Equidistribution in non-archimedean parameter curves towards the activity measures

By Reimi IROKAWA*) and Yûsuke OKUYAMA**)

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Abstract: For every pair of an analytic family $f = f_t$ of endomorphisms of degree > 1 of the Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}$ over an algebraically closed and complete non-trivially valued field K and an analytically marked point $a = a(t)$ in $\mathbb{P}^{1,\mathrm{an}}$ both parametrized by a domain V in the Berkovich analytification of a smooth projective algebraic curve C/K, we establish the equidistribution of the averaged pullbacks of any value in $\mathbb{P}^{1,\mathrm{an}}$ but a subset of logarithmic capacity 0 under the sequence of the morphisms $a_n = a_n(t) = f_t^n(a(t)) : V \to \mathbb{P}^{1,\mathrm{an}}$, towards the activity measure $\mu_{(f,a)}$ on V associated with f and a.

Key words: Analytic family of morphisms; analitically marked point; activity measure; potential theory on Berkovich curves; equidistribution; Varilon exceptional set.

1. Introduction. Let K be an algebraically closed field that is complete with respect to a nontrivial and non-archimedean absolute value. Let V be a (topological) domain in the analytification (Berkovich [3]) of a smooth projective algebraic curve C/K. To the pair (f, a), where f is an analytic family

$$f_t(z) = f(t, z) : V \times_K \mathbb{P}^{1,\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}} = \mathbb{P}^{1,\mathrm{an}}_K$$

of endomorphisms of $\mathbb{P}^{1,\mathrm{an}}$ of degree $d > 1$ analytically parametrized by V and $a : V \to \mathbb{P}^{1,\mathrm{an}}$ is a marked point in $\mathbb{P}^{1,\mathrm{an}}$ analytically parametrized by V (i.e., an analytic map from V to $\mathbb{P}^{1,\mathrm{an}}$), the activity measure

$$\mu_{(f,a)} := \lim_{n\to\infty} \frac{(a_n)^*\delta_{\zeta_{0,1}}}{d^n} \quad \text{weakly on } V \tag{1.1}$$

is associated ([6]), where for every $n \in \mathbb{N}$,

$$a_n(t) := f_t^n(a(t)), \quad t \in V \tag{1.2}$$

is a marked point in $\mathbb{P}^{1,\mathrm{an}}$ analytically parametrized by V, and $\zeta_{0,1}$ is the Gauss (or canonical) point in $\mathbb{P}^{1,\mathrm{an}}$ and δ_ζ is the Dirac measure on $\mathbb{P}^{1,\mathrm{an}}$ at a point $\zeta \in \mathbb{P}^{1,\mathrm{an}}$. The activity measures of especially marked critical points play a key role in pluripotential theoretic studies (since [4]) of bifurcation and (un)stability in complex dynamics (foundationally [7–9]). A non-archimedean version of bifurcation and (un)stability including a non-archimedean λ-lemma has been studied by Thomas Silverman [11].

Our principal result is the following, which has been expected in [6].

Theorem 1. *Let K be an algebraically closed field that is complete with respect to a nontrivial and non-archimedean absolute value, and V be a domain in the (Berkovich) analytification of a smooth projective algebraic curve C/K. Let $f : V \times_K \mathbb{P}^{1,\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ be a family of endomorphisms of $\mathbb{P}^{1,\mathrm{an}}$ of degree $d > 1$ and $a : V \to \mathbb{P}^{1,\mathrm{an}}$ be a marked point in $\mathbb{P}^{1,\mathrm{an}}$, both analytically parametrized by V. Then for every $\zeta \in \mathbb{P}^{1,\mathrm{an}}$ but a subset of logarithmic capacity* 0,

$$\lim_{n\to\infty} \frac{(a_n)^*\delta_\zeta}{d^n} = \mu_{(f,a)} \quad \textit{weakly on } V.$$

Indeed, we would establish the following, which is motivated by Nevanlinna theory.

Theorem 2. *Under the same assumption in Theorem* 1*, the* Valiron *exceptional set*

$$E_V(f, a) := \left\{ \zeta \in \mathbb{P}^{1,\mathrm{an}} \;\middle|\; \begin{array}{l} \text{there is } x \in V \setminus V(K) \text{ such that} \\ \log \|a_n(x), \zeta\| \neq o(d^n) \text{ as } n \to \infty \end{array} \right\}$$

associated with the pair (f, a) is of logarithmic capacity 0 *in* $\mathbb{P}^{1,\mathrm{an}}$.

