_{p:\ \text{prime}} (1 - (p^a + p^b)p^{-s} + p^{c-2s})^{-1}$$

in the family of Euler products

$$\mathfrak{Z}^{ab} = \{Z^{abc}(s) \mid c \in i\mathbf{R}\}.$$

This family $\mathfrak{Z}^{ab}$ contains

$$\zeta(s-a)\zeta(s-b) = Z^{ab(a+b)}(s),$$

which is a meromorphic function in all $s \in \mathbf{C}$ (with a functional equation under $s \longleftrightarrow 1 + a + b - s$). We prove the converse:

Theorem A. *If $Z^{abc}(s)$ is meromorphic in all $s \in \mathbf{C}$, we have $a + b = c$ and*

$$Z^{abc}(s) = \zeta(s-a)\zeta(s-b).$$

This shows rigidity of the Euler product $\zeta(s-a)\zeta(s-b)$ in the family $\mathfrak{Z}^{ab}$ concerning the meromorphy on the entire **C**.

The next result gives a detailed meromorphy for $a + b \neq c$.

Theorem B. *If $a + b \neq c$, then $Z^{abc}(s)$ has an analytic continuation to $\mathrm{Re}(s) > 0$ as a meromorphic function with the natural boundary $\mathrm{Re}(s) = 0$. More precisely, each point on $\mathrm{Re}(s) = 0$ is a limit point of poles of $Z^{abc}(s)$ in $\mathrm{Re}(s) > 0$.*

We notice generalizations in §4 in the text. Our theorems follow from results of Kurokawa [4–6] extending results of Estermann [1].

We remark that our result characterizes $\zeta(s-a)\zeta(s-b)$ by the meromorphy in all $s \in \mathbf{C}$ only in contrast to usual "converse theorems" originated by Hamburger [2] and Hecke [3] where the functional equation and the attached automorphic form are important; $\zeta(s-a)\zeta(s-b)$ corresponds to a Maass wave form studied by Maass [7].

1. Euler datum. We use the triple $E = (P, \mathbf{R}, \alpha)$, where P is the set of all prime numbers, $\mathbf{R}$ denotes the real numbers, and α is the map $\alpha : P \to \mathbf{R}$ given by $\alpha(p) = \log p$. Such a triple is a simple example of Euler datum studied in [4–6]; generalized Euler data are treated there with general "primes" P and general topological groups G instead of $\mathbf{R}$.

Let $R(\mathbf{R})$ be the virtual character ring of $\mathbf{R}$ defined as

$$R(\mathbf{R}) = \left\{ \sum_{a \in i\mathbf{R}} m(a)\chi_a \;\middle|\; m(a) \in \mathbf{Z}, m(a) = 0 \text{ except for finitely many } a \right\},$$

where χ_a is a (continuous) unitary character $\chi_a : \mathbf{R} \to U(1)$ given by $\chi_a(x) = e^{ax}$ for $x \in \mathbf{R}$.

For a polynomial

$$H(T) = \sum_{m=0}^{n} h_m T^m \in 1 + TR(\mathbf{R})[T]$$

we denote by $L(s, E, H)$ the Euler product

$$L(s, E, H) = \prod_{p \in P} H_{\alpha(p)}(p^{-s})^{-1},$$

where

$$H_x(T) = \sum_{m=0}^{n} h_m(x) T^m \in 1 + T\mathbf{C}[T].$$

For example, let $a,\ b,\ c \in i\mathbf{R}$, then the polynomial

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$$H^{abc}(T) = 1 - (\chi_a + \chi_b)T + \chi_c T^2$$

in $1 + TR(\mathbf{R})[T]$ gives the Euler product

$$L(s, E, H^{abc}) = \prod_{p \in P} (1 - (p^a + p^b)p^{-s} + p^{c-2s})^{-1}$$

since

$$\begin{aligned} H^{abc}_{\alpha(p)} &= 1 - (\chi_a(\log p) + \chi_b(\log p))p^{-s} + \chi_c(\log p)p^{-2s} \\ &= 1 - (p^a + p^b)p^{-s} + p^{c-2s}. \end{aligned}$$

2. Unitariness and meromorphy. Let $E = (P, \mathbf{R}, \alpha)$ as in §1 and take a polynomial $H(T)$ in $1 + TR(\mathbf{R})[T]$ of degree n. We say that $H(T)$ is *unitary* when there exist functions $\theta_j; \mathbf{R} \to \mathbf{R}$ satisfying

$$H_x(T) = (1 - e^{i\theta_1(x)}T) \cdots (1 - e^{i\theta_n(x)}T)$$

for all x.

