Lipschitz characterization for exponentially weighted Bergman spaces of the unit ball

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Abstract: The paper concerns the weighted Bergman spaces of the complex unit ball with exponential weights. We characterize the space with respect to Lipschitz type conditions using norm equivalence lemma.

Key words: Exponentially weighted Bergman space; Lipschitz type characterization; norm equivalence.

1. Introduction. Let $\mathbf{B}_n$ be the open unit ball in the n-dimensional complex space $\mathbf{C}^n$ and dV be the normalized Lebesgue volume measure on $\mathbf{B}_n$. Let $H(\mathbf{B}_n)$ be the set of holomorphic functions on $\mathbf{B}_n$. For $\alpha \in \mathbf{R}$ and $\beta > 0$, the weight function $\omega_{\alpha,\beta}$ is given by

$$\omega(z) = \omega_{\alpha,\beta}(z) := (1-|z|)^{\alpha} \exp\left(\frac{-\beta}{1-|z|}\right)$$

for $z \in \mathbf{B}_n$. The volume measure with the weight $\omega_{\alpha,\beta}$ is denoted by

$$dV_{\alpha,\beta}(z) := \omega_{\alpha,\beta}(z)dV(z).$$

Let $0 < p < \infty$. The function space $A^p_{\alpha,\beta}(\mathbf{B}_n) := H(\mathbf{B}_n) \cap L^p(\mathbf{B}_n, dV_{\alpha,\beta})$ is the space of holomorphic functions whose L^p-norm with the measure $dV_{\alpha,\beta}$ is bounded, namely,

$$\|f\|_{\alpha,\beta} = \left[\int_{\mathbf{B}_n} |f(z)|^p dV_{\alpha,\beta}(z)\right]^{\frac{1}{p}} < +\infty.$$

We present characterizations of the space $A^p_{\alpha,\beta}(\mathbf{B}_n)$ by means of Lipschitz type conditions;

Main Theorem. *Let $0 < p < \infty$, $\alpha \in \mathbf{R}$ and $\beta > 0$. Suppose f is holomorphic in $\mathbf{B}_n$, then the following statements are equivalent*:

(a) $f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$;

(b) *There exists a continuous function g belonging to $L^p(\mathbf{B}_n, dV_{\alpha,\beta})$ such that*

$$|f(z) - f(w)| \le d_{\psi}(z,w)(g(z) + g(w));$$

(c) *There exists a continuous function g belonging to $L^p(\mathbf{B}_n, dV_{\alpha+2p,\beta})$ such that*

$$|f(z) - f(w)| \le |z-w|(g(z) + g(w)).$$

Here, d_{ψ} denotes the distance induced by the metric $\psi(z)^{-2} \sum_{j=1}^n dz_j \otimes d\bar{z}_j$:

$$d_{\psi}(z,w) = \inf_{\gamma} \int_0^1 \frac{|\gamma'(t)|}{\psi(\gamma(t))}\, dt$$

where $\psi(z) = (1-|z|)^2$ and $\gamma : [0,1] \to \mathbf{B}_n$ is a parametrization of a piecewise C^1 curve with $\gamma(0) = z$ and $\gamma(1) = w$.

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$. For the standard (weighted) Bergman spaces (in both case of $\mathbf{D}$ and $\mathbf{B}_n$), Wulan and Zhu [8] have characterized the spaces with Lipschitz type conditions. For the exponentially weighted Bergman spaces, we [4] have shown similar characterizations in $\mathbf{D}$. In this paper, we extend the result to $\mathbf{B}_n$.

The weighted Bergman spaces with exponential type weights have been studied in many papers ([1], [2], [6], [7]). The weight $\omega_{\alpha,\beta}$ is an example of exponential type weight, precisely, $\omega_{\alpha,\beta}(z) = e^{-\varphi(z)}$ where $\varphi(z) = \alpha \log\left(\frac{1}{1-|z|}\right) + \frac{\beta}{1-|z|}$. In addition, the function $\psi(r) = (1-r)^2$ comes from the distortion function which is originally defined by

$$\psi_{\omega}(r) = \frac{1}{\omega(r)} \int_r^1 \omega(x)dx$$

due to Siskakis in [7].

