A local characterization of B_2 regular crystals

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Abstract: Stembridge characterized regular crystals associated with a simply-laced generalized Cartan matrix (GCM) in terms of local graph-theoretic quantities. We give a similar axiomatization for B_2 regular crystals and thus for regular crystals associated with a finite GCM except G_2 and an affine GCM except $A_1^{(1)}, G_2^{(1)}, A_2^{(2)}, D_4^{(3)}$.

Key words: Kashiwara crystals; quantum groups; local characterization.

1. Introduction.

1.1. Kashiwara crystals. Let $A=(a_{ij})_{i,j\in I}$ be a symmetrizable GCM and fix a Cartan datum $(P,P^\vee,\Pi,\Pi^\vee)$ [6, §2.1]. A Kashiwara crystal is a 6-tuple $(B,\mathsf{wt},(\tilde{e}_i)_{i\in I},(\tilde{f}_i)_{i\in I},(\varepsilon_i)_{i\in I},(\varphi_i)_{i\in I})$, where B is a set and $\mathsf{wt}:B\to P,\varepsilon_i,\varphi_i:B\to \mathbf{Z}\sqcup\{-\infty\}$, $\tilde{e}_i,\tilde{f}_i:B\to B\sqcup\{\mathbf{0}\}$ are functions that satisfy the axioms [6, (7.1)–(7.5)].

1.2. Highest weight crystals and regular crystals. For a dominant integral weight $\lambda\in P^+$, Kashiwara proved the existence and uniqueness of the crystal basis $B(\lambda)$ (called the highest weight crystal) of the integrable highest weight module $V(\lambda)$ of the quantum group $U_q(A)$ [5]. Under a condition [7, (2.4.1)], regular crystal is a disjoint union of the highest weight crystals [7, Proposition 2.4.4].

1.3. Crystal graphs. A Kashiwara crystal gives an I-colored directed graph (called the crystal graph) by the rule: there is an i-colored arrow from x to y if and only if $\tilde{f}_ix=y$.

Definition 1.1. An I-colored directed graph X is *good* if for any $x\in X$ and $i\in I$

(G1) there is at most one i-colored arrow from x,
(G2) there is at most one i-colored arrow to x,
(G3) the length of the i-string through x is finite.

When there is an i-colored arrow from x to y in a good I-colored directed graph X, we define as $\tilde{f}_ix=y$ and $\tilde{e}_iy=x$. $\tilde{f}_ix=\mathbf{0}$ (resp. $\tilde{e}_ix=\mathbf{0}$) means that there is no i-colored arrow from x (resp. to x). Thanks to the axioms, $\varphi_i(x)=\max\{m\geq 0\mid \tilde{f}_i^mx\neq\mathbf{0}\}$ and $\varepsilon_i(x)=\max\{m\geq 0\mid \tilde{e}_i^mx\neq\mathbf{0}\}$ are well-defined. The crystal graph of $B(\lambda)$ is good and the quantities ε_i,φ_i are the same as above [5, (2.4.1)].

Definition 1.2. Let X be a good I-colored directed graph. We say that $x_0\in X$ is *maximum* if

(M1) for $i\in I$ we have $\tilde{e}_ix_0=\mathbf{0}$ (i.e., $\varepsilon_i(x_0)=0$),
(M2) for $x\in X$ there exists $s\geq 0$ and $(i_1,\cdots,i_s)\in I^s$ such that $\tilde{f}_{i_1}\cdots\tilde{f}_{i_s}x_0=x$.

Definition 1.3. Let X be a good I-colored directed graph. For $g\in\{e,f\},\beta\in\{\varepsilon,\varphi\}$ and $x\in X$, $i,j\in I$ with $\tilde{g}_ix\neq\mathbf{0}$, we define

$$\Delta^g_\beta(i,j,x)=\beta_j(\tilde{g}_ix)-\beta_j(x).$$

1.4. Stembridge crystals.

Theorem 1.4 ([9, Definition 1.1, Theorem 2.4]). *Let $A=(a_{ij})_{i,j\in I}$ be a symmetrizable GCM. For a dominant integral weight $\lambda\in P^+$, the highest weight crystal $B(\lambda)$ is an A-regular graph (defined by the axioms* (S1)–(S5) *below) having a maximum $b_\lambda\in B(\lambda)$ with $\varphi_i(b_\lambda)=\langle h_i,\lambda\rangle$ for all $i\in I$.*

(S1) X is a good I-colored directed graph in the sense of Definition 1.1.
(S2) $\forall x\in X,\forall i\in I,\tilde{e}_ix\neq\mathbf{0}\Rightarrow\forall j\in I\setminus\{i\}$, $\Delta^e_\varphi(i,j,x)-\Delta^e_\varepsilon(i,j,x)=a_{ji}$.
(S3) $\forall x\in X,\forall i\in I,\tilde{e}_ix\neq\mathbf{0}\Rightarrow\forall j\in I\setminus\{i\}$, $\Delta^e_\varphi(i,j,x)\leq 0\leq\Delta^e_\varepsilon(i,j,x)$.
(S4) $\forall i\neq\forall j\in I,\forall x\in X,\quad \tilde{e}_ix\neq\mathbf{0}\neq\tilde{e}_jx\Rightarrow\quad(\mathrm{A}^-_{i,j})$, (B^-).
(S5) $\forall i\neq\forall j\in I,\forall x\in X,\quad \tilde{f}_ix\neq\mathbf{0}\neq\tilde{f}_jx\Rightarrow\quad(\mathrm{A}^+_{i,j})$, (B^+).

$(\mathrm{A}^-_{k,\ell})$ $\Delta^e_\varepsilon(k,\ell,x)=0\Rightarrow\exists z=\tilde{e}_\ell\tilde{e}_kx=\tilde{e}_k\tilde{e}_\ell x$, $\Delta^f_\varphi(\ell,k,z)=0$.

(B^-) $(\Delta^e_\varepsilon(i,j,x),\Delta^e_\varepsilon(j,i,x))=(1,1)\Rightarrow\exists z=\tilde{e}_i\tilde{e}_j^2\tilde{e}_ix=\tilde{e}_j\tilde{e}_i^2\tilde{e}_jx,(\Delta^f_\varphi(i,j,z),\Delta^f_\varphi(j,i,z))=(1,1)$.

