On Brauer–Manin obstructions and analogs of Cassels–Tate's exact sequence for connected reductive groups over global function fields

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Abstract: We show that the Brauer–Manin obstructions to the Hasse principle and weak approximation for homogeneous spaces under connected reductive groups over global function fields with connected reductive stabilizers are the only ones, extending some of Borovoi's results (and thus also proving a partial case of a conjecture of Colliot-Thélène) in this regard. Along the way, we extend some perfect pairings and an important local-global exact sequence (an analog of a Cassels–Tate's exact sequence) proved by Sansuc for connected linear algebraic groups defined over number fields, to the case of connected reductive groups over global function fields and beyond.

Key words: Brauer groups; weak approximation; Galois cohomology; Tate–Shafarevich kernel; local and global field; reductive group.

Introduction. Let k be a field, k_s a separable closure of k, and let $\Gamma := Gal(k_s/k)$ be the absolute Galois group of k, and for V the set of all places of k, let k_v be the completion of k at $v \in V$. Let X be a smooth, geometrically integral k-variety. We say that X satisfies the Hasse principle with respect to V, if $X(k) \neq \emptyset$ once we have $\prod_{v\in V} X(k_v) \neq \emptyset$. We say that X has the weak approximation property with respect to a finite subset $S \subset V$ if $X(k)$ is dense in the product $\prod_{v\in S} X(k_v)$ via the diagonal embedding and that X has the weak approximation property over k if the above holds for any finite set $S \subset V$.

As is well-known, the Brauer–Manin obstructions to various local-global relations between the sets of rational points on varieties play an important role in the arithmetic of algebraic varieties. One of classical results in this direction was obtained by Borovoi [Bo96], which states that over number fields, the Brauer–Manin obstructions to the Hasse principle (resp. weak approximation) of homogeneous spaces under connected linear algebraic groups with connected stabilizers are the only ones.

The aim of the present paper is to extend some of Borovoi's above results to the global function field case, namely by showing that the Brauer–Manin obstructions to the Hasse principle and weak approximation in homogeneous spaces under connected reductive groups with stabilizers, which are either connected reductive or of some specific type are the only ones. Besides, we extend some formulas due to Borovoi which compute an obstruction to the weak approximation in the case of non-connected stabilizers. There are some intensive investigations of Brauer–Manin obstructions for varieties over global function fields (cf. e.g. [CV], [HV], [PV] and references there). Notice that in the case of homogeneous spaces over fields of characteristic > 0, we have to restrict to the case G is connected and reductive.

It is necessarily also to restrict to stabilizers which are connected or of specific type (see below).

To achieve this goal, first we need to extend some of basic results of Sansuc (cf. [Sa, 8.9–8.14]) to the case of connected reductive groups defined over global function fields. Then we apply them to the Brauer–Manin obstructions to the Hasse principle and weak approximation, first for principal homogeneous spaces and then, following the approach by Borovoi, we establish that the Brauer–Manin obstructions to the Hasse principle and weak approximation are the only ones for homogeneous spaces with connected reductive stabilizers. The main tool we use is flasque and co-flasque resolu-

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tions for connected reductive groups constructed in [CT08] combined with an embedding trick and the fibration method used by Borovoi in order to investigate the Brauer–Manin obstructions to the Hasse principle and weak approximation for homogeneous spaces. The details of the proofs will appear elsewhere.

1. Preliminaries. If X is a k-scheme, for a field extension K/k we denote $X \times_k K$ the base change of X from k to K and if $K = k_s, X_s = X \times_k k_s$. Let $Br(X) := \mathrm{H}^2_{et}(X, \mathbf{G}_m)$ denote the cohomological Brauer group of X. Then we have natural homomorphisms $Br(k) \to Br(X) \to Br(X_s)$, for which one may define $Br_1(X) := \mathrm{Ker}(Br(X) \to Br(X_s))$, $Br_0(X) := \mathrm{Im}(Br(k) \to Br(X))$, $Br_a(X) := Br_1(X)/Br_0(X)$, and finally $\text{Б}(X) := \{x \in Br_a(X) \mid x_v = 0 \text{ for all } v \in V\}$, where for $x \in Br_a(X)$, we denote by x_v the image of x in $Br_a(X_v)$. For a finite subset of places $S \subset V$, we denote $\text{Б}_S(X) := \mathrm{Ker}(Br_a(X) \to \prod_{v \notin S} Br_a(X_v))$, then $\text{Б}_\omega(X) = \bigcup_S \text{Б}_S(X)$.

