Braid group action on the module category of quantum affine algebras

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Abstract: Let $\mathfrak{g}_0$ be a simple Lie algebra of type ADE and let $U_q'(\mathfrak{g})$ be the corresponding untwisted quantum affine algebra. We show that there exists an action of the braid group $B(\mathfrak{g}_0)$ on the quantum Grothendieck ring $\mathcal{K}_t(\mathfrak{g})$ of Hernandez-Leclerc's category $\mathscr{C}_{\mathfrak{g}}^0$. Focused on the case of type A_{N-1}, we construct a family of monoidal autofunctors $\{\mathscr{S}_i\}_{i\in\mathbf{Z}}$ on a localization $\mathcal{T}_N$ of the category of finite-dimensional graded modules over the quiver Hecke algebra of type A_∞. Under an isomorphism between the Grothendieck ring $K(\mathcal{T}_N)$ of $\mathcal{T}_N$ and the quantum Grothendieck ring $\mathcal{K}_t(A_{N-1}^{(1)})$, the functors $\{\mathscr{S}_i\}_{1\le i\le N-1}$ recover the action of the braid group $B(A_{N-1})$. We investigate further properties of these functors.

Key words: Quantum affine algebra; quantum Grothendieck ring; braid group action; quiver Hecke algebra; R-matrix.

1. Introduction. The monoidal category $\mathscr{C}_{\mathfrak{g}}$ of finite-dimensional representations of a quantum affine algebra $U_q'(\mathfrak{g})$ has been extensively investigated because of its rich structure. Among various approaches, Nakajima [14], Varagnolo-Vasserot [16], and Hernandez [3] studied t-deformations of the Grothendieck ring of $\mathscr{C}_{\mathfrak{g}}$. These t-deformations are interesting, because they provide a way to calculate the q-character of simple representations. There is a full subcategory $\mathscr{C}_{\mathfrak{g}}^0$ of $\mathscr{C}_{\mathfrak{g}}$, introduced by Hernandez and Leclerc in [4], which contains an essential information on $\mathscr{C}_{\mathfrak{g}}$ but has a smaller set of the classes of simple modules. The Grothendieck ring of $\mathscr{C}_{\mathfrak{g}}^0$ is isomorphic to the polynomial ring in countably many variables, while that of $\mathscr{C}_{\mathfrak{g}}$ is the one in uncountably many variables. For the cases where $\mathfrak{g}$ is one of untwisted ADE types, a t-deformation $\mathcal{K}_t(\mathfrak{g})$ of the Grothendieck ring of $\mathscr{C}_{\mathfrak{g}}^0$, called the *quantum Grothendieck ring*, was investigated from a ring theoretic point of view in [5]. It turns out that the $\mathbf{C}(t^{1/2})$-algebra $\mathcal{K}_t(\mathfrak{g})$ has an interesting presentation: there is a set of generators consisting of a countable infinite number of copies of Drinfeld-Jimbo generators of a half of the quantum group $U_t(\mathfrak{g}_0)$, and they satisfy the quantum Serre relations in a copy, t-boson relations between adjacent copies, and t-commutation relations between non-adjacent copies. This presentation reflects the following feature of the category $\mathscr{C}_{\mathfrak{g}}^0$: for each choice of a Dynkin quiver Q with an additional data, they defined a monoidal subcategory $\mathscr{C}_Q$ of $\mathscr{C}_{\mathfrak{g}}^0$ such that the quantum Grothendieck ring of $\mathscr{C}_Q$ is isomorphic to the half $U_t^-(\mathfrak{g}_0)$ of the quantum group $U_t(\mathfrak{g}_0)$, and all the fundamental representations in $\mathscr{C}_{\mathfrak{g}}^0$ can be obtained from those in $\mathscr{C}_Q$ by taking functors $\mathscr{D}^m$ ($m \in \mathbf{Z}$). Here $\mathscr{D}$ is the contravariant functor taking the right dual.

One of main results of this paper is that there exists an action of the braid group $B(\mathfrak{g}_0)$ of type $\mathfrak{g}_0$ on the quantum Grothendieck ring $\mathcal{K}_t(\mathfrak{g})$ (Theorem 2.3). Since we give the action explicitly, the braid relations can be checked by the presentation of $\mathcal{K}_t(\mathfrak{g})$. Recall that the blocks of the category $\mathscr{C}_{\mathfrak{g}}^0$ is parameterized by the root lattice of $\mathfrak{g}_0$ and the tensor product is compatible with the addition on the root lattice [10]. It turns out that the action of the generators σ_i of $B(\mathfrak{g}_0)$ on $\mathcal{K}_t(\mathfrak{g})$ correspond to the reflections with respect to the simple roots α_i on the

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root lattice. Indeed, the action of σ_i is related with Saito's reflection functor as seen in Theorem 2.4.