$(\mathrm{A}^+_{k,\ell})$ $\Delta^f_\varphi(k,\ell,x)=0\Rightarrow\exists z=\tilde{f}_\ell\tilde{f}_kx=\tilde{f}_k\tilde{f}_\ell x$, $\Delta^e_\varepsilon(\ell,k,z)=0$.

(B^+) $(\Delta^f_\varphi(i,j,x),\Delta^f_\varphi(j,i,x))=(1,1)\Rightarrow\exists z=\tilde{f}_i\tilde{f}_j^2\tilde{f}_ix=\tilde{f}_j\tilde{f}_i^2\tilde{f}_jx,(\Delta^e_\varepsilon(i,j,z),\Delta^e_\varepsilon(j,i,z))=(1,1)$.

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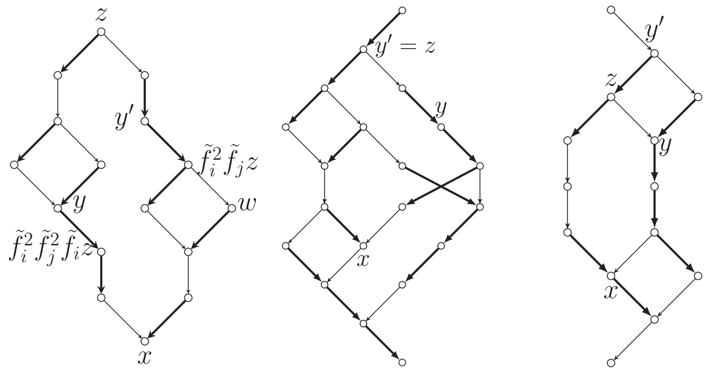

Fig. 1. B_2 crystals $B(\Lambda_1+\Lambda_2), B(3\Lambda_1), B(2\Lambda_2)$ from left to right.

Remark 1.5. As in [9, p. 4810], (B^-) (and $(\mathrm{A}^{\pm}_{k,\ell})$) has a redundancy in that some are forced. However we will not consider minimization of axioms and use abbreviations involving $\exists$.

Theorem 1.6 ([9, Proposition 1.4, Theorem 3.3]). *Let $A=(a_{ij})_{i,j\in I}$ be a simply-laced GCM and let X be an A-regular graph with a maximum $x_0\in X$. Then, there exists a unique I-colored directed graph isomorphism between X and $B(\lambda)$, where $\lambda\in P^+$ satisfies $\langle h_i,\lambda\rangle=\varphi_i(x_0)$ for all $i\in I$.*

Example 1.7. The left (resp.right) figure below is an A_2-crystal which gives a visualization of $(\mathrm{A}^-_{1,2})$ (resp.(B^-)). Here, thick arrows are 1-arrows.

1.5. The main result.

Theorem 1.8. *Let $A=(a_{ij})_{i,j\in I}$ be a symmetrizable GCM with $\forall i\neq\forall j\in I, A|_{i,j}=A_1\oplus A_1, A_2, B_2, {}^tB_2$ and let X be an A-regular graph with a maximum $x_0\in X$ that further satisfies*

$$\forall i\neq\forall j\in I, A|_{i,j}=B_2\Rightarrow \text{(S6),(S7),(S8),(S9)}.$$

Then, there exists a unique I-colored directed graph isomorphism between X and $B(\lambda)$, where $\lambda\in P^+$ satisfies $\langle h_i,\lambda\rangle=\varphi_i(x_0)$ for all $i\in I$.

(S6) $\forall x\in X, \tilde e_i x\neq\mathbf{0}\neq\tilde e_j x, \Delta(x)=(1,2)\Rightarrow(\mathrm{D}^-)$.
(S7) $\forall x\in X, \tilde f_i x\neq\mathbf{0}\neq\tilde f_j x, \Delta'(x)=(1,2)\Rightarrow(\mathrm{D}^+)$.
(S8) $\forall x\in X, \tilde f_i x\neq\mathbf{0}\neq\tilde f_j x, \Delta'(x)=(1,1), \varphi_i(x)\geq 2\Rightarrow(\mathrm{C}^+_1)$.
(S9) $\forall x\in X, \tilde f_i x\neq\mathbf{0}\neq\tilde f_j x, \Delta'(x)=(0,2)$, $\tilde f_j\tilde f_i^2 x\neq\mathbf{0}, \Delta^f_\varphi(j,i,\tilde f_i^2 x)=0\Rightarrow(\mathrm{C}^+_1)$.
(D^-) $y:=\tilde e_i^2\tilde e_j x$, $\exists y'=\tilde e_i^2\tilde e_j^2\tilde e_i x$, (P^-_1), (Q^-_1), (R^-), $(\Delta^f_\varphi(i,j,y),\Delta^f_\varphi(i,j,y'))\neq(1,0)$.
(D^+) $y:=\tilde f_i^2\tilde f_j x$, $\exists y'=\tilde f_i^2\tilde f_j^2\tilde f_i x$, $(\mathrm{Q}^{+\prime}_1)$.
(C^+_1) $\exists z=\tilde f_i\tilde f_j^2\tilde f_i^2 x=\tilde f_j\tilde f_i^3\tilde f_j x$.
(P^-_1) $(\Delta^f_\varphi(i,j,y),\Delta^f_\varphi(i,j,y'))=(1,1)\Rightarrow\tilde f_j y'=\tilde e_i y$, $\Delta^f_\varphi(j,i,y')=1$.
(Q^-_1) $(\Delta^f_\varphi(i,j,y),\Delta^f_\varphi(i,j,y'))=(0,1)\Rightarrow\exists z=\tilde e_j\tilde e_i^3\tilde e_j^2\tilde e_i x=\tilde e_i\tilde e_j^2\tilde e_i^3\tilde e_j x, \Delta'(z)=(1,2)$.
(R^-) $(\Delta^f_\varphi(i,j,y),\Delta^f_\varphi(i,j,y'))=(0,0)\Rightarrow\tilde f_j y'=\tilde e_i y$, $\Delta^f_\varphi(j,i,y')=2, \Delta^f_\varphi(j,i,\tilde f_i^2 y')=0$.
$(\mathrm{Q}^{+\prime}_1)$ $(\Delta^e_\varepsilon(i,j,y),(\Delta^e_\varepsilon(i,j,y'))=(0,1)\Rightarrow\exists z=\tilde f_j\tilde f_i^3\tilde f_j^2\tilde f_i x=\tilde f_i\tilde f_j^2\tilde f_i^3\tilde f_j x$.