For an affine algebraic k-group scheme G, let $\mathrm{H}^i_{\mathrm{fppf}}(k, G) := \mathrm{H}^i_{\mathrm{fppf}}(\bar{k}/k, G(\bar{k}))$ be the flat cohomology in degree i (≤ 1 if G is non-commutative) of G and let $\text{Ш}^i(G) := \mathrm{Ker}\,(\mathrm{H}^i_{\mathrm{fppf}}(k, G) \to \prod_{v \in V} \mathrm{H}^i_{\mathrm{fppf}}(k_v, G))$, be the *Tate–Shafarevich kernel in degree i* of G. One denotes by $\mathrm{A}(G) := \prod_{v \in V} G(k_v)/\overline{G(k)}$ (resp. $\mathrm{A}(S, G) := \prod_{v \in S} G(k_v)/\overline{G(k)}$) the *defect* (or *obstruction*) *to the weak approximation property of G over k* (resp. *obstruction to weak approximation at S*), where $\overline{G(k)}$ denotes the closure of $G(k)$ in the product of $G(k_v)$ (resp. in $\prod_{v \in S} G(k_v)$).

For a connected reductive group G defined over a field k, G^{ss} denotes the derived subgroup of G which is the semisimple part of G. Any homogeneous space X under a smooth affine group G is understood as a left homogeneous space. Denote by $Y^D = \mathrm{Hom}(Y, \mathbf{Q}/\mathbf{Z})$, the Pontrjagin dual of an abelian group Y. If M is a Γ-module, which is a free $\mathbf{Z}$-module of finite type, denote by M^* the dual module $\mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$. M is called a *permutation* Γ-module if there is a $\mathbf{Z}$-basis of M which is permuted by Γ. M is called a *flasque* (resp. *co-flasque*) Γ-module, if for every open subgroup $\Theta \subset \Gamma$, we have $\mathrm{H}^1(\Theta, M^*) = 0$ (resp. $\mathrm{H}^1(\Theta, M) = 0$).

For a k-torus T, we denote its character module by $\hat{T} := X^*(T)$ and its co-character module by $X_*(T)$. A k-torus T is called *induced* (resp. *flasque*, *co-flasque*), if $\hat{T}$ is a permutation (resp. flasque, co-flasque) Γ-module. If H is a smooth affine algebraic group, that is, a linear algebraic group, let H° be the connected component of H, $R_u(H)$ the unipotent radical of H, $H^{red} := H^\circ/R_u(H)$ the largest reductive quotient, $H^{ss} := H^{red}/[H^{red}, H^{red}]$ the semisimple part of H^{red}, $H^{tor} := H^{red}/H^{ss}$ the maximal torus quotient of H° and let $H^{ssu} := \mathrm{Ker}\,(H^\circ \to H^{tor})$. This last subgroup is normal in both H and H° and one denotes the quotient $H^{(m)} := H/H^{ssu}$.

1.1. Flasque resolutions and special coverings. Let H be a connected linear algebraic group defined over a field k, supposed reductive if $char.k > 0$. Then H is called *quasi-trivial* (after Colliot-Thélène [CT08, Sec. 2]), if $k_s[H]^*/k_s^*$ is a permutation Γ-module and the Picard group $Pic(H_{k_s}) = 0$, where $k_s[H]$ stands for the affine algebra of H, and A^* stands for the group of invertible elements of the ring A. Then if H^{tor} denotes the toric quotient of H, $P := H^{tor}$ is an induced k-torus.