Let f be a holomorphic function. The radial derivative $\mathcal{R}f$ denotes

$$\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z) = \lim_{t\to 0} \frac{f(z+tz) - f(z)}{t}$$

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where t is real parameter. The (complex) gradient of f at z is

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z)\right).$$

Theorem 2.2 in [3] gives that for an holomorphic function f,

$$f \in A^p_{\alpha,\beta}(\mathbf{B}_n) \text{ iff } (1-|z|)^2 \mathcal{R} f \in L^p(\mathbf{B}_n, dV_{\alpha,\beta}).$$

It has been proved in the case of $p = 2$ in original, but the result can be extended to the case of $0 < p < \infty$. In addition, we show that $f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$ is equivalent to the assertion

$$(1-|z|)^2 |\nabla f| \in L^p(\mathbf{B}_n, dV_{\alpha,\beta});$$

see Lemma 2.5. It makes the proof of the main theorem accessible.

Throughout this paper, C will be a symbol of a positive constant. The value of the constant can be changed often. The expression $A \lesssim B$ indicates $A \le CB$, and $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries. Let $\psi(z) = (1-|z|)^2$. For $z \in \mathbf{B}_n$ and $r > 0$, we put

$$B_\psi(z,r) := \{\zeta \in \mathbf{B}_n : d_\psi(z,\zeta) < r\}.$$

We also define another ball denoted by

$$E_r(z) := \left\{\zeta \in \mathbf{B}_n : \frac{|z-\zeta|}{(1-|z|)^2} < r\right\}.$$

Lemma 2.1. *Let $r > 0$ be a small number, then there exists a positive constant C (it depends on r) such that*

$$C^{-1} \le \frac{1-|z|}{1-|w|} \le C$$

for all z and w with $w \in E_r(z)$.

There is a relation between two types of balls following Proposition 5 in [5];

Lemma 2.2. *For a small $r > 0$, there are two positive numbers R_1 and R_2 such that*

$$E_{R_1}(z) \subseteq B_\psi(z,r) \subseteq E_{R_2}(z)$$

for $z \in \mathbf{B}_n$.

Moreover, $d_\psi(z,w) \asymp \frac{|z-w|}{(1-|z|)^2}$ when two points z and w are close sufficiently.

Note that Lemma 2.1 gives that

$$1-|z| \asymp 1-|w|$$

when $d_\psi(z,w) < r$ with aid of Lemma 2.2. It also gives that

$$\operatorname{vol}(B_\psi(z,r)) \asymp r^{2n}(1-|z|)^{4n}.$$

Lemma 2.3. *For $\beta \in \mathbf{R}$ and a small $r > 0$ there exists a positive constant C such that*

$$C^{-1} \le \frac{\exp\left(-\frac{\beta}{1-|z|}\right)}{\exp\left(-\frac{\beta}{1-|w|}\right)} \le C$$

for all z and w with $d_\psi(z,w) < r$.

Proof. Suppose $d_\psi(z,w) < r$, then there is a positive number r' satisfying

$$\frac{|z-w|}{(1-|z|)(1-|w|)} < r'$$

by Lemma 2.1 and Lemma 2.2. We have

$$(2.1) \qquad |(1-|w|)-(1-|z|)| \le |z-w| \le r'(1-|z|)(1-|w|).$$

Dividing both sides of (2.1) by $(1-|z|)(1-|w|)$ yields that

$$\left|\frac{1}{1-|z|} - \frac{1}{1-|w|}\right| < r'$$

which implies the result. □

Sub-mean-value inequalities using $B_\psi(z,r)$ are given;