For the definition of the generalized Hsia kernel $\|\zeta, \xi\|$ on $\mathbb{P}^{1,\mathrm{an}}$ with respect to $\zeta_{0,1}$ and that of a

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*) Graduate School of Science, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan.
**) Division of Mathematics, Kyoto Institute of Technology, Matsugasaki, Sakyo-ku, Kyoto 606-8585, Japan.

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subset of $\mathbb{P}^{1,\mathrm{an}}$ of logarithmic capacity 0, see Section 2 below. A standard argument from potential theory on Berkovich curves (using a functoriality and a continuity of the Laplacian dd^c on Berkovich curves (see Thuillier [13, §3.2, §3.3])) shows that for every $\zeta \in \mathbb{P}^{1,\mathrm{an}} \setminus E_V(f,a)$,

$$\lim_{n\to\infty} \frac{(a_n)^*(\delta_{\zeta_{0,1}} - \delta_\zeta)}{d^n} = 0 \quad \text{weakly on } V.$$

Hence Theorem 2 together with (1.1) yields Theorem 1.

Organization of this paper. In Section 2, we gather some background materials and notations from potential theory on the Berkovich projective line and curves. In Section 3, we show Theorem 2 when V is separable. In Section 4, we show Theorem 2 in full generality.

2. Background. The Berkovich projective line $\mathbb{P}^{1,\mathrm{an}} = \mathbb{P}^{1,\mathrm{an}}_K$ is a compact augmentation of $\mathbb{P}^1 = \mathbb{P}^1_K$, and a typical point in $\mathbb{P}^{1,\mathrm{an}}$ is written as $\zeta_{a,r}$ when it is represented by a K-closed disk $B(a,r) := \{z \in K : |z-a| \le r\}$ in K for some $a \in K$ and $r \ge 0$; $K = \mathbb{P}^1 \setminus \{\infty\}$ is identified with the set of all points in $\mathbb{P}^{1,\mathrm{an}}$ written as $\zeta_{a,0}$ for some $a \in K$. The point $\zeta_{0,1}$ is called the Gauss (or canonical) point in $\mathbb{P}^{1,\mathrm{an}}$. The chordal distance $\|z,w\|$ on $\mathbb{P}^1$ normalized as $\|0,\infty\| = 1$ extends to a unique upper semicontinuous and separately continuous function on $\mathbb{P}^{1,\mathrm{an}} \times \mathbb{P}^{1,\mathrm{an}}$, which is still denoted by $\|\zeta,\xi\|$ and is called the generalized Hsia kernel function on $\mathbb{P}^{1,\mathrm{an}}$ with respect to $\zeta_{0,1}$; in particular, $\|\zeta,\zeta\| = 0$ if and only if $\zeta \in \mathbb{P}^1$. For every point $\zeta_0 \in \mathbb{P}^{1,\mathrm{an}}$, the function $\log\|\cdot,\zeta_0\|$ on $\mathbb{P}^{1,\mathrm{an}}$ is locally constant except for the closed interval $[\zeta_0, \zeta_{0,1}]$ between ζ_0 and $\zeta_{0,1}$ in $\mathbb{P}^{1,\mathrm{an}}$. An analytic map h from a domain D in the Berkovich analytification of a smooth projective curve C/K to $\mathbb{P}^{1,\mathrm{an}}$ induces a canonical pullback operator h^* from the space of Radon measures on $\mathbb{P}^{1,\mathrm{an}}$ to that of Radon measures on D (for more details on $\mathbb{P}^{1,\mathrm{an}}$, see [5, Chapter 4], [5, §3.4]).

We adopt the following sign convention on the Laplacian dd^c on Berkovich curves; for every $\zeta \in \mathbb{P}^{1,\mathrm{an}}$,

$$-\mathrm{dd}^c \log\|\cdot,\zeta\| = \delta_\zeta - \delta_{\zeta_{0,1}} \quad \text{on } \mathbb{P}^{1,\mathrm{an}}.$$

We call a function

$$u_{\zeta_{0,1},\rho}(\cdot) := -\int_{\mathbb{P}^{1,\mathrm{an}}} \log\|\cdot,\zeta\|\rho(\zeta) \in [0,+\infty]$$

$$\left(\text{so that } \mathrm{dd}^c u_{\zeta_{0,1},\rho} = \rho - \rho(\mathbb{P}^{1,\mathrm{an}})\zeta_{0,1} \text{ on } \mathbb{P}^{1,\mathrm{an}}\right)$$

