Limiting cases of Sobolev inequalities on stratified groups

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Abstract: In this paper we present critical Gagliardo-Nirenberg, Trudinger-type and Brezis-Gallouet-Wainger inequalities concerning the limiting cases of the embedding theorems for Sobolev spaces on stratified groups. Moreover, using the critical Gagliardo-Nirenberg inequality the existence of least energy solutions of the nonlinear Schrödinger type equations can be obtained. We also express the best constant in the critical Gagliardo-Nirenberg inequality in the variational form as well as in terms of the ground state solutions of the corresponding nonlinear subelliptic equations.

Key words: Trudinger inequality; Gagliardo-Nirenberg inequality; Sobolev inequality; stratified group; sub-Laplacian.

1. Introduction. The theory of Sobolev spaces associated to the sub-Laplacians on the stratified groups and their embedding theorems go back to Folland [Fol75]. In this paper we discuss critical case of such embedding theorems. In particular, we present three cases of such critical inequalities: Moser-Trudinger type inequality, Gagliardo-Nirenberg inequality, and Brezis-Gallouet-Wainger inequality. We are also interested in expressions for best constants in such inequalities, their equivalence, and applications to the solvability of nonlinear partial differential equations associated to the sub-Laplacian.

To start with, consider the following Moser-Trudinger inequality

$$(1)\quad \int_{\mathbf{R}^n} (\exp(\alpha|f(x)|^{\frac{p}{p-1}}) - 1)dx \leq C, \quad 1 < p < \infty,$$

for $f \in L^p_{n/p}(\mathbf{R}^n) = (1-\triangle)^{-n/2p}L^p(\mathbf{R}^n)$ with $\|f\|_{L^p_{n/p}} \leq 1$ and for some positive constants C and α, where $L^p_{n/p}(\mathbf{R}^n)$ is the Sobolev space of order n/p over L^p. This inequality has been generalised in many directions. In bounded domains of $\mathbf{R}^n$ with $p = n \geq 2$, we can also refer to [Mos79], [Ada88], [CC86], [Flu92], [MP89], [Tru67] for finding the best exponents in (1), and to [AS07] for the singular version of this inequality. In unbounded domains, we can refer to [Ada88], [Ada75], [OO91], [Oza95], [Str72], [Oga90], [AT99], [Oza97] for Sobolev spaces of fractional order and of higher order. On the Heisenberg group, we refer to [CL01], [LLT12] for an analogue of inequality (1) on domains of finite measure, and to [LL12] and [LLT14] on the entire Heisenberg group.

In this paper, we are interested in obtaining this inequality on general stratified groups, or rather its refinement, in the spirit of the Euclidean estimate in [Oza95]. We are also interested in critical Gagliardo-Nirenberg and Brezis-Gallouet-Wainger inequalities. Consequently, we give applications of these inequalities to the nonlinear subelliptic equations.

The paper is organised as follows: In Section 2 we briefly recall main concepts of stratified Lie groups and fix the notation. The critical Gagliardo-Nirenberg inequality and Trudinger-type inequality (1) on stratified groups are presented in Section 3, where the constant C is also given more explicitly. In Section 4, we present the Brezis-Gallouet-Wainger inequalities on stratified groups. Finally, applications are given to the nonlinear Schrödinger type equations in Section 5, together with expressions for a best constant.

2. Preliminaries. In this section let us very briefly recall the necessary notation concerning the setting of stratified groups.

We recall that $\mathbf{G} = (\mathbf{R}^n, \circ)$ is a stratified group (or a homogeneous Carnot group) if it satisfies

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the following conditions:

– For natural numbers $N_1, \ldots, N_r$ with $N = N_1$, the decomposition $\mathbf{R}^n = \mathbf{R}^N \times \ldots \times \mathbf{R}^{N_r}$ is valid, and for each positive λ the dilation $\delta_\lambda : \mathbf{R}^n \to \mathbf{R}^n$ given by
$$\delta_\lambda(x) = \delta_\lambda(x', x^{(2)}, \ldots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)})$$
is an automorphism of the group $\mathbf{G}$, where $x' \equiv x^{(1)} \in \mathbf{R}^N$ and $x^{(k)} \in \mathbf{R}^{N_k}$ for $k = 2, \ldots, r$.