Here, we define $\Delta(x)=(\Delta^e_\varepsilon(i,j,x),\Delta^e_\varepsilon(j,i,x))$ and $\Delta'(w)=(\Delta^f_\varphi(i,j,w),\Delta^f_\varphi(j,i,w))$ for $w=x,z$. We adapt a convention for B_2 that α_1 (resp.α_2) is short (resp. long). Note that y in (D^-) (resp.(D^+)) is just defined. The existence is not a part of the axiom because it follows from $\Delta^e_\varepsilon(j,i,x)=2$ (resp.$\Delta^f_\varphi(j,i,x)=2$). Note also that we have $\tilde e_i y\neq\mathbf{0}$ in (P^-_1),(R^-) by $\Delta^e_\varepsilon(j,i,x)=2$ and $\varepsilon_i(x)\geq 1$.

Example 1.9. We duplicate [9, Figure 5] as Figure 1, where thick arrows are 1-arrows. We can see an appearance of (Q^-_1),(P^-_1),(R^-) from left to right, (S7) (resp. (S8)) in the left (resp. middle) graph at z, and (S9) in the right graph at y'.

1.6. Variants of axioms. By Proposition 2.1, we can replace (P^-_1), (Q^-_1) with
(P^-) $(\Delta^f_\varphi(i,j,y),\Delta^f_\varphi(i,j,y'))=(1,1)\Rightarrow y'=\tilde e_i\tilde e_j\tilde e_i\tilde e_j\tilde e_i x=\tilde e_j\tilde e_i^3\tilde e_j x, \Delta^f_\varphi(j,i,y')=1$.
(Q^-) $(\Delta^f_\varphi(i,j,y),\Delta^f_\varphi(i,j,y'))=(0,1)\Rightarrow\exists z=\tilde e_j\tilde e_i^2\tilde e_j\tilde e_i\tilde e_j\tilde e_i x=\tilde e_j\tilde e_i^3\tilde e_j^2\tilde e_i x=\tilde e_i\tilde e_j^2\tilde e_i^3\tilde e_j x=\tilde e_i\tilde e_j\tilde e_i\tilde e_j\tilde e_i^2\tilde e_j x$, $\Delta'(z)=(1,2)$.
respectively (and independently). A reason why the shorter version works is that Proposition 3.1 that is used in the proof of Theorem 1.8 just needs weak Church-Rosser (a.k.a. local confluence) property.

Definition 1.10 (see [1, §2.7])**.** Let X be a good I-colored directed graph. We say that X has a *homogeneous local confluence* property if for $x \in X$ and $i \neq j \in I$ with $\tilde{e}_i x \neq \mathbf{0} \neq \tilde{e}_j x$ there exists $s \geq 2$ and $(i_1, \cdots, i_s), (i'_1, \cdots, i'_s) \in I^s$ such that

$$i_s = i, i'_s = j, \exists z = \tilde{e}_{i_1} \cdots \tilde{e}_{i_s} x = \tilde{e}_{i'_1} \cdots \tilde{e}_{i'_s} x$$

and $\{i_k \mid 1 \leq k \leq s\} = \{i'_k \mid 1 \leq k \leq s\}$ as multisets.

Remark 1.11. In (Q_1^-), $\tilde{f}_i^2 \tilde{f}_j z = \tilde{f}_i y'$, $\tilde{f}_i^2 \tilde{f}_j^2 \tilde{f}_i z = \tilde{f}_i y$ (see Figure 1) and (S2) imply $(\Delta_\varepsilon^e(i, j, \tilde{f}_i^2 \tilde{f}_j z), \Delta_\varepsilon^e(i, j, \tilde{f}_i^2 \tilde{f}_j^2 \tilde{f}_i z)) = (0, 1)$.

1.7. Comparison with previous studies. Finding a local characterization of B_2 regular crystals has been a well-known open problem since [9].

Comparison with [10]. The confluence relations in (P^-),(Q^-) (and (R^-) that implies $\tilde{e}_i \tilde{e}_j^2 \tilde{e}_i x = \tilde{e}_j \tilde{e}_i^2 \tilde{e}_j x$ by (S4)) were observed in [9, p. 4822] and were proved in [10]. To determine which occurs for x with $\Delta(x) = (1, 2)$ from the local structure of x, existences of y and y' in (D^-) are crucial.

Remark 1.12. In this paper, "local condition" for $x \in X$ is an axiom that involves only $\Delta_\beta^g(k, \ell, y), \beta_k(y)$ and $=$ between y's, where $k, \ell \in I$, $g \in \{e, f\}, \beta \in \{\varepsilon, \varphi\}$ and y is "near" x. It means that we can go back and forth between x and y at most N arrows, where N is a constant. In Stembridge's axiom $N = 4$ and in ours $N = 7$. Note that the existence of a (unique) maximum element in Theorem 1.4 and Theorem 1.8 is not a local condition.

Other missing axioms play the following role.

(S8) compensates the symmetry breaking in (P_1^-) in that $\Delta'(z) = (1, 1)$ instead of $\Delta'(z) = (1, 2)$, where $z = \tilde{e}_i^2 \tilde{e}_j^2 \tilde{e}_i x = \tilde{e}_j \tilde{e}_i^3 \tilde{e}_j x (= y')$,

(S9) handles the fact $\tilde{f}_i^2 \tilde{f}_j^2 \tilde{f}_i z$ is "under" or "below" x in (R^-), where $z = \tilde{e}_i \tilde{e}_j^2 \tilde{e}_i x = \tilde{e}_j \tilde{e}_i^2 \tilde{e}_j x$ notwithstanding $\Delta'(z) = (1, 2)$.

Remark 1.13. As [9, Remark 1.5], Theorem 1.6 gives an iterative algorithm that draws simply-laced highest weight crystals (the proof of [9, Proposition 1.4] provides an algorithm). Especially thanks to (S9), it is similarly applied to Theorem 1.8 (the proof of Proposition 3.3 provides an algorithm).