The main theorem proved in [5] gives in this particular situation the following result:

Theorem 1.

(1) *If $H(T)$ is unitary, then $L(s, E, H)$ is meromorphic in all $s \in \mathbf{C}$.*

(2) *If $H(T)$ is not unitary, then $L(s, E, H)$ is meromorphic in* $\mathrm{Re}(s) > 0$ *with the natural boundary. Moreover, each point on* $\mathrm{Re}(s) = 0$ *is a limit point of poles of $L(s, E, H)$ in* $\mathrm{Re}(s) > 0$.

This theorem was proved in [5] (p. 45, §8, Theorem 1) since our $E = (P, \mathbf{R}, \alpha)$ is nothing but $\overline{E_0(\mathbf{Q}/\mathbf{Q})}$ there.

3. Proof of rigidity. After looking Theorem 1 recalled in §2 we see that Theorems A and B in Introduction are both derived from the following result:

Theorem 2. *Let a, b, $c \in i\mathbf{R}$ and*

$$H^{abc}(T) = 1 - (\chi_a + \chi_b)T + \chi_c T^2 \in 1 + TR(\mathbf{R})[T].$$

Then the following conditions are equivalent.

(1) *$a + b = c$.*

(2) *$H(T)$ is unitary.*

Proof. (1)$\Longrightarrow$(2): From $a + b = c$ we get

$$\begin{aligned} H^{abc}(T) &= 1 - (\chi_a + \chi_b)T + \chi_a\chi_b T^2 \\ &= (1 - \chi_a T)(1 - \chi_b T). \end{aligned}$$

This gives

$$\begin{aligned} H^{abc}_x(T) &= (1 - \chi_a(x)T)(1 - \chi_b(x)T) \\ &= (1 - e^{ax}T)(1 - e^{bx}T) \end{aligned}$$

for $x \in \mathbf{R}$. Hence $H^{abc}(T)$ is unitary, by

$$|e^{ax}| = |e^{bx}| = 1.$$

(2)$\Longrightarrow$(1): Assume that $H^{abc}(T)$ is unitary, and set

$$H^{abc}_x(T) = (1 - e^{i\theta_1(x)}T)(1 - e^{i\theta_2(x)}T)$$

with $\theta_j : \mathbf{R} \to \mathbf{R}$. Then comparing with

$$H^{abc}_x(T) = 1 - (e^{ax} + e^{bx})T + e^{cx}T^2$$

we obtain

$$e^{ax} + e^{bx} = e^{i\theta_1(x)} + e^{i\theta_2(x)}, \tag{3.1}$$

$$e^{cx} = e^{i(\theta_1(x) + \theta_2(x))}. \tag{3.2}$$

Note that the complex conjugation of (3.1) gives

$$e^{-ax} + e^{-bx} = e^{-i\theta_1(x)} + e^{-i\theta_2(x)}.$$

Since

$$e^{-ax} + e^{-bx} = e^{-(a+b)x}(e^{ax} + e^{bx})$$

and

$$e^{-i\theta_1(x)} + e^{-i\theta_2(x)} = e^{-i(\theta_1(x) + \theta_2(x))}(e^{i\theta_1(x)} + e^{i\theta_2(x)})$$

we obtain the equality

$$e^{-(a+b)x}(e^{ax} + e^{bx}) = e^{-cx}(e^{ax} + e^{bx})$$

by using (3.1) and (3.2).
Hence we get

$$(e^{(a+b-c)x} - 1)(e^{ax} + e^{bx}) = 0$$

for all $x \in \mathbf{R}$. Especially

$$\frac{e^{(a+b-c)x} - 1}{x}(e^{ax} + e^{bx}) = 0$$

for all $x \in \mathbf{R} \setminus \{0\}$. Thus letting $x \to 0$ we obtain the desired equality

$$a + b - c = 0.$$

$\square$

4. Generalizations. From the proof above it would be easy to see that we have generalizations of Theorems A and B by using results of [4–6]. Hence we notice simple results only.