Lemma 2.4. *Let $0 < p < \infty$ and $f \in H(\mathbf{B}_n)$. Then for a small $r > 0$, there are constants $C_1 > 0$ and $C_2 > 0$ such that*

$$|f(w)|^p \le C_1 \frac{1}{r^{2n}(1-|w|)^{4n}} \int_{B_\psi(w,r)} |f(\zeta)|^p dV(\zeta)$$

and

$$|f(w)|^p \omega_{\alpha,\beta}(w) \le C_2 \frac{1}{r^{2n}(1-|w|)^{4n}} \int_{B_\psi(w,r)} |f(\zeta)|^p \omega_{\alpha,\beta}(\zeta) dV(\zeta).$$

Especially, if $u \in B_\psi(z,r)$, then there is positive number $C_3 > 0$ such that

$$(2.2) \quad |f(u)|^p \le C_3 \frac{1}{r^{2n}(1-|z|)^{4n}} \int_{B_\psi(z,2r)} |f(\zeta)|^p dV(\zeta).$$

Note that the right hand side of the inequality (2.2) is independent of u.

Proof. The proof is similar to Lemma 2.4 in [4]. The difference is the exponent of the volume of the ball. □

Lemma 2.5. *Let $0 < p < \infty$, $\alpha \in \mathbf{R}$ and $\beta > 0$. Suppose f is holomorphic in $\mathbf{B}_n$, then the following statements are equivalent:*

(a) $f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$;
(b) $(1-|z|)^2|\nabla f| \in L^p(\mathbf{B}_n, dV_{\alpha,\beta})$;
(c) $(1-|z|)^2\mathcal{R}f \in L^p(\mathbf{B}_n, dV_{\alpha,\beta})$.

Proof. (a) $\Rightarrow$ (b). Let $r>0$ be a small number, we choose a number r' which satisfies $0<r'<r$. By Cauchy's estimates for $B_\psi(z,r')$,

$$\left|\frac{\partial f}{\partial z_j}(z)\right|^p \lesssim \frac{1}{(1-|z|)^{2p}} \sup\{|f(\zeta)|^p : |\zeta - z| = r'(1-|z|)^2\}.$$

The subharmonic inequality (2.2) yields

$$\left|\frac{\partial f}{\partial z_j}(z)\right|^p \lesssim \frac{1}{(1-|z|)^{2p+4n}} \int_{B_\psi(z,r)} |f(u)|^p dV(u). \tag{2.3}$$

For each $j = 1,\dots,n$, the right hand side of the inequality (2.3) is independent of j. It gives

$$|\nabla f(z)|^p \lesssim \frac{1}{(1-|z|)^{2p+4n}} \int_{B_\psi(z,r)} |f(u)|^p dV(u). \tag{2.4}$$

An upper bound of $\|(1-|z|)^2|\nabla f|\|^p_{\alpha,\beta}$ is obtained from the inequality (2.4);

$$\begin{aligned}\int_{\mathbf{B}_n} &|\nabla f(z)|^p (1-|z|)^{2p} \omega_{\alpha,\beta}(z) dV(z)\\ &\lesssim \int_{\mathbf{B}_n} \frac{1}{(1-|z|)^{4n}} \int_{B_\psi(z,r)} |f(u)|^p dV(u) \omega_{\alpha,\beta}(z) dV(z).\end{aligned}$$

Let $\chi_{(z,r)}$ denote the characteristic function of the set $B_\psi(z,r)$. One can see that for any points $z, u \in \mathbf{B}_n$, $\chi_{(z,r)}(u) = \chi_{(u,r)}(z)$. Then we have