Comparison with [2]. In [2], they gave a set of axioms and claimed that it characterizes B_2 regular crystals (see the first paragraph of [2, §3]. In [3], they gave a set of axioms for graphs $G = (V, E)$ equipped with labels $\ell(v) \in \{L, C, R\}$ on the vertices $v \in V$). Their idea in [2] is different from [9] while this paper is a small modification of [9] as in Remarks 1.12 and 1.13. For example, it is not clear how the axioms of [2] are translated to an iterative algorithm mentioned in Remark 1.13.

2. Proof of Theorem 1.8: $B(\lambda)$ satisfies the axioms in Theorem 1.8.

2.1. A reduction to $A = B_2$. Combined with Theorem 1.4, to prove that $B(\lambda)$ satisfies the axioms in Theorem 1.8, it is enough to prove that B_2 highest weight crystals satisfy (S6),(S7),(S8),(S9) putting $i = 1, j = 2$. In the rest of §2, we assume $A = B_2$ (indexed by $I = \{1, 2\}$, where α_1 is short) as §1.5 and prove Proposition 2.1, Proposition 2.2, Proposition 2.3 in §2.4, §2.5, §2.6 that imply ((S6),(S7)),(S8),(S9) respectively thanks to Proposition 2.4, which is a version of the Lusztig involution.

Proposition 2.1. *Fix $\lambda \in P^+$ and take $x \in B(\lambda)$. If $\tilde{e}_1 x \neq \mathbf{0} \neq \tilde{e}_2 x$ and $(\Delta_\varepsilon^e(1, 2, x), \Delta_\varepsilon^e(2, 1, x)) = (1, 2)$, then $\exists y' = \tilde{e}_1^2 \tilde{e}_2^2 \tilde{e}_1 x$ and we have exactly (i.e., exclusively) one of the following 3 cases. Here $\Delta' = (\Delta_\varphi^f(1, 2, z), \Delta_\varphi^f(2, 1, z))$ and $\Delta'' = (\Delta_\varphi^f(1, 2, y), \Delta_\varphi^f(1, 2, y'))$, $y = \tilde{e}_1^2 \tilde{e}_2 x$.*

(case $\Delta'' = (1, 1)$)

$y' = \tilde{e}_1 \tilde{e}_2 \tilde{e}_1 \tilde{e}_2 \tilde{e}_1 x = \tilde{e}_2 \tilde{e}_1^3 \tilde{e}_2 x, \Delta_\varphi^f(2, 1, y') = 1$.

(case $\Delta'' = (0, 1)$)

$\exists z = \tilde{e}_2 \tilde{e}_1^2 \tilde{e}_2 \tilde{e}_1 \tilde{e}_2 \tilde{e}_1 x = \tilde{e}_2 \tilde{e}_1^3 \tilde{e}_2^2 \tilde{e}_1 x = \tilde{e}_1 \tilde{e}_2^2 \tilde{e}_1^3 \tilde{e}_2 x = \tilde{e}_1 \tilde{e}_2 \tilde{e}_1 \tilde{e}_2 \tilde{e}_1^2 \tilde{e}_2 x, \Delta' = (1, 2)$.

(case $\Delta'' = (0, 0)$)

$\tilde{f}_2 y' = \tilde{e}_1 y, \Delta_\varphi^f(2, 1, y') = 2, \Delta_\varphi^f(2, 1, \tilde{f}_1^2 y') = 0$.

Proposition 2.2. *Fix $\lambda \in P^+$ and take $x \in B(\lambda)$. If $\tilde{e}_1 x \neq \mathbf{0} \neq \tilde{e}_2 x$ and $\varepsilon_1(x) \geq 2$, $(\Delta_\varepsilon^e(1, 2, x), \Delta_\varepsilon^e(2, 1, x)) = (1, 1)$, then $\exists z = \tilde{e}_1 \tilde{e}_2^2 \tilde{e}_1^2 x = \tilde{e}_1 \tilde{e}_2 \tilde{e}_1 \tilde{e}_2 \tilde{e}_1 x = \tilde{e}_2 \tilde{e}_1^3 \tilde{e}_2 x$.*

Proposition 2.3. *Fix $\lambda \in P^+$ and take $x \in B(\lambda)$. If $\tilde{e}_1 x \neq \mathbf{0} \neq \tilde{e}_2 x$ and $(\Delta_\varepsilon^e(1, 2, x), \Delta_\varepsilon^e(2, 1, x)) = (0, 2)$, $\tilde{e}_2 \tilde{e}_1^2 x \neq \mathbf{0}, \Delta_\varepsilon^e(2, 1, \tilde{e}_1^2 x) = 0$, then $\exists z = \tilde{e}_2 \tilde{e}_1^3 \tilde{e}_2 z = \tilde{e}_2 \tilde{e}_1^2 \tilde{e}_2 \tilde{e}_1 x = \tilde{e}_1 \tilde{e}_2^2 \tilde{e}_1^2 x$.*

Proposition 2.4 (see [6, §7.4])**.** *For $\lambda \in P^+$, there is an involution $w : B(\lambda) \xrightarrow{\sim} B(\lambda)$ such that*

(a) $\forall b \in B(\lambda), \forall i \in I, \varepsilon_i(b) = \varphi_i(w(b))$,

(b) $\forall b \in B(\lambda), \forall i \in I, \tilde{e}_i b \neq \mathbf{0} \Rightarrow w(\tilde{e}_i b) = \tilde{f}_i(w(b))$.

2.2. A realization of B_2 highest weight crystals. The choice $\boldsymbol{i} = s_1 s_2 s_1 s_2$ (resp. $\boldsymbol{j} = s_2 s_1 s_2 s_1$) of a reduced expression of the longest element w_0 gives the convex order on the positive roots. Lusztig's PBW parameterization associated with $\boldsymbol{k} \in \{\boldsymbol{i}, \boldsymbol{j}\}$ gives a realization of $B(\infty)$ on $\mathbf{N}^4$, where $4 = \ell(w_0)$. The function R switches the two parameterizations [4, §3].