(1) Dedekind case. Let $\zeta_F(s)$ be the Dedekind zeta function of a finite extension field F of the rational number field $\mathbf{Q}$. Let a, b, $c \in i\mathbf{R}$ and

$$\begin{aligned} Z^{abc}_F(s) = \prod_{P \in \mathrm{Specm}(O_F)} &(1 - (N(P)^a + N(P)^b)N(P)^{-s} \\ &+ N(P)^{c-2s})^{-1}, \end{aligned}$$

where P runs over the set $\mathrm{Specm}(O_F)$ of maximal ideals of the integer ring O_F of F. Then we have exactly the same Theorems A and B chracterizing $\zeta_F(s-a)\zeta_F(s-b)$ among $Z_F^{abc}(s)$ by using Theorem 1 of [5, §8] for $\overline{E_0(F/F)}$.

(2) Selberg case. Let $\zeta_M(s)$ be the Selberg (or Ruelle) zeta function

$$\zeta_M(s)=\prod_{P\in\mathrm{Prim}(M)}(1-N(P)^{-s})^{-1}$$

of a compact Riemann surface M of genus $g\geq 2$, where $\mathrm{Prim}(M)$ denotes the prime geodesics on M with $N(P)=\exp(\mathrm{length}(P))$. Let a, b, $c\in i\mathbf{R}$ and

$$Z_M^{abc}(s)=\prod_{P\in\mathrm{Prim}(M)}(1-(N(P)^a+N(P)^b)N(P)^{-s}+N(P)^{c-2s})^{-1}.$$

Then we have the same Theorems A and B characterizing $\zeta_M(s-a)\zeta_M(s-b)$ among $Z_M^{abc}(s)$ by using Theorem 9 of [6, p. 232].

(3) More parameters. It is possible to generalize the situation with more parameters (or representations). For example, let a, b, c, $d\in i\mathbf{R}$ and

$$Z^{abcd}(s)=\prod_{p\text{: prime}}(1-(p^a+p^b+p^c)p^{-s}+(p^{a+b}+p^{b+c}+p^{c+a})p^{-2s}-p^{d-3s})^{-1}.$$

Then we have the following result by a similar proof: $Z^{abcd}(s)$ is meromorphic in all $s\in\mathbf{C}$ if and only if $a+b+c=d$. This result characterizes $\zeta(s-a)\zeta(s-b)\zeta(s-c)$ among $Z^{abcd}(s)$.

Moreover, we have the following Theorem C generalizing Theorems A and B. This characterizes $\zeta(s-a_1)\cdots\zeta(s-a_n)$ for $a_1,\ldots,a_n\in i\mathbf{R}$ with $n\geq 2$.

Theorem C. *For $n\geq 2$ and $a_1,\ldots,a_n,b\in i\mathbf{R}$, let*

$$Z(s)=\prod_{p\text{: prime}}((1-p^{a_1-s})\cdots(1-p^{a_n-s})+(-1)^n(p^b-p^{a_1+\cdots+a_n})p^{-ns})^{-1}.$$

Then $Z(s)$ has an analytic continuation to all $s\in\mathbf{C}$ as a meromorphic function if and only if $a_1+\cdots+a_n=b$ that is $Z(s)=\zeta(s-a_1)\cdots\zeta(s-a_n)$. When $a_1+\cdots+a_n\neq b$, it holds that $Z(s)$ is meromorphic in $\mathrm{Re}(s)>0$ with the natural boundary $\mathrm{Re}(s)=0$.