– Let $X_1, \ldots, X_N$ be the left invariant vector fields on $\mathbf{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \ldots, N$, where N is as above. Then we have
$$\operatorname{rank}(\operatorname{Lie}\{X_1, \ldots, X_N\}) = n,$$
for each $x \in \mathbf{R}^n$, that is, the iterated commutators of $X_1, \ldots, X_N$ span the Lie algebra of $\mathbf{G}$.

The stratified groups have been thoroughly investigated by Folland [Fol75]. We also refer to [FS82], and to [FR16] for more detailed discussions from the point of view of more general graded Lie groups.

We define the homogeneous dimension of $\mathbf{G}$ by
$$Q = \sum_{k=1}^{r} k N_k, \quad N_1 = N.$$
We also recall that the standard Lebesgue measure dx on $\mathbf{R}^n$ is the Haar measure for $\mathbf{G}$ (see, for example [FR16, Proposition 1.6.6]). The (canonical) sub-Laplacian on the stratified group $\mathbf{G}$ is defined by
$$\mathfrak{L} = \sum_{k=1}^{N} X_k^2. \tag{2}$$

3. Critical Gagliardo-Nirenberg and Trudinger-type inequalities. In this section we give the critical Gagliardo-Nirenberg and Trudinger-type inequalities on stratified groups, and then we note that these inequalities are actually equivalent.

We refer to [Fol75, Section 4] for the definition of the Sobolev spaces $L^p_\alpha(\mathbf{G})$ on stratified groups for $\alpha \geq 0$ and $1 < p < \infty$, which is equipped with the norm
$$\|f\|_{L^p_\alpha(\mathbf{G})} = \|f\|_{L^p(\mathbf{G})} + \|(-\mathfrak{L})^{\alpha/2} f\|_{L^p(\mathbf{G})}.$$
Let us start with the critical version of the Gagliardo-Nirenberg inequalities.

Theorem 3.1. *Let $\mathbf{G}$ be a stratified group of homogeneous dimension Q and let $1 < p < \infty$. Then we have*
$$\|f\|_{L^q(\mathbf{G})} \leq C_1 q^{1-1/p} \|(-\mathfrak{L})^{\frac{Q}{2p}} f\|^{1-p/q}_{L^p(\mathbf{G})} \|f\|^{p/q}_{L^p(\mathbf{G})} \tag{3}$$
for any q with $p \leq q < \infty$ and for any function $f \in L^p_{Q/p}(\mathbf{G})$, where the constant C_1 depends only on p and Q.

We note that by [Fol75, Theorem 4.17], we have the embedding $L^p_\alpha(\mathbf{G}) \hookrightarrow L^q(\mathbf{G})$ for $1/q = 1/p - \alpha/Q$ and $0 < \alpha < Q/p$ with $1 < p < q < \infty$. In the critical case $\alpha = Q/p$, the obtained inequality (3) allows us to obtain the embedding $L^p_{Q/p}(\mathbf{G}) \hookrightarrow L^q(\mathbf{G})$ for $1 < p \leq q < \infty$. That is why we can call this inequality the critical Gagliardo-Nirenberg inequality.

In the Euclidean case $\mathbf{G} = (\mathbf{R}^n, +)$ and $\mathfrak{L} = \triangle$ being the Laplacian, the inequality (3) was obtained in [OO91] for $p = 2$, in [KOS92] for $p = n$, and in [Oza95] for general p as in Theorem 3.1.

Now we state the Trudinger-type inequality with the remainder estimate on stratified groups.

