Some remarks on finite submodules of the unramified Iwasawa module of totally real fields

Dedicated to Professor Keiichi Komatsu on the Occasion of his 70th birthday

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Abstract: Let k be a number field and p a prime number. It is conjectured by Greenberg that the Iwasawa λ- and μ-invariants of the cyclotomic $\mathbf{Z}_p$-extension of k always vanish if k is totally real. In this article, we will discuss a weak version of Greenberg's conjecture, and give results analogous to Greenberg's and Ozaki's results.

Key words: Unramified Iwasawa modules; finite submodules.

1. Introduction. Let p be a prime number and $k/\mathbf{Q}$ a finite extension, where denote by $\mathbf{Q}$ the field of rational numbers. Let $\mathbf{Z}_p$ be the ring of p-adic integers. An algebraic extension K/k is called a $\mathbf{Z}_p$-extension if K/k is a Galois extension and $\mathrm{Gal}(K/k) \simeq \mathbf{Z}_p$ as topological groups. Let K/k be a $\mathbf{Z}_p$-extension. For each non-negative integer n, let k_n denote the unique intermediate field of K/k such that $[k_n : k] = p^n$. For a number field F, let A_F be the p-part of the ideal class group of F. Then, it is shown by Iwasawa that there exist non-negative integers λ, μ and an integer ν depending only on K/k such that the normalized additive p-adic valuation of the order of A_{k_n} is described as $\lambda n + \mu p^n + \nu$ for all sufficiently large n. It is conjectured by Greenberg [1] that λ and μ always vanish for the cyclotomic $\mathbf{Z}_p$-extension k_∞/k of each totally real field k (for short we say GC). In the same paper [1], Greenberg showed the following

Theorem (Greenberg [1]). *Let k be a totally real field such that k_∞ has the unique prime lying above p and that k_∞/k is totally ramified at the ramified prime. Then the following two conditions are equivalent.*

(a) $\lambda = \mu = 0$.

(b) *The lifting mapping $A_k \to A_{k_n}$ is trivial for some $n \geq 1$.*

For a $\mathbf{Z}_p$-extension K/k, let X_K be the Galois group of the maximal unramified abelian pro-p extension L_K/K, which is called the unramified Iwasawa module of K/k. The module X_K is also defined to be the projective limit $\varprojlim_n A_{k_n}$ with respect to the norm maps. Then the complete group ring $\mathbf{Z}_p[[\mathrm{Gal}(K/k)]]$ with coefficients in $\mathbf{Z}_p$ acts on X_K. Let X_K^0 be the maximal finite $\mathbf{Z}_p[[\mathrm{Gal}(K/k)]]$-submodule of X_K. Then $\lambda = \mu = 0$ if and only if $X_K = X_K^0$. From this point of view, we can consider a weak version of Greenberg's conjecture as follows: When k is a totally real field, if $X_{k_\infty} \neq 0$, then does it hold that $X_{k_\infty}^0 \neq 0$? (for short we say WGC). For a prime number p and a totally real field k, if GC holds for p and k then WGC holds for p and k. The author thinks that the study of WGC is important to study GC. Ozaki [3] obtained the following result concerning WGC. We shall introduce a slightly modified statement.

Theorem (Ozaki [3]). *Let p be an odd prime number and k a totally real field in which p splits completely. Suppose that Leopoldt's conjecture for p and k holds. Then the following two conditions are equivalent.*

(a) $X_{k_\infty}^0 \neq 0$.

(b) $L_{k_\infty} \neq M_{k_\infty}$.

Here, M_{k_∞}/k_∞ stands for the maximal abelian pro-p extension unramified outside all primes lying above p.

In this article, we will study WGC and give analogous results to Greenberg's and Ozaki's theorems. From here to the end of this article, we assume the following

Assumption (A). Let p be a prime number and k a totally real field such that k_∞ has the unique

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prime lying above p and that k_∞/k is totally ramified at the ramified prime.

The first result is a WGC analogue of Greenberg's theorem.

Theorem 1. *Under the assumption* (A), *the following two conditions are equivalent.*

(a) $X^0_{k_\infty} \neq 0$.

(b) *The lifting mapping* $A_k \to A_{k_n}$ *is not injective for some* $n \geq 1$.

The second result is analogous to Ozaki's result when p does not split in $k/\mathbf{Q}$.

Theorem 2. *Suppose that the assumption* (A) *is satisfied. Let* U^1 *be the local principal unit group of* k *with respect to the unique prime* $\mathfrak{p}$ *lying above* p, *and* E^1 *the completion the group of units of* k *congruent to* 1 *modulo* $\mathfrak{p}$ *in* U^1. *Suppose that Leopoldt's conjecture for* p *and* k *holds, and that* U^1/E^1 *is free over* $\mathbf{Z}_p$. *Then the following two conditions are equivalent.*

(a) $X^0_{k_\infty} \neq 0$.

(b) $L_{k_\infty} \neq M_{k_\infty}$.

We must mention here that, on Theorem 2, the oddness of p is not needed.

We shall set some notations of this article. For a $\mathbf{Z}_p$-module A, let $\mathrm{Tor}_{\mathbf{Z}_p}A$ be the submodule of A which consists of all $\mathbf{Z}_p$-torsion elements of A. For a topological group G and a topological G-module M, let M^G and M_G be the maximal submodule and the maximal quotient module of M on which G acts trivially.