We conjecture that the braid group action can be lifted to the action on the monoidal category $\mathscr{C}^0_{\mathfrak{g}}$. We show that it is the case when $\mathfrak{g}$ is of type $A^{(1)}_{N-1}$. A key point of view is the use of a rigid monoidal category $\mathcal{T}_N$ which is constructed out of the category $\mathcal{A}$ of finite-dimensional graded modules over the quiver Hecke algebra R^{A_∞} of type A_∞ [7]. It is a certain localization of $\mathcal{A}$ and there is a monoidal functor $\mathcal{F}_N$ from $\mathcal{T}_N$ to $\mathscr{C}^0_{A^{(1)}_{N-1}}$ which sends simple objects to simple objects. Moreover this functor induces an isomorphism between the Grothendieck ring $K(\mathcal{T}_N)$ and the quantum Grothendieck ring $\mathcal{K}_t(A^{(1)}_{N-1})$. It is summarized by the diagram

$$K(\mathcal{T}_N) \xrightarrow{\sim} \mathcal{K}_t(A^{(1)}_{N-1}) \xrightarrow{t=1} K(\mathscr{C}^0_{A^{(1)}_{N-1}}).$$

Hence the category $\mathcal{T}_N$ can be understood as a *graded lift* of $\mathscr{C}^0_{A^{(1)}_{N-1}}$ as a rigid monoidal category.

We show that there is a family of monoidal autofunctors $\{\mathscr{S}_i\}_{1\le i\le N-1}$ on the category $\mathcal{T}_N$ which recover the action of the braid group $B(A_{N-1})$ under the isomorphism between $K(\mathcal{T}_N)$ and $\mathcal{K}_t(A^{(1)}_{N-1})$ (Theorem 3.1, Theorem 3.3). There is a general procedure, developed in [11], to construct monoidal functors between the categories of modules over quiver Hecke algebras, and a similar procedure can be applied for the category $\mathcal{T}_N$. This is a main advantage in working on the category $\mathcal{T}_N$ rather than the category $\mathscr{C}^0_{A^{(1)}_{N-1}}$.

Finally we provide several consequences of the existence of such functors $\mathscr{S}_i$. For a simple object L which belongs to an orbit of $L(i)$ for some i under the action $B(A_{N-1})$, one can define an automorphism s_L which has similar properties with the automorphisms s_i (Theorem 4.2). Moreover $s_{L(i)}$ coincides with s_i.

This paper is an announcement whose details will appear elsewhere.

Convention 1.1. Throughout this paper, we keep the following conventions.

(a) For a statement P, $\delta(\mathsf{P})$ is 1 or 0 according that P is true or not.

(b) For $k, \ell \in \mathbf{Z}$ and $s \in \mathbf{Z}_{\ge 1}$, we write $k \equiv_s \ell$ if s divides $k-\ell$ and $k \not\equiv_s \ell$, otherwise.

2. Braid group action on the quantum Grothendieck rings. Let $\mathfrak{g}_0$ be a finite-dimensional simple Lie algebra of simply-laced type with a Cartan matrix $\mathbf{A} = (a_{ij})_{i,j\in I_0}$, $\mathfrak{g}$ the untwisted affine Kac-Moody algebra associated with $\mathfrak{g}_0$, and $U'_q(\mathfrak{g})$ the quantum affine algebra associated with $\mathfrak{g}$. We take the algebraic closure of $\mathbf{C}(q)$ inside $\bigcup_{m>0}\mathbf{C}((q^{1/m}))$ as the base field $\mathbf{k}$ for $U'_q(\mathfrak{g})$. Let $\mathscr{C}_{\mathfrak{g}}$ be the category of finite-dimensional integrable modules over $U'_q(\mathfrak{g})$. There is a family $\{V(\varpi_i)_c \mid i \in I_0, c \in \mathbf{k}^\times\}$ in $\mathscr{C}_{\mathfrak{g}}$, called the *fundamental representations*.