Theorem 3.2. *Let $\mathbf{G}$ be a stratified group of homogeneous dimension Q and let $1 < p < \infty$. Then there exist positive constants α and C_2 such that the inequality*
$$\int_{\mathbf{G}} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \leq k < p-1,\, k \in \mathbf{N}} \frac{1}{k!} (\alpha |f(x)|^{p'})^k \right) dx \leq C_2 \|f\|^p_{L^p(\mathbf{G})} \tag{4}$$
holds for any function $f \in L^p_{Q/p}(\mathbf{G})$ with $\|(-\mathfrak{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbf{G})} \leq 1$, where $1/p + 1/p' = 1$.

In the case $\mathbf{G} = (\mathbf{R}^n, +)$ and $\mathfrak{L} = \triangle$, the inequality (4) was obtained in [Oza95]. In this abelian case, we also refer to [Oga90] for $p = 2$, [OO91] for $p = n = 2$, and to [AT99] for $p = n \geq 2$.

The following result shows that the presented critical Gagliardo-Nirenberg and Trudinger-type inequality are actually equivalent on general stratified groups, the fact that is known in $\mathbf{R}^n$, see [Oza97].

Theorem 3.3. *The inequalities* (3) *and* (4) *are equivalent. Furthermore, we have*
$$\frac{1}{\widetilde{\alpha} p' e} = A^{p'} = B^{p'}, \tag{5}$$
where
$$\widetilde{\alpha} = \sup\{\alpha > 0; \exists C_2 = C_2(\alpha) : (4) \text{ holds } \forall f \in L^p_{Q/p}(\mathbf{G}) \text{ with } \|(-\mathfrak{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbf{G})} \leq 1\},$$

$$A = \inf\{C_1 > 0; \exists r = r(C_1) \text{ with } r \geq p : (3) \text{ holds } \forall f \in L^p_{Q/p}(\mathbf{G}), \forall q \text{ with } r \leq q < \infty\},$$

$$B = \limsup_{q\to\infty} \frac{\|f\|_{L^q(\mathbf{G})}}{q^{1-1/p}\|(-\mathfrak{L})^{\frac{Q}{2p}}f\|^{1-p/q}_{L^p(\mathbf{G})}\|f\|^{p/q}_{L^p(\mathbf{G})}}.$$

On stratified groups, we refer to [RY19] and [RSY17] for the weighted versions of Trudinger-Moser and Gagliardo-Nirenberg inequalities, and to [RY17] and [RY18] for their hypoelliptic versions.

4. Brezis-Gallouet-Wainger inequalities. In this section we present Brezis-Gallouet-Wainger type inequalities, which concern the limiting case of the Sobolev estimates from another point of view (see [Bre82], [BG80] and [BW80]).

Theorem 4.1. *Let $\mathbf{G}$ be a stratified group of homogeneous dimension Q and let $a, p, q \in \mathbf{R}$ be such that $1 < p, q < \infty$ and $m > Q/q$. Then we have*

$$(6) \quad \|f\|_{L^\infty(\mathbf{G})} \leq C_3(1 + \log(1 + \|(-\mathfrak{L})^{m/2}f\|_{L^q(\mathbf{G})}))^{1/p'}$$

for all functions $f \in L^p_{Q/p}(\mathbf{G}) \cap L^q_m(\mathbf{G})$ with $\|f\|_{L^p_{Q/p}(\mathbf{G})} \leq 1$, where $1/p + 1/p' = 1$.

In the case $\mathbf{G} = (\mathbf{R}^n, +)$ and $\mathfrak{L} = \triangle$, the inequality (6) was obtained in [BW80] by using the Fourier transform techniques, and in [Eng89] for $n/p, m \in \mathbf{Z}$ and in [Oza95] for general case without using the Fourier transform.

We can also obtain the following estimate using Theorem 3.1:

Theorem 4.2. *Let $\mathbf{G}$ be a stratified group of homogeneous dimension Q and let $1 < p < \infty$. Then we have*

$$(7) \quad \int_\Omega |f(x)|dx \leq C_4\|f\|_{L^p_{Q/p}(\mathbf{G})}|\Omega|(1 + |\log|\Omega||)^{1/p'}$$

for any function $f \in L^p_{Q/p}(\mathbf{G})$ and for any Lebesgue measurable set Ω with $|\Omega| < \infty$, where the constant C_4 depends only on p and Q, $1/p + 1/p' = 1$, and $|\Omega|$ denotes the Lebesgue measure of Ω.