2. Preliminaries. We need the following lemmas.

Lemma 3 (Ozaki [2]). *Let* K/k *be a* $\mathbf{Z}_p$*-extension which is totally ramified at all ramified primes. Then we have* $\mathrm{Ker}(A_{k_n} \to \varinjlim_m A_{k_m}) = \mathrm{Im}(X^0_K \to A_{k_n})$.

Lemma 4 (See Lemma 13.15 of [4]). *Let* k *be a number field and* K/k *a* $\mathbf{Z}_p$*-extension such that* K/k *is ramified at only one prime* $\mathfrak{p}$ *and is totally ramified at* $\mathfrak{p}$. *Then* $(X_K)_{\mathrm{Gal}(K/k)} \simeq A_k$.

3. Proof of Theorem 1. Let $\Gamma = \mathrm{Gal}(k_\infty/k)$, $X = X_{k_\infty}$ and $X^0 = X^0_{k_\infty}$ for simplicity. By lemma 4, we have $X_\Gamma \simeq A_k$, and hence $(X/X^0)_\Gamma$ is finite. It follows from the exact sequence

$$0 \to (X/X^0)^\Gamma \to X/X^0 \to X/X^0 \to (X/X^0)_\Gamma \to 0$$

that the characteristic ideal of $(X/X^0)^\Gamma$ is trivial. Thus, since X/X^0 has no non-trivial finite submodule, it holds that $(X/X^0)^\Gamma = 0$. Hence we have the following exact sequence and commutative diagram

$$\begin{array}{ccccccc} 0 \to & X^0_\Gamma & \to & X_\Gamma & \to (X/X^0)_\Gamma & \to 0 \\ & \uparrow & & \wr\! | & & \\ & X^0 & \to & A_k & & \end{array}.$$

By Nakayama's lemma, we find that $X^0 \neq 0$ if and only if $X^0_\Gamma \neq 0$. By Lemma 3 and the above exact sequence, $X^0 \neq 0$ if and only if $A_k \to A_{k_n}$ is not injective for some $n \geq 1$. □

Greenberg's theorem is also shown from the above arguments. In fact, we know that $\lambda = \mu = 0$ if and only if $X^0 = X$. By Nakayama's lemma, $X^0 = X$ if and only if $X^0_\Gamma = X_\Gamma \simeq A_k$. This assertion is equivalent to that $A_k \to A_{k_n}$ is trivial for some $n \geq 1$ by Lemma 3.

4. Proof of Theorem 2. For a number field F, let M_F/F be the maximal abelian pro-p extension unramified outside all primes lying above p and $\mathfrak{X}_F$ its Galois group. Since Leopoldt's conjecture holds for p and k, k_∞ is the unique $\mathbf{Z}_p$-extension of k. Thus, it holds that $\mathfrak{X}^\Gamma_{k_\infty} = 0$ and that $(\mathfrak{X}_{k_\infty})_\Gamma$ is finite. Then it also holds that $(\mathfrak{X}_{k_\infty})_\Gamma \simeq \mathrm{Tor}_{\mathbf{Z}_p}\mathfrak{X}_k$ since $\mathrm{Gal}(k_\infty/k)$ is $\mathbf{Z}_p$-free. By class field theory, we have an exact sequence $0 \to U^1/E^1 \to \mathfrak{X}_k \to A_k \to 0$. Similarly, there is a natural surjective map $\mathfrak{X}_{k_\infty} \to X$ of $\mathbf{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-modules. Since there exist surjective and isomorphic mappings

$$(\mathfrak{X}_{k_\infty})_\Gamma \twoheadrightarrow X_\Gamma \simeq A_k,$$

and since $(\mathfrak{X}_{k_\infty})_\Gamma = \mathrm{Tor}_{\mathbf{Z}_p}\mathfrak{X}_k$, we have the following exact sequence

$$0 \to \mathrm{Tor}_{\mathbf{Z}_p}(U^1/E^1) \to (\mathfrak{X}_{k_\infty})_\Gamma \to A_k \to 0$$

of finite abelian groups. By our assumption, it holds that $\mathrm{Tor}_{\mathbf{Z}_p}(U^1/E^1) = 0$, and hence $(\mathfrak{X}_{k_\infty})_\Gamma \simeq X_\Gamma \simeq A_k$. Put $I = \mathrm{Gal}(M_{k_\infty}/L_{k_\infty})$. Then we have the following exact sequence

$$0 \to I \to \mathfrak{X}_\infty \to X \to 0$$

of $\mathbf{Z}_p[[\mathrm{Gal}(k_\infty)/k]]$-modules. By taking the invariant and the co-invariant sequence, we have an exact sequence

$$\mathfrak{X}^\Gamma_{k_\infty} \to X^\Gamma \to I_\Gamma \to (\mathfrak{X}_{k_\infty})_\Gamma \to X_\Gamma \to 0.$$

Since we have checked that $\mathfrak{X}^\Gamma_{k_\infty} = 0$ and $(\mathfrak{X}_{k_\infty})_\Gamma \simeq X_\Gamma$, and since X^Γ is finite, it holds that $X^\Gamma \simeq I_\Gamma$ and $X^\Gamma = (X^0)^\Gamma$. This shows that $X^0 \neq 0$ if and only if $I \neq 0$ by a property of finite p-groups and by Nakayama's lemma. This completes the proof of Theorem 2. □

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