Then for any given connected linear algebraic k-group G, supposed reductive if $char.k > 0$, there exist a quasi-trivial k-group, supposed reductive if $char.k > 0$, a flasque k-torus T and an exact sequence of k-groups $1 \to F \to H \to G \to 1$. The flasque torus F plays an important role in the arithmetic and geometry of G, so it will be called *flasque kernel of G* in the sequel. Though they are not uniquely determined, but they are almost uniquely, in the sense, that their character groups, as Galois modules, are defined uniquely up to similarity (see [CT08]).

Let G be a connected reductive k-group, $\tilde{G}$ the simply connected covering of G^{ss}, all are defined over k. Any central k-isogeny $\pi : H_1 \to G$ of connected reductive groups, with $H_1 \simeq_k \tilde{G} \times P$, where P is an induced k-torus, is called a *special covering of G* ([Sa, p. 14]). Such covering may not exist in general, but for any connected reductive group G, there exists a power of G, which has a special covering (see [Sa, Lem. 1.10]). Notice that in [Sa], several important arithmetic results have been proved by reducing the computation of some arithmetic invariants to that of kernels of the isogenies appearing in special coverings. The quasi-trivial group H defined above enjoys many (especially cohomological) nice properties over local and global fields, as it was demonstrated in [CT08] and in [Th13]. For this reason, one may think of H as a special kind of "universal covering" of G, and thus of F as a special kind of "fundamental group" of G.

Let k be a field. A surjective k-homomorphism of connected reductive k-groups $\pi : G_1 \to G$ is called a *quasi-trivial covering* of G if $B := \mathrm{Ker}\,(\pi)$ is a central k-subgroup (which is necessary of multiplicative type) of G_1 and G_1 is quasi-trivial. We regard special coverings as a special kind of quasi-trivial coverings, and the kernel B can be regarded (though it may not be finite) as a special kind of fundamental group, called later for short *quasi-fundamental group*.

Let k be a global field, S a subset of V, and let M be a commutative k-group scheme of finite type. We consider the localization maps $f_{i,S} : \mathrm{H}^i_{\mathrm{fppf}}(k, M) \to \prod_{v\in S} \mathrm{H}^1_{\mathrm{fppf}}(k_v, M)$, $f_i := f_{i,V}$ and set $Ш^i_S(M) := \mathrm{Ker}\,(f_{i,V\setminus S})$. Denote $Ч^i(M) := \mathrm{Coker}\,(f_i)$, $Ч^i_S(M) := \mathrm{Coker}\,(f_{i,S})$, and finally denote $Ш^i_\omega(M) := \varinjlim_S Ш^i_S(M)$ and $Ч^i_\omega(M) := \varprojlim_S (\mathrm{Coker}\,(f_{i,S})$ where S runs over finite subsets of V.

1.2. Quasi-fundamental groups versus flasque kernels. We show below some interesting arithmetic analogies between the quasi-fundamental groups and flasque kernels (cf. [CT08], [Th13]) which shows that for computing some arithmetic invariants of connected reductive algebraic groups, the methods of using either the quasi-fundamental groups or the flasque kernels, a priori, agree with each other.

1.3. Proposition (Cf. [Sa, Thm. 3.3], [CT08, Thm. 9.4 (i)], [Th13, Thm. 2.3]). *Let k be a global field, S a finite set of places of k and let G be a connected reductive k-group having a quasi-trivial covering with quasi-fundamental group B.*
(1) *There is a bijection of finite abelian groups* $\mathrm{A}(S,G) \simeq Ч^1_S(B)$.
(2) *There is a bijection of finite abelian groups* $\mathrm{A}(G) \simeq Ч^1_\omega(B)$.

Notice that the assertion that $Ч^1_S(B)$, $Ч^1_\omega(B)$ are finite abelian groups do not, a priori, follow from the proof given in char. 0 case, since in the case of a global function field k, the abelian groups $\mathrm{H}^1_{\mathrm{fppf}}(k,B)$, $\mathrm{H}^1_{\mathrm{fppf}}(k_v,B)$ may be infinite. We have the following

1.4. Proposition (Cf. [Sa, Thm. 4.3], [CT08, Thm. 9.4 (ii)], [Th13, Thm. 3.6 (1)]). *Let k be a global field and let the connected reductive k-group G have a quasi-trivial covering with quasi-fundamental group B. Then we have an isomorphism of finite abelian groups* $Ш^1(G) \simeq Ш^2(B)$.