Following [4], we denote by $\mathscr{C}^0_{\mathfrak{g}}$ the smallest full subcategory of the category $\mathscr{C}_{\mathfrak{g}}$ which is stable under taking subquotients, extensions, tensor products and contains

$$\{V(\varpi_i)_{(-q)^p} \mid i \in I_0,\ p \equiv d(1,i) \mod 2\},$$

where $d(i,j)$ is the distance between the vertices i and j in the Dynkin diagram of $\mathfrak{g}_0$. Here $1 \in I_0$ is an arbitrarily chosen element. Then the complexified Grothendieck ring $\mathbf{C}\otimes_{\mathbf{Z}} K(\mathscr{C}^0_{\mathfrak{g}})$ of $\mathscr{C}^0_{\mathfrak{g}}$ has a t-deformation $\mathcal{K}_t(\mathfrak{g})$, called the *quantum Grothendieck ring* of $\mathscr{C}^0_{\mathfrak{g}}$. To each simple module S in $\mathscr{C}^0_{\mathfrak{g}}$, we can associate an element $[S]_t$ of $\mathcal{K}_t(\mathfrak{g})$ and we have $\mathcal{K}_t(\mathfrak{g}) = \bigoplus_S \mathbf{C}(t^{1/2})[S]_t$. Here S ranges over the set of the isomorphism classes of simple modules in $\mathscr{C}^0_{\mathfrak{g}}$.

Let Q be a Dynkin quiver with type $\mathfrak{g}_0$, and let ϕ_Q be a height function, i.e., it associates an integer $\phi_Q(i)$ to each vertex i of Q such that $\phi_Q(i) = \phi_Q(j)+1$ if $i \to j$. We assume further that $\phi_Q(1) \in 2\mathbf{Z}$. A pair $Q = (Q, \phi_Q)$ is called a Q-data.

For a sink i of Q, let $s_iQ := (s_iQ, \phi_{s_iQ})$ be the Q-data consisting of the Dynkin quiver s_iQ obtained from Q by reversing the arrows of Q adjacent to i and the height function ϕ_{s_iQ} of s_iQ given by $\phi_{s_iQ}(j) = \phi_Q(j) + 2\delta_{i,j}$.

To a Q-data Q, Hernandez-Leclerc [5] associated a full monoidal subcategory $\mathscr{C}_Q$ of $\mathscr{C}^0_{\mathfrak{g}}$, and a monoidal functor $\mathcal{F}_Q\colon R_{\mathfrak{g}_0}\text{-mod} \to \mathscr{C}_Q$ is constructed in [2,6], and Fujita [1,2] proved that $\mathcal{F}_Q$ is an equivalence of categories. Here, $R_{\mathfrak{g}_0}$-mod is the monoidal category of finite-dimensional modules (with nilpotent actions of the generators x_k) over the quiver Hecke algebra $R_{\mathfrak{g}_0}$ associated with $\mathfrak{g}_0$. Note that $\mathcal{F}_Q(L(i))$ is a fundamental module for any $i \in I_0$, where $L(i) \in R_{\mathfrak{g}_0}$-mod is the simple module associated with i.

Then, for a Q-data Q, we have an embedding of $\mathbf{Z}[t^{\pm1}]$-algebras

$$j_Q\colon K(R_{\mathfrak{g}_0}\text{-gmod}) \hookrightarrow \mathcal{K}_t(\mathfrak{g})$$

induced by $\mathcal{F}_Q$.

Let $\mathbf{K}_t(\mathfrak{g}_0)$ be the $\mathbf{C}(t^{1/2})$-algebra generated by

$\{y_{i,m} \mid i \in I_0, m \in \mathbf{Z}\}$ with the defining relations: For $m \in \mathbf{Z}$ and $i, j \in I_0$,

(a) $y_{i,m}y_{j,m} = y_{j,m}y_{i,m}$ if $a_{ij} = 0$,
$y_{i,m}^2 y_{j,m} - (t + t^{-1})y_{i,m}y_{j,m}y_{i,m} + y_{j,m}y_{i,m}^2 = 0$ if $a_{ij} = -1$,

(b) $y_{i,m}y_{j,m+1} = t^{a_{ij}}y_{j,m+1}y_{i,m} + \delta_{ij}(1 - t^2)$,

(c) $y_{i,m}y_{j,p} = t^{(-1)^{p-m+1}a_{ij}}y_{j,p}y_{i,m}$, for $p > m + 1$.

Remark 2.1. We change t into t^{-1} in the presentation in [5].