on $\mathbb{P}^{1,\mathrm{an}}$ the logarithmic potential function of a positive Radon measure ρ on $\mathbb{P}^{1,\mathrm{an}}$ with respect to $\zeta_{0,1}$. The logarithmic capacity (with respect to $\zeta_{0,1}$) of a subset E of $\mathbb{P}^{1,\mathrm{an}} \setminus \{\zeta_{0,1}\}$ is

$$\operatorname{Cap} E := \exp\left(-\inf_\rho \int_{\mathbb{P}^{1,\mathrm{an}}} u_{\zeta_{0,1},\rho}(\xi)\rho(\xi)\right) \in [0,1],$$

where ρ ranges over all probability Radon measures on $\mathbb{P}^{1,\mathrm{an}}$ supported by E, and we say E is of logarithmic capacity 0 (with respect to $\zeta_{0,1}$) if $\operatorname{Cap} E = 0$. If E is not of logarithmic capacity 0, then there is a compact subset of E which is not of logarithmic capacity 0. If a compact subset $\mathcal{C}$ of $\mathbb{P}^{1,\mathrm{an}} \setminus \{\zeta_{0,1}\}$ is not of logarithmic capacity 0, then a unique equilibrium mass distribution $\rho_{\mathcal{C}}$ on $\mathcal{C}$ with respect to $\zeta_{0,1}$ (i.e., a probability Radon measure ρ on $\mathbb{P}^{1,\mathrm{an}}$ supported by $\mathcal{C}$ and satisfying $\operatorname{Cap}\mathcal{C} = \exp(-\int_{\mathbb{P}^{1,\mathrm{an}}} u_{\zeta_{0,1},\rho}(\xi)\rho(\xi)))$ exists. For more details on the logarithmic capacity theory on $\mathbb{P}^{1,\mathrm{an}}$, see [2, Chapter 6].

3. Proof of Theorem 2: separable domain case. Let V be a domain in the (Berkovich) analytification of a smooth projective algebraic curve C/K. Let $f : V \times_K \mathbb{P}^{1,\mathrm{an}} \to \mathbb{P}^{1,\mathrm{an}}$ be a family of endomorphisms of $\mathbb{P}^{1,\mathrm{an}}$ of degree $d > 1$ and $a : V \to \mathbb{P}^{1,\mathrm{an}}$ be a marked point in $\mathbb{P}^{1,\mathrm{an}}$, both analytically parametrized by V. Recall the definition (1.2) of the marked point a_n for each $n \in \mathbb{N}$.

For every $x \in V \setminus V(K)$, the subset

$$\text{(3.1)} \quad \mathcal{E}_x = \mathcal{E}_x(f,a) := \bigcup_{j\in\mathbb{N}} \bigcap_{N\in\mathbb{N}} \bigcup_{n\ge N} \left\{\zeta \in \mathbb{P}^{1,\mathrm{an}} \;\middle|\; \|a_n(x),\zeta\| < e^{\frac{-d^n}{j}}\right\}$$

of $\mathbb{P}^{1,\mathrm{an}}$ is the countable union of subsets of $\mathbb{P}^1$ of finite Hyllengren measures for the increasing sequence $(d^n)_n$ (cf. [12]), so in particular is of logarithmic capacity 0 (for a more direct argument, see e.g. [10, Proof of Lemma 2.1]).

Suppose now that V is separable, that is, $V \setminus V(K)$ contains a countable dense subset S (as in e.g. the case that $K = \mathbb{C}_p$). Then the subset

$$\mathcal{E}_S := \bigcup_{x\in S} \mathcal{E}_x$$

of $\mathbb{P}^{1,\mathrm{an}}$ is of logarithmic capacity 0. We claim that $E_V(f,a) \subset \mathcal{E}_S$; for, otherwise, there are $\zeta_0 \in \mathbb{P}^{1,\mathrm{an}} \setminus \mathcal{E}_S$, $x_0 \in V \setminus V(K)$, and a sequence (n_j) in $\mathbb{N}$ tending to ∞ such that

$$\lim_{j\to\infty}\frac{\log\|a_{n_j}(x_0),\zeta_0\|}{d^{n_j}}<0.$$

From [6], we can pick a connected open affinoid neighborhood U of x_0 in V so small that for a non-degenerate homogeneous polynomial lift $F:U\times_K\mathbb{A}^{2,\mathrm{an}}\to\mathbb{A}^{2,\mathrm{an}}=\mathbb{A}^{2,\mathrm{an}}_K$ of $f:U\times_K\mathbb{P}^{1,\mathrm{an}}\to\mathbb{P}^{1,\mathrm{an}}$ and an analytic lift $A:U\to\mathbb{A}^{2,\mathrm{an}}\setminus\{0_{\mathbb{A}^2_K}\}$ of $a:U\to\mathbb{P}^{1,\mathrm{an}}$, writing $F^n=(F_0^{(n)},F_1^{(n)})$ for each $n\in\mathbb{N}$, the function