Proof. The method is quite similar to the case of $n=2$ treated in the proofs of Theorems A and B.

We define

$$H(T)\in 1+TR(\mathbf{R})[T]$$

by

$$\begin{aligned}H(T)&=(1-\chi_{a_1}T)\cdots(1-\chi_{a_n}T)\\&\quad+(-1)^n(\chi_b-\chi_{a_1+\cdots+a_n})T^n\\&=1-(\chi_{a_1}+\cdots+\chi_{a_n})T+\cdots+(-1)^n\chi_bT^n.\end{aligned}$$

Then it is sufficient to show the equivalence of
(1) $a_1+\cdots+a_n=b$,
and
(2) $H(T)$ is unitary.
(1)$\Longrightarrow$(2): If $a_1+\cdots+a_n=b$, then $H(T)=(1-\chi_{a_1}T)\cdots(1-\chi_{a_n}T)$, which is unitary.
(2)$\Longrightarrow$(1): Suppose that $H(T)$ is unitary. Then, for $x\in\mathbf{R}$

$$\begin{aligned}H_x(T)&=(1-e^{i\theta_1(x)}T)\cdots(1-e^{i\theta_n(x)}T)\\&=1-(e^{i\theta_1(x)}+\cdots+e^{i\theta_n(x)})T+\cdots\\&\quad+(-1)^ne^{i(\theta_1(x)+\cdots+\theta_n(x))}T^n\end{aligned}$$

with $\theta_j:\mathbf{R}\to\mathbf{R}$. By comparing with

$$H_x(T)=1-(e^{a_1x}+\cdots+e^{a_nx})T+\cdots+(-1)^ne^{bx}T^n$$

we obtain the following identities for all $x\in\mathbf{R}$:

$$(4.1)\qquad e^{a_1x}+\cdots+e^{a_nx}=e^{i\theta_1(x)}+\cdots+e^{i\theta_n(x)},$$

$$(4.\,n-1)\qquad \begin{aligned}&e^{(a_1+\cdots+a_{n-1})x}+\cdots+e^{(a_2+\cdots+a_n)x}\\&=e^{i(\theta_1(x)+\cdots+\theta_{n-1}(x))}+\cdots\\&\quad+e^{i(\theta_2(x)+\cdots+\theta_n(x))},\end{aligned}$$

$$(4.\,n)\qquad e^{bx}=e^{i(\theta_1(x)+\cdots+\theta_n(x))},$$

where $(4.\,k)$ indicates the coefficients of T^k in both sides for $k=1,n-1,n$.

Now, the complex conjugate of $(4.\,1)$ gives

$$(\alpha)\qquad e^{-a_1x}+\cdots+e^{-a_nx}=e^{-i\theta_1(x)}+\cdots+e^{-i\theta_n(x)}.$$

Dividing $(4.\,n-1)$ by $(4.\,n)$ we get

$$\begin{aligned}&e^{(a_1+\cdots+a_{n-1}-b)x}+\cdots+e^{(a_2+\cdots+a_n-b)x}\\&=e^{-i\theta_1(x)}+\cdots+e^{-i\theta_n(x)}\end{aligned}$$

that is

$$(\beta)\qquad \begin{aligned}&e^{(a_1+\cdots+a_n-b)x}(e^{-a_1x}+\cdots+e^{-a_nx})\\&=e^{-i\theta_1(x)}+\cdots+e^{-i\theta_n(x)}.\end{aligned}$$

Then $(\beta)-(\alpha)$ implies

$$(e^{(a_1+\cdots+a_n-b)x}-1)(e^{-a_1x}+\cdots+e^{-a_nx})=0$$

for all $x\in\mathbf{R}$. Hence we obtain

$$\frac{e^{(a_1+\cdots+a_n-b)x}-1}{x}(e^{-a_1x}+\cdots+e^{-a_nx})=0$$

for all $x\in\mathbf{R}\setminus\{0\}$. Thus, letting $x\to 0$ we have $a_1+\cdots+a_n=b$. $\square$

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