1.5. Theorem (Cf. [Sa, Thm. 5.1, Corol. 7.4], [CT08, Thm. 7.2], [Th13, Thm. 3.8], [BSch, Thm. 5.1] for number field case). *Let k be a global field, G a quasi-trivial reductive k-group, and let B be a k-subgroup of G such that B° is reductive, $B^{(m)}$ is of multiplicative type and let $X := G/B$.*
(1) (Cf. [BSch, Thm. 5.1]) *We have the following isomorphisms of finite abelian groups* $Ш^1_S(\hat{B}) \simeq Б_S(X)$, $Ш^1_\omega(\hat{B}) \simeq Б_\omega(X)$, $Ш^1(\hat{B}) \simeq Б(X)$.
(2) (Cf. [BSch, Thm. 0.4]) *We have a perfect duality of finite abelian groups between the pairs* $Ч^1_S(B)$ *and* $Ш^1_S(\hat{B})/Ш^1(\hat{B})$ *and* $Ч^1_\omega(B)$ *and* $Ш^1_\omega(\hat{B})/Ш^1(\hat{B})$.

The proof of Theorem 1.5 uses the flasque resolution in essential way.

2. Extension of some perfect pairings and some exact sequences of Sansuc to global function fields. We extend some perfect pairings considered in [Sa, Sec. 8] and an exact sequence of Sansuc [Sa, Corol. 8.15]) to the global function field case for connected reductive groups. Here we show the existence of a perfect duality mentioned above, thus the functoriality of the whole exact sequence above (consisting of objects which are functorial in G). Remark that the proof given in [Sa] (namely that of Lemma 8.11) does not seem to be extended to the case of char.$k > 0$. Moreover, the proof we give is also valid for any global field.

2.1. Theorem ([Sa. Thm. 8.12, Corol. 8.14] for number fields). *Let k be a global field, S a finite set of places of k, and let G be a connected reductive k-group.*
(1) *There exists a perfect duality of finite abelian groups between* $\mathrm{A}(G)$ *and* $Б_\omega(G)/Б(G)$ *and between* $\mathrm{A}(S,G)$ *and* $Б_S(G)/Б(G)$, *which induce natural (functorial in G) isomorphisms of finite abelian groups* $\mathrm{A}(G) \simeq (Б_\omega(G)/Б(G))^D$, $\mathrm{A}(S,G) \simeq (Б_S(G)/Б(G))^D$.
(2) *We have the following exact sequence of finite abelian groups*

(S) $\quad 1 \to \mathrm{A}(G) \to Б_\omega(G)^D \to Ш^1(G) \to 1,$

(S′) $\quad 1 \to \mathrm{A}(S,G) \to Б_S(G)^D \to Ш^1(G) \to 1,$

which, as sequences depending on G, are additive and functorial in G.

Notice that the exact sequence (S) is crucial in the proof of the validity of the Sansuc–Voskresenskii exact sequence (S-V) (see Theorem 2.4 below).

If one would attempt to prove the existence of the exact sequence (S-V) also in the case of function field, then naturally there would be raised a problem of constructing a *smooth* k-compactification $\mathscr{G}$ for any given connected reductive k-group G which might be depending on the resolution of singularities in characteristic $p > 0$. A milder solution has been proposed in [Th13, Theorem 3.7, (2)], which is based on the existence of a smooth compactification for any k-torus.

The following theorem complements Theorem 1.5 and gives a connection between the pairings between $\mathrm{A}(G)$ and $\text{Б}_\omega(G)$ (or $\text{Б}_\omega(G)/\text{Б}(G)$), $\mathrm{A}(S,G)$ and $\text{Б}_S(G)$ (or $\text{Б}_S(G)/\text{Б}(G)$), $\text{Ч}^1_\omega(B)$ and $\text{Ш}^1_\omega(\hat{B})$ (or $\text{Ш}^1_\omega(\hat{B})/\text{Ш}^1(\hat{B})$) and between $\text{Ч}^1_S(B)$ and $\text{Ш}^1_S(\hat{B})$ (or $\text{Ш}^1_S(\hat{B})/\text{Ш}^1(\hat{B})$) (compare [Sa, 8.10–8.14] for number field case).