$$h^{(n)}(t):=\frac{\log\max\{|F_0^{(n)}(t,A(t))|,|F_1^{(n)}(t,A(t))|\}}{d^n}$$

on $U(K)$ extends continuously and subharmonically to U so that $-\mathrm{dd}^c h^{(n)}=a_n^*\delta_{\zeta_{0,1}}/d^n$ on U, and the uniform limit $h_{(f,a)}=\lim_{n\to\infty}h^{(n)}$ on U exists and is a continuous and subharmonic function on U. Then for every $j\in\mathbb{N}$, the function

$$\frac{\log\|a_{n_j}(\cdot),\zeta_0\|}{d^{n_j}}+h^{(n_j)}\quad\text{on } U$$

is subharmonic (using a functoriality of dd^c), and shrinking U if necessary, the family of those subharmonic functions on U is also uniformly bounded from above on U. Then using a non-archimedean version of Hörmander's version of Hartogs's lemma (see [2, Proposition 8.54]), either

$$\lim_{j\to\infty}\frac{\log\|a_{n_j}(\cdot),\zeta_0\|}{d^{n_j}}=-\infty$$

uniformly on any compact subset of U or, taking a subsequence of (n_j) if necessary, there is an upper semicontinuous function ϕ on U such that

$$\phi=\lim_{j\to\infty}\frac{\log\|a_{n_j}(\cdot),\zeta_0\|}{d^{n_j}}\quad\text{on } U\setminus U(K)$$

(so $\{\phi<0\}$ in U is open and contains x_0). In the former case, we must have $\zeta_0\in\mathcal{E}_S$ since $U\cap S\neq\emptyset$. In the latter case, we must still have $\zeta_0\in\mathcal{E}_S$ since $\{\phi<0\}\cap S\neq\emptyset$. In any case, this is a contradiction.

Hence $E_V(f,a)\subset\mathcal{E}_S$, which completes the proof of Theorem 2 in this case. □

4. Proof of Theorem 2: general case. Let V be a domain in the (Berkovich) analytification C^{an} of a smooth projective algebraic curve C/K. Let $f:V\times_K\mathbb{P}^{1,\mathrm{an}}\to\mathbb{P}^{1,\mathrm{an}}$ be a family of endomorphisms of $\mathbb{P}^{1,\mathrm{an}}$ of degree $d>1$ and $a:V\to\mathbb{P}^{1,\mathrm{an}}$ be a marked point in $\mathbb{P}^{1,\mathrm{an}}$, both analytically parametrized by V. Recall the definition (1.2) of the marked point a_n for each $n\in\mathbb{N}$.

Lemma 4.1. *For any probability Radon measure ρ on $\mathbb{P}^{1,\mathrm{an}}$, we have* $\mathrm{Cap}(E_V(f,a)\cap(\operatorname{supp}\rho))=0$.

Proof. Pick a probability Radon measure ρ on $\mathbb{P}^{1,\mathrm{an}}$.

Recall the profinite graph (more precisely, the inverse limit of skeletons) structure of the Berkovich curve C^{an} and the (extended) skeletal metric on $C^{\mathrm{an}}\setminus C(K)$ (see [1, §5]). Here, a skeleton Σ of C^{an} is a finite subgraph in C^{an} so that all connected components of $C^{\mathrm{an}}\setminus\Sigma$ are open balls in C^{an} (see [1, §3]). In particular, by the connectedness of V and the compactness of the topological boundary ∂V of V in C^{an}, there is a sequence $(U_j)_j$ of relatively compact subdomains in V increasing to V such that for any $j\in\mathbb{N}$, U_j is a connected component of the complement in C^{an} of a finite subset of C^{an}.

Let Γ_0 be the union of all paths in V joining distinct two points in ∂V. Then for every $j\in\mathbb{N}$, $\Gamma_0^{(j)}:=\Gamma_0\cap U_j$ is an at most finitely branched and connected subgraph in U_j. For any $j,n\in\mathbb{N}$, the signed measure $(a_n^*(\rho-\delta_{\zeta_{0,1}}))|\overline{U_j}$ is finite, so for every $j\in\mathbb{N}$, by the argument in [2, Proof of Lemma 5.7], there is an increasing sequence $(\Gamma_k^{(j)})_{k\in\mathbb{N}}$ of finite and connected subgraphs in U_j such that $\bigcup_{n\in\mathbb{N}}(\operatorname{supp}(a_n^*(\rho-\delta_{\zeta_{0,1}})))\cap U_j$ is contained in the closure of $\bigcup_{k\in\mathbb{N}}\Gamma_k^{(j)}$ in V. Noting that for $j\gg1$, U_j contains the union L of all (finitely many non-trivial) loops in V, for every $j\gg1$ and every $k\in\mathbb{N}\cup\{0\}$, we replace $\Gamma_k^{(j)}$ with an at most finitely branched and connected subgraph in U_j containing $\Gamma_k^{(j)}\cup L$. Then letting