Definition 2.5. Let $R:\mathbf{N}^4\to\mathbf{N}^4$, $(a,b,c,d)\mapsto(n_1,\mu-n_2,n_2+n_3-\mu,n_4-2n_3+\mu)$ be a bijection with $R^{-1}:\mathbf{N}^4\to\mathbf{N}^4$, $(a,b,c,d)\mapsto(p_1,\nu-p_2,2p_2+p_3-2\nu,p_4-p_3+\nu)$.

$$n_1=\max(b,\max(b,d)+c-a),$$
$$p_1=\max(b,\max(b,d)+2(c-a)),$$
$$n_2=\max(a,c)+2b,\quad p_2=\max(a,c)+b,$$
$$n_3=\min(c+d,a+\min(b,d)),$$
$$p_3=\min(2c+d,2a+\min(b,d)),$$
$$n_4=\min(a,c),\quad p_4=\min(a,c),$$
$$\mu=\max(2n_3,n_2+n_4),\quad \nu=\max(p_3,p_2+p_4).$$

In $B(\infty)\otimes T_\lambda$, thanks to [6, Proposition 8.2], $B(\lambda)$ is isomorphic to

$$\{b\otimes t_\lambda\mid b\in B(\infty),\forall i\in I,\varepsilon_i^*(b)\le\langle h_i,\lambda\rangle\},$$

where T_λ is given as [6, Example 7.3]. Though we do not explain the $*$-structure (see [6, §8.3]), we use the fact $\varepsilon_1^*(\boldsymbol{x})=x_4$ (resp. $\varepsilon_2^*(\boldsymbol{a})=a_4$) (see [8, §2.11]) for $\boldsymbol{x}\in\mathbf{N}^4$ (resp. $\boldsymbol{a}\in\mathbf{N}^4$) in the parameterization associated with $\boldsymbol{j}$ (resp. $\boldsymbol{i}$). Thus:

Proposition 2.6. *For $\lambda\in P^+$, $B(\lambda)$ is realized as $(B(\lambda),\mathsf{wt},(\tilde{e}_i)_{i\in I},(\tilde{f}_i)_{i\in I},(\varepsilon_i)_{i\in I},(\varphi_i)_{i\in I})$.*

$$B(\lambda)=\{(\boldsymbol{a},\boldsymbol{x})\in\mathbf{N}^4\times\mathbf{N}^4\mid R(\boldsymbol{a})=\boldsymbol{x},\ x_4\le\langle h_1,\lambda\rangle,a_4\le\langle h_2,\lambda\rangle\},$$
$$\mathsf{wt}(\boldsymbol{a},\boldsymbol{x})=\lambda-(x_2+2x_3+x_4)\alpha_1-(x_1+x_2+x_3)\alpha_2,$$
$$\varepsilon_1(\boldsymbol{a},\boldsymbol{x})=a_1,\quad \varepsilon_2(\boldsymbol{a},\boldsymbol{x})=x_1,$$
$$\varphi_i(\boldsymbol{a},\boldsymbol{x})=\varepsilon_i(\boldsymbol{a},\boldsymbol{x})+\langle h_i,\mathsf{wt}(\boldsymbol{a},\boldsymbol{x})\rangle,$$
$$\tilde{e}_1(\boldsymbol{a},\boldsymbol{x})=\begin{cases}((a_1-1,a_2,a_3,a_4),R(a_1-1,a_2,a_3,a_4))\\ \mathbf{0}\end{cases},$$
$$\tilde{e}_2(\boldsymbol{a},\boldsymbol{x})=\begin{cases}(R^{-1}(x_1-1,x_2,x_3,x_4),(x_1-1,x_2,x_3,x_4))\\ \mathbf{0}\end{cases},$$
$$\tilde{f}_1(\boldsymbol{a},\boldsymbol{x})=\begin{cases}((a_1+1,a_2,a_3,a_4),R(a_1+1,a_2,a_3,a_4))\\ \mathbf{0}\end{cases},$$
$$\tilde{f}_2(\boldsymbol{a},\boldsymbol{x})=\begin{cases}(R^{-1}(x_1+1,x_2,x_3,x_4),(x_1+1,x_2,x_3,x_4))\\ \mathbf{0}\end{cases}.$$

Here, $\tilde{e}_i(\boldsymbol{a},\boldsymbol{x})=\mathbf{0}$ (resp. $\tilde{f}_i(\boldsymbol{a},\boldsymbol{x})=\mathbf{0}$) if and only if $\varepsilon_i(\boldsymbol{a},\boldsymbol{x})=0$ (resp. $\varphi_i(\boldsymbol{a},\boldsymbol{x})=0$) for $i=1,2$.

2.3. Auxiliary formulas.

Lemma 2.7. *For $\boldsymbol{a}=(a_1,a_2,a_3,a_4)\in\mathbf{N}^4$ with $a_3\ge a_1$, $R(\boldsymbol{a})$ is given by* $(\max(a_2,a_4)+a_3-a_1,a_1,\min(a_2,a_4),a_3+2a_2-2\min(a_2,a_4))$.

Corollary 2.8. *For $\lambda\in P^+$, take $m=((a_1,a_2,a_3,a_4),(x_1,x_2,x_3,x_4))\in B(\lambda)$. If $a_3\ge a_1$ and $x_1\ge 1$, then $\Delta_\varepsilon^e(2,1,m)=\max(0,2+a_1-a_3+2a_2-2\max(a_2,a_4))$.*

Lemma 2.9. *For $\boldsymbol{x}=(x_1,x_2,x_3,x_4)\in\mathbf{N}^4$ with $x_3\ge x_1$, $R^{-1}(\boldsymbol{x})$ is given by* $(\max(x_2,x_4)+2(x_3-x_1),x_1,\min(x_2,x_4),x_3+x_2-\min(x_2,x_4))$.