$$\begin{aligned}\int_{\mathbf{B}_n} &|\nabla f(z)|^p (1-|z|)^{2p} \omega_{\alpha,\beta}(z) dV(z)\\ &\lesssim \int_{\mathbf{B}_n} (1-|z|)^{\alpha-4n} e^{\frac{-\beta}{1-|z|}} \int_{\mathbf{B}_n} |f(u)|^p \chi_{(z,r)}(u) dV(u) dV(z)\\ &= \int_{\mathbf{B}_n}\int_{\mathbf{B}_n} (1-|z|)^{\alpha-4n} e^{\frac{-\beta}{1-|z|}} |f(u)|^p \chi_{(u,r)}(z) dV(u) dV(z)\\ &= \int_{\mathbf{B}_n} |f(u)|^p \int_{B_\psi(u,r)} (1-|z|)^{\alpha-4n} e^{\frac{-\beta}{1-|z|}} dV(z) dV(u)\\ &\lesssim \int_{\mathbf{B}_n} |f(u)|^p (1-|u|)^{\alpha-4n} e^{\frac{-\beta}{1-|u|}} \int_{B_\psi(u,r)} dV(z) dV(u)\\ &\asymp \|f\|^p_{\alpha,\beta}\end{aligned} \tag{2.5}$$

by Fubini's theorem, Lemma 2.1 and Lemma 2.3. Hence f belongs to $A^p_{\alpha,\beta}(\mathbf{B}_n)$ implies that $(1-|z|)^2|\nabla f|$ belongs to $L^p(\mathbf{B}_n, dV_{\alpha,\beta})$.

(b) $\Rightarrow$ (c). It is well-known fact that

$$|\mathcal{R}f(z)| \le |z||\nabla f(z)| \le |\nabla f(z)|,$$

which gives the result.

(a) $\Leftrightarrow$ (c). It is due to Cho and Park [3]. □

3. Lipschitz Characterizations.

Theorem 3.1. *Let $0<p<\infty$, $\alpha \in \mathbf{R}$ and $\beta>0$. Suppose f is holomorphic in $\mathbf{B}_n$, then the following statements are equivalent*:

(a) $f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$;
(b) *There exists a continuous function g belonging to $L^p(\mathbf{B}_n, dV_{\alpha,\beta})$ such that*

$$|f(z) - f(w)| \le d_\psi(z,w)(g(z)+g(w)).$$

Proof. (b) $\Rightarrow$ (a). Let f be a holomorphic function. Suppose that $|f(z)-f(w)| \le d_\psi(z,w)(g(z)+g(w))$ for some positive function $g \in L^p(\mathbf{B}_n, dV_{\alpha,\beta})$. For any $z \in \mathbf{B}_n$, let $w = z + tz$ where t is a scalar. It gives

$$\begin{aligned}|\mathcal{R}f(z)| &= \lim_{t\to 0} \frac{|f(z+tz)-f(z)|}{|t|}\\ &\le \lim_{w\to z} \frac{|z||f(z)-f(w)|}{|z-w|}\\ &\le \lim_{w\to z} \frac{d_\psi(z,w)}{|z-w|}(g(z)+g(w))\\ &\lesssim \frac{2g(z)}{(1-|z|)^2}\end{aligned}$$

with aid of Lemma 2.2. We have

$$|\mathcal{R}f(z)|(1-|z|)^2 \le 2g(z)$$

which implies

$$\int_{\mathbf{B}_n} |\mathcal{R}f(z)|^p (1-|z|)^{2p} \omega_{\alpha,\beta}(z) dV(z) < +\infty.$$

Thus we can get $f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$ by Lemma 2.5.

(a) $\Rightarrow$ (b). Suppose that f belongs to $A^p_{\alpha,\beta}(\mathbf{B}_n)$. Let $z, w \in \mathbf{B}_n$. For a fixed radius $r>0$, we first assume $d_\psi(z,w) < r$. It is given that

$$|f(z)-f(w)| \le |z-w| \int_0^1 |\nabla f(tz + (1-t)w)| dt$$

by the fundamental theorem of calculus. Since the line segment belongs to $B_\psi(z,r)$, we have

$$|f(z)-f(w)| \le |z-w| \sup\{|\nabla f(u)| : u \in B_\psi(z,r)\}.$$

Lemma 2.1 yields

$$\begin{aligned}&|f(z)-f(w)|\\ &\le \frac{|z-w|}{(1-|z|)^2} C \sup\{(1-|u|)^2|\nabla f(u)| : u \in B_\psi(z,r)\}.\end{aligned}$$