Theorem 2.2 ([5], Theorem 7.3)**.** *Let Q be a Q-data. Then there is an isomorphism $\iota_Q\colon \mathbf{K}_t(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{K}_t(\mathfrak{g})$ such that $\iota_Q(y_{i,m})$ is equal to $[\mathscr{D}^m \mathcal{F}_Q(L(i))]_t$, where $L(i)$ is the simple module in $R_{\mathfrak{g}_0}$-mod corresponding to $i \in I_0$.*

Let $B(\mathfrak{g}_0)$ be the Braid group associated with $\mathfrak{g}_0$. It is generated by $\{\sigma_i \mid i \in I_0\}$ with the defining relations

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{if } a_{ij} = -1,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } a_{ij} = 0.$$

One of our main theorems is the following

Theorem 2.3. *The Braid group $B(\mathfrak{g}_0)$ acts on $\mathbf{K}_t(\mathfrak{g}_0)$ by the following formulas:*

$$\sigma_i(y_{j,m}) = \begin{cases} y_{j,m+\delta_{ij}} & \text{if } a_{ij} \neq -1, \\ \dfrac{t^{1/2}y_{j,m}y_{i,m} - t^{-1/2}y_{i,m}y_{j,m}}{t - t^{-1}} & \text{if } a_{ij} = -1, \end{cases}$$

$$\sigma_i^{-1}(y_{j,m}) = \begin{cases} y_{j,m-\delta_{ij}} & \text{if } a_{ij} \neq -1, \\ \dfrac{t^{1/2}y_{i,m}y_{j,m} - t^{-1/2}y_{j,m}y_{i,m}}{t - t^{-1}} & \text{if } a_{ij} = -1. \end{cases}$$

Theorem 2.4. *Let i be a sink of a Q-data Q. Then the following diagrams commute:*

$$\begin{array}{ccc} K(R_{\mathfrak{g}_0}\text{-gmod}) & \xrightarrow{\ j_Q\ } & \mathcal{K}_t(\mathfrak{g}) \underset{\iota_Q}{\simeq} \mathbf{K}_t(\mathfrak{g}_0) \\ & \searrow^{j_{s_iQ}} & \downarrow \sigma_i \\ & & \mathcal{K}_t(\mathfrak{g}) \underset{\iota_Q}{\simeq} \mathbf{K}_t(\mathfrak{g}_0), \end{array}$$

$$\begin{array}{ccccc} K({}_iR_{\mathfrak{g}_0}\text{-gmod}) & \hookrightarrow & K(R_{\mathfrak{g}_0}\text{-gmod}) & \searrow^{j_Q} & \\ \downarrow \mathbf{T}_i & & & & \mathcal{K}_t(\mathfrak{g})\ . \\ K({}^iR_{\mathfrak{g}_0}\text{-gmod}) & \hookrightarrow & K(R_{\mathfrak{g}_0}\text{-gmod}) & \nearrow_{j_{s_iQ}} & \end{array}$$

Here, ${}_iR_{\mathfrak{g}_0}$-gmod (resp. ${}^iR_{\mathfrak{g}_0}$-gmod) is the full subcategory of $R_{\mathfrak{g}_0}$-gmod consisting of graded modules M with $E_i^*M = 0$ (resp. $E_iM = 0$), and $\mathbf{T}_i$ is the reflection functor due to S. Kato [12,13] (cf. Y. Saito [15]). For E_i and E_i^*, see for example, [8].

3. The category $\mathcal{T}_N$ and reflection functors. Let J be the index set of simple roots of the root system A_∞. One can identify J with $\mathbf{Z}$ and the root lattice Q is the subspace of $\bigoplus_{a\in\mathbf{Z}} \mathbf{Z}\varepsilon_a$ generated by $\alpha_a = \varepsilon_a - \varepsilon_{a+1}$ for $a \in \mathbf{Z}$. Let R^{A_∞} be the symmetric quiver Hecke algebra of type A_∞ over $\mathbf{k}$ with the choice of parameters

$$Q_{ij}(u, v) = \delta(i \neq j)(u - v)^{\delta(j=i+1)}(v - u)^{\delta(j=i-1)}$$