Corollary 2.10. *For $\lambda\in P^+$, take $m=((a_1,a_2,a_3,a_4),(x_1,x_2,x_3,x_4))\in B(\lambda)$. If $x_3\ge x_1$ and $a_1\ge 1$, then $\Delta_\varepsilon^e(1,2,m)=\max(0,1+x_1-x_3+x_2-\max(x_2,x_4))$.*

Lemma 2.11. *For $\boldsymbol{a}=(a_1,a_2,a_3,a_4)\in\mathbf{N}^4$ with $a_3\le a_1$, $R(\boldsymbol{a})$ is given by*

$$\begin{cases}(a_2,a_3,a_4,a_1+2a_2-2a_4)\\ \quad\text{if } a_2\ge a_4+(a_3-a_1)/2,\\ (a_2,2a_3+2a_4-a_1-2a_2,a_1+2a_2-(a_3+a_4),a_3)\\ \quad\text{if } a_4+a_3-a_1\le a_2\le a_4+(a_3-a_1)/2,\\ (a_4+a_3-a_1,a_1,a_2,a_3)\\ \quad\text{if } a_2\le a_4+a_3-a_1.\end{cases}$$

Lemma 2.12. *For $\boldsymbol{x}=(x_1,x_2,x_3,x_4)\in\mathbf{N}^4$ with $x_3\le x_1$, $R^{-1}(\boldsymbol{x})$ is given by*

$$\begin{cases}(x_2,x_3,x_4,x_1+x_2-x_4)\\ \quad\text{if } x_2\ge x_4+x_3-x_1,\\ (x_2,2x_3+x_4-x_1-x_2,2x_1+2x_2-2x_3-x_4,x_3)\\ \quad\text{if } x_4+2(x_3-x_1)\le x_2\le x_4+x_3-x_1,\\ (x_4+2(x_3-x_1),x_1,x_2,x_3)\\ \quad\text{if } x_2\le x_4+2(x_3-x_1).\end{cases}$$

Corollary 2.13. *For $\lambda\in P^+$, take $m=((a_1,a_2,a_3,a_4),(x_1,x_2,x_3,x_4))\in B(\lambda)$. If $a_1>a_3$ and $x_1>x_3$, then $\Delta_\varepsilon^e(1,2,m)\Delta_\varepsilon^e(2,1,m)=0$.*

Proof. By Lemma 2.11, $x_1>x_3$ implies $a_2\ge a_4+(a_3-a_1)/2$ or $a_2\le a_4+a_3-a_1$. In the former, $a_2\ge a_4+(a_3-(a_1-1))/2$ holds by $a_2=x_1>x_3=a_4$ and $a_1>a_3$. This implies $\Delta_\varepsilon^e(1,2,m)=a_2-a_2=0$. The latter is similar by Lemma 2.12. □

2.4. Proof of Proposition 2.1.

Put

$$Y:=\{m\in B(\lambda)\mid\varepsilon_1(m),\varepsilon_2(m)>0,\ (\Delta_\varepsilon^e(1,2,m),\Delta_\varepsilon^e(2,1,m))=(1,2)\},$$
$$X_1:=\{((a,b,a,b),(b,a,b,a))\mid a,b\ge 1\}\cap B(\lambda),$$
$$X_2:=\{((a,b,a,c),(b,a,c,a+2b-2c))\mid a\ge 1,0\le c<b\}\cap B(\lambda),$$
$$X_3:=\{((a,b,c,a+b-c),(b,a,b,c))\mid b\ge 1,0\le c<a\}\cap B(\lambda).$$

We show $Y=X_1\sqcup X_2\sqcup X_3$. Since the inclusion $\supseteq$ is verified by direct calculation, take $m=((a_1,a_2,a_3,a_4),(x_1,x_2,x_3,x_4))\in Y$. By Corollaries 2.8 and 2.10, we have $a_1\ge a_3,x_1\ge x_3$ and thus we get $a_1=a_3$ or $x_1=x_3$ by Corollary

2.13. By Corollaries 2.8 and 2.10, this implies $a_2 \geq a_4$ (i.e., $m \in X_1 \sqcup X_2$) or $x_2 \geq x_4$ (i.e., $m \in X_1 \sqcup X_3$).

By direct calculation, one can check $x \in X_i$ satisfies the formula in case $\Delta'' = (0,1), (1,1), (0,0)$ depending on $i = 1, 2, 3$ respectively.

2.5. Proof of Proposition 2.2. Put

$$L = \{m \in B(\lambda) \mid \varepsilon_1(m) \geq 2, \varepsilon_2(m) > 0, (\Delta^e_\varepsilon(1,2,m), \Delta^e_\varepsilon(2,1,m)) = (1,1)\},$$
$$M = \{((a,b,a+1,c),(b+1,a,c,a+2b-2c+1)) \mid a \geq 2, 0 \leq c \leq b\} \cap B(\lambda).$$

It is enough to show $L = M$ since one can check $\tilde{e}_1\tilde{e}_2^2\tilde{e}_1^2 m = \tilde{e}_1\tilde{e}_2\tilde{e}_1\tilde{e}_2\tilde{e}_1 m = \tilde{e}_2\tilde{e}_1^3\tilde{e}_2 m$ for $m \in M$.

A direct calculation verifies $L \supseteq M$. To prove $L \subseteq M$, it is enough to show $a_3 \geq a_1$ for $m = ((a_1,a_2,a_3,a_4),(x_1,x_2,x_3,x_4)) \in L$ by Corollary 2.8. Assume $a_1 > a_3$. Corollary 2.13 implies $x_1 \leq x_3$ and Corollary 2.10 implies $x_1 = x_3, x_2 \geq x_4$ that means $m \in X_1 \sqcup X_3$. This contradicts $\Delta^e_\varepsilon(2,1,m) = 1$.

2.6. Proof of Proposition 2.3. Put

$$S = \{m \in B(\lambda) \mid \varepsilon_1(m) \geq 2, \varepsilon_2(m) > 0, \varepsilon_2(\tilde{e}_1^2 m) > 0, \Delta^e_\varepsilon(2,1,\tilde{e}_1^2 m) = 0, (\Delta^e_\varepsilon(1,2,m), \Delta^e_\varepsilon(2,1,m)) = (0,2)\},$$
$$T = \{((a,b,c,a+b-c-1),(b,a-2,b+1,c)) \mid a \geq 2, b \geq 1, 0 \leq c \leq a-2\} \cap B(\lambda).$$

It is enough to show $S = T$ since one can check $\tilde{e}_1\tilde{e}_2^2\tilde{e}_1^2 m = \tilde{e}_2\tilde{e}_1^2\tilde{e}_2\tilde{e}_1 m = \tilde{e}_2\tilde{e}_1^3\tilde{e}_2 m$ for $m \in T$.