2.2. Theorem. *Let k be a global field, S a finite set of places of k, G a connected reductive k-group, and let $1 \to B \to H \to G \to 1$ be a quasi-trivial covering of G. We have the following diagrams which are anti-commutative, where the rows are perfect pairings*

$$\begin{array}{ccccc}
\mathrm{A}(G) & \times & (\text{Б}_\omega(G)/\text{Б}(G)) & \to & \mathbf{Q}/\mathbf{Z} \\
\downarrow \alpha & & \uparrow \alpha' & & \updownarrow = \\
\text{Ч}^1_\omega(B) & \times & (\text{Ш}^1_\omega(\hat{B})/\text{Ш}(\hat{B})) & \to & \mathbf{Q}/\mathbf{Z} \\
\mathrm{A}(S,G) & \times & \text{Б}_S(G)/\text{Б}(G) & \to & \mathbf{Q}/\mathbf{Z} \\
\downarrow \beta & & \uparrow \beta' & & \updownarrow = \\
\text{Ч}^1_S(B) & \times & \text{Ш}^1_S(\hat{B})/\text{Ш}^1(\hat{B}) & \to & \mathbf{Q}/\mathbf{Z}.
\end{array}$$

2.3. Extension of Sansuc–Voskresenskii exact sequence to global function fields. In this section we give an analog of Sansuc–Voskresenski theorem for connected reductive groups defined over a global function field k. We have

2.4. Theorem. *Let k be a global function field, G a connected reductive k-group. Then*
(a) (Cf. [Sa, Lem. 6.1, Lem. 6.3, Corol. 9.4 and Prop. 9.8] for number fields) *For any smooth k-compactification $\mathscr{G}$ of G, we have $\text{Б}_\omega(G) \simeq \text{Б}_\omega(\mathscr{G}) \simeq \mathrm{H}^1(k, Pic(\mathscr{G}_s))$, $\text{Б}(G) \simeq \text{Б}(\mathscr{G})$. The finite abelian groups $\text{Б}_\omega(G), \text{Б}(G)$ are stably birational invariants of G in the class of connected reductive k-groups.*
(b) (Cf. [Sa, Thm. 9.5(ii)], [BKG04, Corol. 8.12(ii), Thm. 8.16(ii)] for number fields) *The finite abelian groups $\mathrm{A}(G)$ and $\text{Ш}^1(G)$ are k-stably birational invariants of the k-group G in the class of connected reductive k-groups having a smooth k-compactification.*
(c) (Cf. [Sa, Thm. 9.5(iii)], [CT08, Thm. 9.4] for number fields) *For any smooth compactification $\mathscr{G}$ of G, we have the following exact sequence*

$$\text{(S-V)}\quad 1 \to \mathrm{A}(G) \to \mathrm{H}^1(k, Pic(\mathscr{G}_s))^D \to \text{Ш}^1(G) \to 1,$$

called Sansuc–Voskresenskii sequence.

The proof of Theorem 2.1 follows from the proof of Theorem 1.5, while Theorems 2.2 and 2.4 follow from Theorem 2.1.

3. Brauer–Manin obstructions to the Hasse principle and weak approximation for homogeneous spaces with connected stabilizers over global function fields.