for $i, j \in J$. It is a family $\{R^{A_\infty}(\beta)\}_{\beta\in\mathsf{Q}^+}$ of associative $\mathbf{Z}$-graded $\mathbf{k}$-algebras, where $\mathsf{Q}^+ = \sum_{i\in J}\mathbf{Z}_{\geq 0}\alpha_i$ is the positive root lattice of type A_∞. Each $R^{A_\infty}(\beta)$ is generated by $\{e(\nu)\}_{\nu\in J^\beta}$, $\{x_k\}_{1\leq k\leq n}$ and $\{\tau_m\}_{1\leq m\leq n-1}$, where $n = |\beta| := \sum_{i\in I} n_i$ with $\beta = \sum_{i\in J} n_i\alpha_i$, and $J^\beta := \{\nu \in J^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}$. See [7] for a set of defining relations of $R^{A_\infty}(\beta)$. Note that there is an embedding of $R^{A_\infty}(\beta) \otimes R^{A_\infty}(\gamma)$ into $R^{A_\infty}(\beta + \gamma)$. Hence the category $\mathcal{A} = \bigoplus_{\beta\in\mathsf{Q}^+} R^{A_\infty}(\beta)$-gmod of finite-dimensional graded R^{A_∞}-modules is a monoidal category whose tensor product is given by the *convolution product*:

$$M \circ N := R^{A_\infty}(\beta + \gamma) \underset{R^{A_\infty}(\beta)\otimes R^{A_\infty}(\gamma)}{\otimes} (M \otimes N).$$

For $M \in R^{A_\infty}(\beta)$-gmod, we set $\mathrm{wt}(M) := -\beta$.

For each pair of integers a, b with $a \leq b$, let $[a, b]$ be the interval $\{k \in \mathbf{Z} \mid a \leq k \leq b\}$, and call it a *segment*. For each segment $[a, b]$, let $L(a, b)$ be the one-dimensional simple graded R^{A_∞}-module generated by a vector $u(a, b)$ such that $e(\nu)u(a, b) = \delta(\nu = (a, \dots, b))u(a, b)$. We set $L(a) := L(a, a)$ for $a \in \mathbf{Z}$. For each $N \geq 2$, let $\mathcal{S}_N$ be the smallest subcategory of $\mathcal{A}$ which is stable under taking convolution, subquotients, extensions, and containing $\{L(a, b) \mid b - a + 1 > N\}$. Then the quotient category $\mathcal{A}/\mathcal{S}_N$ equips with a new tensor product $\star$ given by

$$X \star Y := t^{B(\mathrm{wt}(X),\mathrm{wt}(Y))} X \circ Y,$$

where $B(x, y) := -\sum_{k>0}(S^k x, y)$ for $x, y \in \bigoplus_{a\in\mathbf{Z}} \mathbf{Z}\varepsilon_a$ and S is an automorphism on $\bigoplus_{a\in\mathbf{Z}} \mathbf{Z}\varepsilon_a$ given by $S(\varepsilon_a) := \varepsilon_{a+N}$. The category $\mathcal{T}_N$ is constructed in [7] as a localization of the monoidal category $(\mathcal{A}/\mathcal{S}_N, \star)$. The objects of $\mathcal{T}_N$ is the same with the ones of $\mathcal{A}/\mathcal{S}_N$. The group of morphisms is given by

$$\mathrm{Hom}_{\mathcal{T}_N}(X,Y) := \varinjlim_{\lambda,\mu} \mathrm{Hom}_{\mathcal{A}/\mathcal{S}_N}(X \circ P^{\lambda}, Y \circ P^{\mu}),$$

where $P^{\nu} := \circ_{a\in\mathbf{Z}} L(a, a+N-1)^{\circ\nu_a}$ for $\nu \in (\mathbf{Z}_{\geq 0})^{\oplus J}$ and the limit runs over all the pairs (λ,μ) such that $\mathrm{wt}(X\circ P^{\lambda}) = \mathrm{wt}(Y \circ P^{\mu})$. It turns out that $\mathcal{T}_N$ is an abelian rigid monoidal category with a tensor product $\star$. We denote the right dual (resp. left dual) of X by $\mathscr{D}(X)$ (resp. $\mathscr{D}^{-1}(X)$). Note that $L(a, a+N-1) \simeq \mathbf{1}$ in $\mathcal{T}_N$ for all $a \in \mathbf{Z}$. We have a chain of functors

$$\mathcal{A} \xrightarrow{\mathcal{Q}_N} \mathcal{A}/\mathcal{S}_N \xrightarrow{\Upsilon_N} \mathcal{T}_N.$$

The composition will be denoted by Ω_N. Note that the Grothendieck ring $K(\mathcal{T}_N)$ is a $\mathbf{Z}[t^{\pm 1}]$-algebra on which t acts by the grading shift.