The inclusion $S \supseteq T$ is verified by direct calculation. To prove the inclusion $S \subseteq T$, it is enough to show $x_3 \geq x_1, x_2 \geq x_4$ for any $m = ((a_1,a_2,a_3,a_4),(x_1,x_2,x_3,x_4)) \in S$ because the following deduces $x_3 = x_1 + 1$.

(a) $x_3 = x_1$, $x_2 \geq x_4$ implies $m \in X_1 \sqcup X_3$ and contradicts $\Delta^e_\varepsilon(1,2,m) = 0$.

(b) Let $x_3 = x_1 + n$ and assume $n \geq 2$ (then, we get a contradiction as (c)–(e)).

(c) By Lemma 2.9, $(a_1,a_2,a_3,a_4) = (x_2+2n, x_1, x_4, x_1+n+x_2-x_4)$.

(d) Because $a_2 - (a_4 + a_3 - (a_1-2)) = n-2 \geq 0$ and $a_4 + (a_3 - (a_1-2))/2 - a_2 = 1 + (x_2 - x_4)/2 \geq 0$, we have $\tilde{e}_1^2 m = ((a_1-2, a_2, a_3, a_4), (x_1, x_2+2, x_1+n-2, x_4))$ by Lemma 2.11.

(e) Because $x_1 - 1, x_1 \leq x_1 + n - 2$ we see $\Delta^e_\varepsilon(2,1,\tilde{e}_1^2 m) = 2$ by Lemma 2.9.

In the rest, we show $x_3 \geq x_1, x_2 \geq x_4$.

First, we show $a_1 > a_3$ as follows: Corollary 2.8 and $\Delta^e_\varepsilon(2,1,m) = 2$ imply $a_3 \leq a_1$. If $a_1 = a_3$, then $a_2 \geq a_4$ by Corollary 2.8. It means $m \in X_1 \sqcup X_2$ and contradicts $\Delta^e_\varepsilon(1,2,m) = 0$.

Next, we show $x_3 \geq x_1$. For this purpose, we assume $x_3 < x_1$ (and $a_1 > a_3$) to draw contradictions. By Lemma 2.12, $a_1 > a_3$ only happens when $x_2 \geq x_4 + x_3 - x_1$ or $x_2 \leq x_4 + 2(x_3 - x_1)$. In the former case, $x_2 \geq x_4 + x_3 - (x_1 - 1)$ also holds because $x_2 = a_1 > a_3 = x_4$ (and $x_1 > x_3$). Again, Lemma 2.12 implies $\Delta^e_\varepsilon(2,1,m) = x_2 - x_2 = 0$. In the latter case, we may assume $a_1 - 2 > a_3$ because otherwise

$$\Delta^e_\varepsilon(2,1,\tilde{e}_1^2 m) = \max(0, 2 + (a_1-2) - a_3 + 2a_2 - 2\max(a_2,a_4)) = a_1 - a_3 > 0$$

follows from Corollary 2.8 and $a_4 = x_3 < x_1 = a_2$. Thus, we know $\tilde{e}_1^2 m = ((a_1-2,a_2,a_3,a_4),(x_1,x_2,x_3,x_4-2))$ by Lemma 2.11 and $a_4 + (a_3 - (a_1-2))/2 - a_2 = (x_2 - x_4 + 2)/2 \leq 0$. This implies $\Delta^e_\varepsilon(2,1,\tilde{e}_1^2 m) = 2$ since $x_2 \leq (x_4 - 2) + 2(x_3 - (x_1 - 1))$ and Lemma 2.12. In both cases, we arrived at contradictions.

Finally, we show $x_2 \geq x_4$. For this purpose, we assume $x_2 < x_4$ (and $x_3 \geq x_1, a_1 > a_3$) to draw contradictions. Note that in Lemma 2.11 $x_2 < x_4$ only occurs when $a_2 > a_4 + (a_3 - a_1)/2$. In each of the following, we arrived at a contradiction.

Assume $a_1 - 2 \geq a_3$. Because $a_2 \geq a_4 + (a_3 - (a_1-2))/2$, again by Lemma 2.11, we have $\tilde{e}_1^2 m = ((a_1-2,a_2,a_3,a_4),(x_1,x_2,x_3,x_4-2))$. Lemma 2.9 and $x_1 - 1, x_1 \leq x_3$ imply $\Delta^e_\varepsilon(2,1,\tilde{e}_1^2 m) = 2$. Assume $a_1 - 2 < a_3$. This only happens when $a_1 = a_3 + 1$. Thanks to Lemma 2.9, m is of the form $m = ((x_2+1,x_1,x_2,x_1),(x_1,x_2,x_1,x_2+1))$. Then, we can check $\Delta^e_\varepsilon(2,1,\tilde{e}_1^2 m) = 1$ by Lemma 2.7.

3. Proof of Theorem 1.8: Uniqueness. We denote by $\mathbf{N}[I]$ the free commutative monoid generated by I. The following is a version of [9, Proposition 1.2, Remark 1.3.(a)], which is easily proved by induction on $d = \mathrm{depth}(x) := \min\{s \geq 0 \mid \exists(i_1,\cdots,i_s) \in I^s, x = \tilde{f}_{i_1}\cdots\tilde{f}_{i_s}x_0\}$.

Proposition 3.1. *Let X be a good I-colored directed graph with a maximum $x_0 \in X$ and with homogeneous local confluence property* (*see Definition* 1.1, 1.2, 1.10). *Then, for $x = \tilde{f}_{i_1}\cdots\tilde{f}_{i_s}x_0$, $\mathrm{wt}_0(x) = \sum_{k=1}^{s} i_k \in \mathbf{N}[I]$ is well-defined.*

Remark 3.2. In Proposition 3.1 and assume that X satisfies (S2) further. Fix $\lambda \in P^+$ such that $\forall i \in I, \langle h_i, \lambda\rangle = \varphi_i(x_0)$. By induction on $\mathrm{depth}(x)$, Proposition 3.1 implies $\varphi_i(x) = \varepsilon_i(x) + \langle h_i, \mathrm{wt}(x)\rangle$ for $i \in I, x \in X$ by defining $\mathrm{wt}(x) = \lambda - U(\mathrm{wt}_0(x))$ for $x \in X$, where $U: \mathbf{N}[I] \to P, \sum_k i_k \mapsto \sum_k \alpha_{i_k}$.