When $\mathbf{G} = (\mathbf{R}^n, +)$ and $\mathfrak{L} = \triangle$, the inequality (7) was obtained in [BW80, Lemma 2] and in [Oza95, Theorem 3]. In [BW80], using this estimate and Morrey's technique the authors proved the Brezis-Wainger inequality: for any function $f \in L^p_{n/p+1}(\mathbf{R}^n)$ and for each $x, y \in \mathbf{R}^n$, the following inequality

$$|f(x) - f(y)| \leq C_5\|f\|_{L^p_{n/p+1}(\mathbf{R}^n)} \times |x-y|(1 + |\log|x-y||)^{1/p'}, \quad 1 < p < \infty,$$

holds true, where the constant C_5 depends only on n and p.

5. Applications to nonlinear Schrödinger type equations. In this section we discuss an application of the critical case of the Gagliardo-Nirenberg inequality (3) to the existence of least energy solutions of nonlinear Schrödinger type equations, and a sharp expression for the smallest positive constant C_1 in (3). For non-critical case on nilpotent Lie groups, similar results were obtained in [CR13] on Heisenberg group, and in [RTY17] on general graded groups.

More precisely, the critical Gagliardo-Nirenberg inequality (3) is related to the following Schrödinger equation with the power nonlinearities:

$$(8) \quad (-\mathfrak{L})^{\frac{Q}{2p}}(|(-\mathfrak{L})^{\frac{Q}{2p}}u|^{p-2}(-\mathfrak{L})^{\frac{Q}{2p}}u) + |u|^{p-2}u = |u|^{q-2}u, \quad u \in L^p_{Q/p}(\mathbf{G}).$$

Moreover, this inequality is related to the variational problem

$$(9) \qquad d = \inf_{\substack{u \in L^p_{Q/p}(\mathbf{G})\setminus\{0\} \\ \mathfrak{I}(u)=0}} \mathfrak{L}(u),$$

for functionals

$$(10) \quad \mathfrak{L}(u) := \frac{1}{p}\int_{\mathbf{G}} |(-\mathfrak{L})^{\frac{Q}{2p}}u(x)|^p dx + \frac{1}{p}\int_{\mathbf{G}} |u(x)|^p dx - \frac{1}{q}\int_{\mathbf{G}} |u(x)|^q dx$$

and

$$(11) \quad \mathfrak{I}(u) := \int_{\mathbf{G}} (|(-\mathfrak{L})^{\frac{Q}{2p}}u(x)|^p + |u(x)|^p - |u(x)|^q)dx.$$

Let us denote $p^* = \frac{pQ}{Q-p}$. Then we have:

Theorem 5.1. *Let $\mathbf{G}$ be a stratified group of homogeneous dimension Q, let $1 < p < Q$ and $p < q < p^*$. Then the Schrödinger type equation (8) has a least energy solution $\phi \in L^p_{Q/p}(\mathbf{G})$. Furthermore, we have $d = \mathfrak{L}(\phi)$.*

Now let us give a sharp expression for the smallest positive constant C_1 in (3).

Theorem 5.2. *Let $\mathbf{G}$ be a stratified group of homogeneous dimension Q. Let $1 < p < Q$ and $p < q < p^*$. Let ϕ be a least energy solution of (8) and let C_{GN} be the smallest positive constant C_1 in (3). Then we have*

$$(12) \qquad C_{GN} = q^{-q+q/p}\frac{q}{p}\left(\frac{q-p}{p}\right)^{\frac{p-q}{p}}\|\phi\|^{p-q}_{L^p(\mathbf{G})}$$

$$= q^{-q+q/p}\,\frac{q}{p}\left(\frac{q-p}{p}\right)^{\frac{p-q}{p}}\left(\frac{p^2}{q-p}d\right)^{\frac{p-q}{p}},$$

where d is defined in (9).

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