From now on, let $\mathfrak{g}$ be the affine Kac-Moody algebra of type $A^{(1)}_{N-1}$. We regard $\mathcal{T}_N$ as a $\mathbf{Z}$-graded lifting of $\mathscr{C}^0_{\mathfrak{g}}$ as a rigid monoidal category. Indeed there exists a monoidal functor $\mathcal{F}_N : \mathcal{T}_N \to \mathscr{C}^0_{\mathfrak{g}}$ which sends simples to simples. It induces an isomorphism of $\mathbf{C}(t^{1/2})$-algebras $[\mathcal{F}_N] : \mathbf{C}(t^{1/2}) \otimes_{\mathbf{Z}[t^{\pm 1}]} K(\mathcal{T}_N) \xrightarrow{\sim} \mathcal{K}_t(\mathfrak{g})$ ([7], Theorem 4.33). Under the isomorphism, the generator $y_{i,m}$ corresponds to $[\mathscr{D}^m L(i)]$ for $i \in \mathbf{Z}$, $m \in \mathbf{Z}$.

For a pair (M,N) of objects in a $\mathbf{k}$-linear abelian monoidal category in which every object has a finite length, we denote by $M\nabla N$ the head of $M\otimes N$ and by $M\Delta N$ the socle of $M\otimes N$, respectively.

We show that there is a family of autofunctors on $\mathcal{T}_N$ which recover the braid group action on the quantum Grothendieck ring $\mathcal{K}_t(\mathfrak{g})$. For this purpose, we adjoin a formal object $t^{1/2}\mathbf{1}$ into $\mathcal{T}_N$ such that $t^{1/2}\mathbf{1} \star t^{1/2}\mathbf{1} \simeq t\mathbf{1}$. Then the *grading shift by* $1/2$ *of* X is given by $X \to t^{1/2}\mathbf{1} \star X$.

Theorem 3.1. *For* $i \in \mathbf{Z}$*, there exists a monoidal functor*

$$\mathscr{S}_i : \mathcal{T}_N \to \mathcal{T}_N$$

satisfying

$$\mathscr{S}_i(L(j)) \simeq \begin{cases} \mathscr{D}L(j) & \text{if } j \equiv_N i, \\ t^{1/2}(L(j\mp 1)\nabla L(j)) & \text{if } j \equiv_N i \pm 1, \\ L(j) & \text{otherwise.} \end{cases}$$

The functor $\mathscr{S}_i$ *has an inverse*

$$\mathscr{S}_i^{-1} : \mathcal{T}_N \to \mathcal{T}_N$$

satisfying

$$\mathscr{S}_i^{-1}(L(j)) \simeq \begin{cases} \mathscr{D}^{-1}L(j) & \text{if } j \equiv_N i, \\ t^{1/2}(L(j)\nabla L(j\mp 1)) & \text{if } j \equiv_N i \pm 1, \\ L(j) & \text{otherwise.} \end{cases}$$

Let us explain briefly how to construct the functors $\mathscr{S}_i$. For each $j \in J$, denote $\bar{M}_j$ the R^{A_∞}-module $t^{-1}L(j+1, j+N-1)$ if $j \equiv_N i$, $t^{1/2}(L(j\mp 1)\nabla L(j))$ if $j \equiv_N i \pm 1$ and $L(j)$ otherwise. For each $\beta \in \mathsf{Q}^+$ and $\mu = (\mu_1, \ldots, \mu_m) \in J^{\beta}$, set

$$\Delta(\mu) = M_{\mu_1} \circ \cdots \circ M_{\mu_m}, \text{ and } \Delta(\beta) = \bigoplus_{\mu \in J^{\beta}} \Delta(\mu),$$

where M_j is the affinization of $\bar{M}_j$. Then along a similar line with ([11], Section 4), one can show that there exists a ring homomorphism

$$(R^{A_\infty}(\beta))^{\mathrm{opp}} \to \mathrm{End}_{\mathcal{A}^{\mathrm{big}}/\mathcal{S}_N^{\mathrm{big}}}(\mathcal{Q}_N(\Delta(\beta))),$$

where $\mathcal{A}^{\mathrm{big}}/\mathcal{S}_N^{\mathrm{big}}$ is a quotient category of the category of graded R^{A_∞}-modules which is defined in a similar way with $\mathcal{A}/\mathcal{S}_N$ (see [7], Section 4.4). Let $\mathcal{R}'_{\beta} : R^{A_\infty}(\beta)\text{-gmod} \to \mathcal{A}/\mathcal{S}_N$ be the restriction of a left adjoint of the functor $\mathrm{Hom}_{\mathcal{A}^{\mathrm{big}}/\mathcal{S}_N^{\mathrm{big}}}(\mathcal{Q}_N(\Delta(\beta)), -)$. Then we obtain a monoidal functor $\mathcal{R} : \mathcal{A} \to \mathcal{T}_N$, the composition

$$\mathcal{A} \xrightarrow{\bigoplus_{\beta\in\mathsf{Q}^+} \mathcal{R}'_{\beta}} \mathcal{A}/\mathcal{S}_N \xrightarrow{\Upsilon_N} \mathcal{T}_N.$$