3.1. Brauer–Manin pairing (Cf. [Sk, Chap. V])**.** Let X be an irreducible, smooth, geometrically integral variety defined over a global field k, and assume that $\prod_v X(k_v) \neq \emptyset$. Consider the following Brauer–Manin pairing $\prod_v X(k_v) \times \text{Б}_\omega(X) \to \mathbf{Q}/\mathbf{Z}$, $\langle (x_v), b\rangle := \sum_v inv_v(b_v(x_v))$ (cf. [Sa, Lem. 6.2]) and write $(x_v) \perp b$ if $\langle (x_v), b\rangle = 0$. Define the Brauer–Manin set $(\prod_v X(k_v))^{\text{Б}_\omega(X)}$ as the set $\{(x_v) \in \prod_v X(k_v) \mid (x_v) \perp \text{Б}_\omega(X)\}$. Then one defines a natural homomorphism $m_H(X) : \text{Б}(X) \to \mathbf{Q}/\mathbf{Z}$, $b \mapsto \langle (x_v), b\rangle$ which does not depend on the choice of (x_v), so it defines an element $m_H(X) \in \text{Б}(X)^D$. If $X(k) \neq \emptyset$ and x is a k-point of X, then an exact sequence of global class field theory shows that $\sum_v inv_v(b_v(x)) = 0$ for all such x, so $m_H(X)$ is trivial and we have

$$X(k) \subset \left(\prod_v X(k_v)\right)^{\text{Б}_\omega(X)}. \tag{3.1.1}$$

The map $m_H(X)$ is regarded as Brauer–Manin obstruction to the Hasse principle and we say that *the Brauer–Manin osbtruction to the Hasse principle is the only one* if we have $X(k) \neq \emptyset$ once we have $(\prod_v X(k_v))^{\text{Б}_\omega(X)} \neq \emptyset$.

Now assume that $X(k) \neq \emptyset$ and let $x \in X(k)$. Then for each finite set S of places of k, the pairing $\prod_{v\in S} X(k_v) \times \text{Б}_S(X) \to \mathbf{Q}/\mathbf{Z}$, $\langle (x_v)_{v\in S}, b\rangle_S := \sum_{v\in S}(inv_v(b_v(x_v)) - inv_v(b_v(x))$, where $b \in \text{Б}_S(X)$ induces a map $\prod_{v\in S} X(k_v) \times (\text{Б}_S(X)/\text{Б}(X)) \to \mathbf{Q}/\mathbf{Z}$, thus also a map $m_{W,S}(X) : \prod_{v\in S} X(k_v) \to (\text{Б}_S(X)/\text{Б}(X))^D$, which is continuous with respect to the topology on $\prod_{v\in S} X(k_v)$ due to the continuity of the pairing. If $x \in X(k)$, then $m_{W,S}(X)(x) = 0$, thus if $X(k)$ is dense in $\prod_{v\in S} X(k_v)$, then $m_{W,S}$ is identically zero due to the continuity, thus it can be regarded as an obstruction, called *Brauer–Manin obstruction to the weak approximation of X in S*. For the weak approximation over k, assume that $X(k) \neq \emptyset$. Then it follows from the continuity of the natural pairing $\prod_v X(k_v) \times Б_\omega(X) \to \mathbf{Q}/\mathbf{Z}$ that if $X(k)$ is dense in $\prod_v X(k_v)$, then for all $(x_v) \in \prod_v X(k_v)$, we have $(x_v) \perp Б_\omega(X)$, thus $m_{W,S}(X)$ is trivial for all S. We then have

$$\overline{X(k)} \subseteq \left(\prod_{v\in S} X(k_v)\right)^{Б_S(X)}, \tag{3.1.2}$$

$$\overline{X(k)} \subseteq \left(\prod_{v} X(k_v)\right)^{Б_\omega(X)}. \tag{3.1.3}$$

If the equality holds in (3.1.2) (resp. (3.1.3)), then we say that *the Brauer–Manin obstruction to the weak approximation in S (resp. over k) is the only one.*

3.2. Brauer–Manin obstructions. From results of Sections 1, 2 we derive the following analogs of [Sa, 8.7, 8.13] for the case of global function fields.

3.3. Theorem (Cf. [Sa, 8.7, 8.13], for torsors and [Bo96, Thms. 2.2 and 2.4] for homogeneous spaces over number fields)**.** *Let k be a global field with no real places and let X be a homogeneous space under a connected reductive k-group G with a connected reductive stabilizer H. Then the Brauer–Manin obstruction to the Hasse principle and weak approximation with respect to S and over k for X are the only ones.*

The method of the proof follows that of Borovoi and also makes use of a method and some of our previous results from [NT14], [NT16].