The following is similar to [9, Proposition 1.4].

Proposition 3.3. *For a symmetrizable GCM $A = (a_{ij})_{i,j\in I}$ with $\forall i \neq \forall j \in I, A|_{i,j} = A_1 \oplus A_1, A_2, B_2, {}^tB_2$, let X, X' be A-regular graphs satisfying*

$$\forall i \neq \forall j \in I, A|_{i,j} = B_2 \Rightarrow \text{(S6),(S7),(S8),(S9)}$$

with maximum $x_0 \in X$, $x'_0 \in X'$ respectively. If $\varphi_i(x_0) = \varphi_i(x'_0)$ for all $i \in I$, there exists a unique I-colored directed graph isomorphism $X \xrightarrow{\sim} X'$.

Proof. Uniqueness is obvious because x_0 exists. To prove existence, by induction on k, we will construct a bijection $h_k : X_k \xrightarrow{\sim} X'_k$ such that

(1_k) $\bigsqcup_{\ell=0}^{k} h_\ell : \bigsqcup_{\ell=0}^{k} X_\ell \xrightarrow{\sim} \bigsqcup_{\ell=0}^{k} X'_\ell$ is an I-colored directed graph isomorphism,

(2_k) $\varphi_i(x) = \varphi_i(h_k(x)), \varepsilon_i(x) = \varepsilon_i(h_k(x))$ for all $x \in X_k$ and $i \in I$.

Here, $\mathcal{X}_k = \{x \in \mathcal{X} \mid \mathrm{depth}(x) = k\}$ for $\mathcal{X} = X, X'$.

For $k = 0$, the only choice is $h_0(x_0) = x'_0$. For $k \geq 1$, we define $h_k(x) = \tilde{f}_i h_{k-1}(\tilde{e}_i x)$ if $\tilde{e}_i x \neq \mathbf{0}$. It is well-defined by (X),(Y),(Z) below.

(X) for any $x \in X_k$ there exists $i \in I$ such that $\tilde{e}_i x \in X_{k-1}$ by Proposition 3.1.

(Y) $\tilde{f}_i h_{k-1}(\tilde{e}_i x) \neq \mathbf{0}$ because $\varphi_i(h_{k-1}(\tilde{e}_i x)) = \varphi_i(\tilde{e}_i x) > 0$ by (2_{k-1}).

(Z) For $i \neq j \in I$ with $\tilde{e}_i x \neq \mathbf{0} \neq \tilde{e}_j x$, we show $\tilde{f}_i h_{k-1}(\tilde{e}_i x) = \tilde{f}_j h_{k-1}(\tilde{e}_j x)$ as follows:

When $A|_{i,j} = A_1 \oplus A_1, A_2$, (Z) is in the proof of [9, Proposition 1.4] (or similar to the arguments below). So let us $A|_{i,j} = B_2$. By (S2),(S3), possibilities of $\Delta(x) = (\Delta^e_\varepsilon(i,j,x), \Delta^e_\varepsilon(j,i,x))$ are $\Delta(x) = (0,0), (1,0), (0,1), (1,1), (0,2), (1,2)$. Among them, cases $\Delta(x) = (0,0), (1,0), (0,1), (1,1), (0,2)$, (Z) is again the same as in the proof of [9, Proposition 1.4] (or similar to the arguments below). Thus, we assume $\Delta(x) = (1,2)$. By (D^-) in (S6), $\exists y = \tilde{e}_i^2 \tilde{e}_j x \in X_{k-3}, \exists y' = \tilde{e}_i^2 \tilde{e}_j^2 \tilde{e}_i x \in X_{k-5}$. Again (S2),(S3) imply $\Delta'' = (\Delta^f_\varphi(i,j,y), \Delta^f_\varphi(i,j,y')) = (0,0), (1,0), (0,1), (1,1)$. Assume $\Delta'' = (0,1)$. By (Q_1^-) in (D^-) in (S6), we have $\exists z = \tilde{e}_j \tilde{e}_i^3 \tilde{e}_j^2 \tilde{e}_i x = \tilde{e}_i \tilde{e}_j^2 \tilde{e}_i^3 \tilde{e}_j x \in X_{k-7}, \Delta'(z) = (1,2)$ and as in Remark 1.11 $(\Delta^e_\varepsilon(i,j,\tilde{f}_i^2 \tilde{f}_j z), \Delta^e_\varepsilon(i,j,\tilde{f}_i^2 \tilde{f}_j^2 \tilde{f}_i z)) = (0,1)$. Then, by induction hypothesis and (S7), we have $\tilde{f}_j \tilde{f}_i^3 \tilde{f}_j^2 \tilde{f}_i h_{k-7}(z) = \tilde{f}_i \tilde{f}_j^2 \tilde{f}_i^3 \tilde{f}_j h_{k-7}(z)$. Since $h_{k-1}(\tilde{e}_i x) = \tilde{f}_j^2 \tilde{f}_i^3 \tilde{f}_j h_{k-7}(z)$ and $h_{k-1}(\tilde{e}_j x) = \tilde{f}_i^3 \tilde{f}_j^2 \tilde{f}_i h_{k-7}(z)$, we are done. The case $\Delta'' = (0,0)$ (resp. $\Delta'' = (1,1)$) is similar using (R^-) (resp.(P_1^-)) in (D^-) in (S6) and (S9) (resp.(S8)). Because $\Delta'' \neq (1,0)$ by (D^-) in (S6), (Z) is proved.

Finally, we show (1_k) and (2_k). h_k is epi by X' version of (X). By symmetry h_k is bijective. For (2_k), by (1_k) we have $\forall x \in X_k, \forall i \in I, \varepsilon_i(x) = \varepsilon_i(h_k(x))$. Then, $\forall x \in X_k, \forall i \in I, \varphi_i(x) = \varphi_i(h_k(x))$ follows from Remark 3.2 because h_k preserves wt_0. □

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