Note that the family $\{\bar{M}_j\}_{j\in J}$ of objects in $\mathcal{T}_N$ satisfies for any $a \in J$ that (1) $\bar{M}_a \star \bar{M}_{a+1} \star \cdots \star \bar{M}_{a+N-1} \simeq \mathbf{1}$, (2) $\mathrm{hd}(\bar{M}_a \star \bar{M}_{a+1} \star \cdots \star \bar{M}_{a+k-1}) \star \bar{M}_{a+k}$ is not simple for $1 \leq k \leq N-1$, and (3) $\mathscr{D}^2(\bar{M}_a) \simeq \bar{M}_{a+N}$. A similar argument as the one in ([9], Section 6.1) shows that there is a monoidal functor $\mathscr{S}_i : \mathcal{T}_N \to \mathcal{T}_N$ such that $\mathcal{R} \simeq \mathscr{S}_i \circ \Omega_N$.

Recall that there is an automorphism $\mathrm{T} : \mathcal{T}_N \to \mathcal{T}_N$ given by $L(j) \mapsto L(j+1)$ for all $j \in \mathbf{Z}$. It satisfies that $\mathrm{T}^N \simeq \mathscr{D}^2$. The functors $\{\mathscr{S}_i \mid i \in \mathbf{Z}\}$ satisfy the following properties.

Proposition 3.2. *We have*
(i) $\mathscr{S}_{i+1} \simeq \mathrm{T} \circ \mathscr{S}_i \circ \mathrm{T}^{-1}$ *for* $i \in \mathbf{Z}$,
(ii) $\mathscr{S}_i \circ \mathscr{D} \simeq \mathscr{D} \circ \mathscr{S}_i$ *for* $i \in \mathbf{Z}$,
(iii) $\mathscr{S}_i \simeq \mathscr{S}_{N+i}$ *for* $i \in \mathbf{Z}$,
(iv) $\mathscr{S}_1 \mathscr{S}_2 \cdots \mathscr{S}_{N-1} \simeq \mathrm{T}$,
(v) $\mathscr{S}_i \circ \mathscr{S}_j \simeq \mathscr{S}_j \circ \mathscr{S}_i$ *for* $|i-j| > 2$,
(vi) $\mathscr{S}_i \circ \mathscr{S}_{i+1} \circ \mathscr{S}_i \simeq \mathscr{S}_{i+1} \circ \mathscr{S}_i \circ \mathscr{S}_{i+1}$ *for* $i \in \mathbf{Z}$.

The family of functors $\{\mathscr{S}_i\}_{1\leq i \leq N-1}$ recovers the braid group action in Theorem 2.3 in the case of type A_{N-1}.

Theorem 3.3. *For each* $1 \leq i \leq N-1$ *the* $\mathbf{Z}[t^{\pm 1/2}]$*-algebra automorphism on* $K(\mathcal{T}_N)$ *induced* *by $\mathscr{S}_i$ is equal to σ_i in Theorem 2.3 under the isomorphism $[\mathcal{F}_N]\colon \mathbf{C}(t^{1/2}) \otimes_{\mathbf{Z}[t^{\pm 1}]} K(\mathcal{T}_N) \xrightarrow{\sim} \mathcal{K}_t(\mathfrak{g})$.*

4. Reflections by root modules. Recall that for each pair of non-zero modules (X, Y) of $\mathcal{A}$, there exists a distinguished nonzero morphism $\mathbf{r}_{X,Y}\colon t^{\Lambda(X,Y)} X \circ Y \to Y \circ X$ called the *r-matrix* [7]. Here, t is the grading shift functor. We have $\Omega_N(\mathbf{r}_{X,Y})\colon t^{\Lambda_N(X,Y)} X \star Y \to Y \star X$ in $\mathcal{T}_N$, where $\Lambda_N(X, Y) = \Lambda(X, Y) - B(\mathrm{wt}(X), \mathrm{wt}(Y)) + B(\mathrm{wt}(Y), \mathrm{wt}(X))$.