4. Formulas for obstruction to weak approximation. Let G be a connected reductive group defined over a global field k, $X = G/H$ where H is a smooth connected k-subgroup of G. In [Bo90]–[Bo99], Borovoi gave various formulas to compute an obstruction to the weak approximation for X in cohomological terms. To study the Brauer–Manin obstruction to the weak approximation we need some extensions of such results to the case of global function fields. Recall that (cf. [La], [Th19]) a smooth affine k-group G is called *quasi-connected* if there is an exact sequence $1 \to G \to G_1 \to T \to 1$, where G_1 is a connected reductive k-group and T is a k-torus. For a quasi-connected k-group H, we may define the group $\mathrm{H}^i_{ab,\mathrm{fppf}}(.,H)$ which is the abelianized (fppf) cohomology of H (see [Bo98], [La], [Th19]). If k is a global field and S is a finite set of places of k, denote $\gamma_S : \mathrm{H}^1_{ab,\mathrm{fppf}}(k,H) \to \prod_{v\in S} \mathrm{H}^1_{ab,\mathrm{fppf}}(k_v,H)$, $Ч^1_{ab,S}(H) = \mathrm{Coker}\,(\gamma_S)$, $\gamma_\omega : \mathrm{H}^1_{ab,\mathrm{fppf}}(k,H) \to \prod_v \mathrm{H}^1_{ab,\mathrm{fppf}}(k_v,H)$, $Ч^1_{ab,\omega}(H) = \mathrm{Coker}\,(\gamma_\omega)$.

Then we have the following

4.1. Theorem (Cf. [Bo99, Thm. 1.3, 1.11] for number fields)**.** *Let k be a global field with no real places, S a finite set of places of k, and let $X = G/H$ be a homogeneous space under a connected reductive k-group G with a quasi-connected k-stabilizer H.*
(1) *If G is quasi-trivial, and either* (*a*) *H is connected and reductive or* (*b*) *H is quasi-connected, reductive, then there is an exact sequence of pointed sets* (*here the closure is taken in $\prod_{v\in S} X(k_v)$*)

$$1 \to \overline{X(k)} \to \prod_{v\in S} X(k_v) \to Ч^1_{ab,S}(H^{(m)}) \to 1,$$

(2) *Assume that either* (*a*) *H is a connected reductive k-group, or* (*b*) *G is quasi-trivial and H is a quasi-connected reductive k-group such that $H^{(m)}$ is of multiplicative type. Then there is an exact sequence of pointed sets where the closure is taken in $\prod_{v\in S} X(k_v)$*

$$1 \to \overline{X(k)} \to \prod_{v\in S} X(k_v) \to (Б_S(X)/Б(X))^D \to 1.$$

In particular, the Brauer–Manin obstructions to the weak approximation in S and over k are the only ones. In the case H is connected, there are isomorphisms $Ч^1_S(H^{(m)}) \simeq (Б_S(X)/Б(X))^D$, $Ч^1_\omega(H^{(m)}) \simeq (Б_\omega(X)/Б(X))^D$.

To prove Theorem 4.1, beside the results obtained above, we need also a function field version of a main result of [BSch] (see Theorem 1.5).

5. Brauer–Manin obstructions for homogeneous spaces with non-connected stabilizers over global function fields. One of our main results of this paper is to extend Borovoi's result [Bo96, Corol. 2.5] to the case of global function fields. The following is the main result of this section.

5.1. Theorem (Cf. [Bo96, Thms. 2.2 and 2.4])**.** *Let k be a global field with no real places, X a k-homogeneous space under a connected reductive group G having simply connected semisimple part* *and with a group H as a stabilizer such that H° is reductive and $H^{(m)}$ is of multiplicaticative type. Then the Brauer–Manin obstructions to the Hasse principle and weak approximation for X are the only ones.*

For the proof, we reduce Theorem 5.1, by using an embedding trick due to Borovoi, to the simpler cases and then use the fibration method.

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Added in the proof. After the paper had been submitted (August 2020), there appeared a preprint by C. Demarche and D. Harari [DH], where, among other things, Theorem 3.3, 4.1 (1a), (2a) were proved by a quite different method.

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