$$X_\rho=X_\rho^{(f,a)}:=\overline{\bigcup_{j\in\mathbb{N}}\left(\bigcup_{k\in\mathbb{N}\cup\{0\}}\Gamma_k^{(j)}\right)}\quad\text{in } V,$$

we have

$$\bigcup_{\zeta\in\operatorname{supp}\rho}\left(\bigcup_{n\in\mathbb{N}}a_n^{-1}([\zeta,\zeta_{0,1}])\right)\subset X_\rho,\tag{4.1}$$

and there is a countable dense subset $\mathcal{Y}_\rho$ of X_ρ. Set

$$\mathcal{E}_{\mathcal{Y}_\rho}:=\bigcup_{y\in\mathcal{Y}_\rho}\mathcal{E}_y,$$

which is still of logarithmic capacity 0 in $\mathbb{P}^{1,\mathrm{an}}$. Here for each $y\in V\setminus V(K)$, the subset $\mathcal{E}_y$ of $\mathbb{P}^{1,\mathrm{an}}$ (indeed of $\mathbb{P}^1$) is defined as in (3.1).

We claim that $E_V(f,a)\cap(\operatorname{supp}\rho)\subset\mathcal{E}_{\mathcal{Y}_\rho}$; for, otherwise, there are $\zeta_0\in((\operatorname{supp}\rho)\cap E_V(f,a))\setminus\mathcal{E}_{\mathcal{Y}_\rho}$,

$x_0 \in V \setminus V(K)$, and a sequence (n_j) in $\mathbb{N}$ tending to ∞ such that

$$\lim_{j\to\infty} \frac{\log \|a_{n_j}(x_0), \zeta_0\|}{d^{n_j}} < 0. \tag{4.2}$$

Suppose that $x_0 \in V \setminus X_\rho$, and let U be the connected component of $V \setminus X_\rho$ containing x_0. Then for every $j \in \mathbb{N}$, the continuous function $(\log \|a_{n_j}(\cdot), \zeta_0\|)/d^{n_j}$ on V is constant on U (for, otherwise, since the generalized Hsia kernel $\|\cdot, \zeta_0\|$ is locally constant on $\mathbb{P}^{1,\mathrm{an}} \setminus [\zeta_0, \zeta_{0,1}]$, by $\zeta_0 \in \operatorname{supp} \rho$ and (4.1), we must have $\emptyset \neq U \cap a_{n_j}^{-1}([\zeta_0, \zeta_{0,1}]) \subset X_\rho$, which is impossible). This with (4.2) yields the convergence

$$\begin{aligned} &\lim_{j\to\infty} \frac{\log \|a_{n_j}(\cdot), \zeta_0\|}{d^{n_j}} \\ &\qquad \equiv \lim_{j\to\infty} \frac{\log \|a_{n_j}(x_0), \zeta_0\|}{d^{n_j}} < 0 \quad \text{on } \overline{U}. \end{aligned}$$

Hence, since $\emptyset \neq \partial U \subset X_\rho \setminus V(K)$, where ∂U is the topological boundary of U in C^{an}, we can replace the original x_0 so that $x_0 \in X_\rho$ (without changing the ζ_0). Then we are done by an argument by contradiction similar to that in the separable case in Section 3 which involves a non-archimedean version of Hörmander's version of Hartogs's lemma. Hence the claim holds.

Once this claim is at our disposal, we are done by $\mathrm{Cap}(\mathcal{E}_{\mathcal{Y}_\rho}) = 0$ and the monotonicity of the capacity function Cap. $\square$

If $E_V(f, a)$ is not of logarithmic capacity 0, then there must exist a probability Radon measure ρ on $\mathbb{P}^{1,\mathrm{an}}$ supported by $E_V(f, a)$ and satisfying $\int_{\mathbb{P}^{1,\mathrm{an}}} u_{\zeta_{0,1},\rho}(\xi)\rho(\xi) < \infty$. By Lemma 4.1, this is impossible (see [2, Lemma 6.16]). Now the proof of Theorem 2 is complete. $\square$

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