For a pair (X, Y) of objects in $\mathcal{T}_N$, set

$$\mathfrak{d}(X, Y) := \frac{1}{2}\left(\Lambda_N(X, Y) + \Lambda_N(Y, X)\right).$$

Note that $\mathfrak{d}(X, Y) = \dfrac{1}{2}(\Lambda(X, Y) + \Lambda(Y, X))$ if $\Omega(\mathbf{r}_{X,Y}) \neq 0$.

A simple object X in an abelian monoidal category is called *real* if $X \otimes X$ is simple. A real simple object L in $\mathcal{T}_N$ is called a *root module* if

$$\mathfrak{d}(L, \mathscr{D}^k(L)) = \delta(k = \pm 1).$$

For example, the objects $L(a, b)$ with $b - a + 1 < N$ are root modules. If L is a root module, then $\mathscr{D}(L)$, $\mathscr{D}^{-1}(L)$ and $\mathscr{S}_i(L)$ for $i \in \mathbf{Z}$ are root modules.

The following is the main theorem of this section.

Theorem 4.1. *Let X be a simple object in $\mathcal{T}_N$. For $i \in \mathbf{Z}$, if*

$$\mathfrak{d}(\mathscr{D}^k(L(i)), X) = n\delta(k = a)$$

for some $n \geq 0$ and $a \in \mathbf{Z}$, then

$$\mathscr{S}_i(X) \simeq (\mathscr{D}^a L(i))^{\circ n} \nabla X$$

up to a multiple of a power of t.

The following is one of the applications of Theorem 4.1.

Theorem 4.2. *Let $[L]$ belongs to the orbit of $L(i)$ for some $1 \leq i \leq N-1$ under the braid group $B(A_{N-1})$ action in Theorem 2.3. Then there is an automorphism s_L on $K(\mathcal{T}_N)$ such that*

(a) *$s_{L(i)} = s_i$ for $1 \leq i \leq N-1$.*
(b) *$s_{(s_L(L'))} = s_L \circ s_{L'} \circ s_L^{-1}$ if L' also satisfies the condition in the theorem.*
(c) *$s_{\mathscr{D}^a L} = s_L$ for all $a \in \mathbf{Z}$.*
(d) *$s_L([X]) = [(\mathscr{D}^a L)^{\circ n} \nabla X]$ up to a power of t, if $\mathfrak{d}(\mathscr{D}^k L, X) = n\delta(k = a)$ for some $n \geq 0$ and $a \in \mathbf{Z}$.*

5. Conjectures. Let $U_q'(\mathfrak{g})$ be an arbitrary quantum affine algebra. We say that a real simple module L in $\mathscr{C}_{\mathfrak{g}}^0$ is a *root module* if $\mathfrak{d}(\mathscr{D}^k M, M) = \delta(k = \pm 1)$ for any k.

Conjecture. For any root module $L \in \mathscr{C}_{\mathfrak{g}}^0$, there exists a monoidal autofunctor $\mathscr{S}_L$ of $\mathscr{C}_{\mathfrak{g}}^0$ which satisfies the following conditions:

(a) $\mathscr{S}_L$ satisfies similar properties in Theorem 4.1 and Theorem 4.2.
(b) (Braid relation) For root modules L and L',
  (1) if $\mathfrak{d}(\mathscr{D}^k L, L') = 0$ for any $k \in \mathbf{Z}$, then
  $$\mathscr{S}_L \circ \mathscr{S}_{L'} \simeq \mathscr{S}_{L'} \circ \mathscr{S}_L,$$
  (2) if $\mathfrak{d}(\mathscr{D}^k L, L') = \delta(k = 0)$ for any $k \in \mathbf{Z}$, then
  $$\mathscr{S}_L \circ \mathscr{S}_{L'} \circ \mathscr{S}_L \simeq \mathscr{S}_{L'} \circ \mathscr{S}_L \circ \mathscr{S}_{L'}.$$
(c) Let Q be a Q-data, and $L := \mathcal{F}_Q(L(i))$. Then the automorphism of $\mathcal{K}_t(\mathfrak{g})$ induced by $\mathscr{S}_L$ coincides with σ_i, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathbf{K}_t(\mathfrak{g}_0) & \xrightarrow[\iota_Q]{\sim} & \mathcal{K}_t(\mathfrak{g}) \\ {\scriptstyle \sigma_i}\downarrow & & \downarrow{\scriptstyle \mathscr{S}_L} \\ \mathbf{K}_t(\mathfrak{g}_0) & \xrightarrow[\iota_Q]{\sim} & \mathcal{K}_t(\mathfrak{g}). \end{array}$$

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