By Lemma 2.2, it can be obtained that

$$|f(z)-f(w)| \le d_\psi(z,w) h(z)$$

where

$$h(z) = C \sup\{(1-|u|)^2 |\nabla f(u)| : u \in B_\psi(z,r)\}.$$

We next assume $d_\psi(z,w) \geq r$. The assumption implies that

$$|f(z) - f(w)| \leq \frac{d_\psi(z,w)}{r}(|f(z)| + |f(w)|)$$

with aid of triangle inequality. It is obvious that the function $\frac{|f(z)|}{r}$ is in $L^p(\mathbf{B}_n, dV_{\alpha,\beta})$.

We have the function $g(z) = h(z) + \frac{|f(z)|}{r}$ as a desired function provided that $h \in L^p(\mathbf{B}_n, dV_{\alpha,\beta})$. It is shown as follows. Since $|\nabla f|^p$ is subharmonic, for $u \in B_\psi(z,r)$,

$$|\nabla f(u)|^p \lesssim \frac{1}{r^{2n}(1-|z|)^{4n}} \int_{B_\psi(z,2r)} |\nabla f(\zeta)|^p dV(\zeta)$$

by the inequality (2.2). It is given that

$$\begin{aligned}|\nabla f(u)|^p (1-|u|)^{2p} \\ \lesssim \frac{1}{r^{2n}}(1-|z|)^{2p-4n} \int_{B_\psi(z,2r)} |\nabla f(\zeta)|^p dV(\zeta)\end{aligned}$$

by Lemma 2.1. This means that

$$|h(z)|^p \lesssim (1-|z|)^{2p-4n} \int_{B_\psi(z,2r)} |\nabla f(\zeta)|^p dV(\zeta). \tag{3.1}$$

Integrating both sides of (3.1) yields

$$\begin{aligned}\|h\|^p_{\alpha,\beta} \\ \lesssim \int_{\mathbf{B}_n} (1-|z|)^{\alpha+2p-4n} e^{\frac{-\beta}{1-|z|}} \int_{B_\psi(z,2r)} |\nabla f(\zeta)|^p dV(\zeta) dV(z).\end{aligned}$$

The same way as (2.5), using the characteristic function and Fubini's theorem, gives that

$$\begin{aligned}\|h\|^p_{\alpha,\beta} \\ \lesssim \int_{\mathbf{B}_n} |\nabla f(\zeta)|^p (1-|\zeta|)^{\alpha+2p-4n} e^{\frac{-\beta}{1-|\zeta|}} \int_{B_\psi(\zeta,2r)} dV(z) dV(\zeta) \\ \asymp \|(1-|z|)^2 |\nabla f|\|^p_{\alpha,\beta}.\end{aligned}$$

Lemma 2.5 shows that the norm is dominated by $\|f\|^p_{\alpha,\beta}$. It completes the proof. □

Theorem 3.2. *Let* $0 < p < \infty$, $\alpha \in \mathbf{R}$ *and* $\beta > 0$. *Suppose* f *is holomorphic in* $\mathbf{B}_n$, *then the following statements are equivalent*:

(a) $f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$;

(b) *There exists a continuous function* g *belonging to* $L^p(\mathbf{B}_n, dV_{\alpha+2p,\beta})$ *such that*

$$|f(z) - f(w)| \leq |z-w|(g(z) + g(w)).$$

Proof. (b) ⇒ (a). Let f be a holomorphic function. Suppose that $|f(z) - f(w)| \leq |z-w|(g(z)+g(w))$ for some positive function $g \in L^p(\mathbf{B}_n, dV_{\alpha+2p,\beta})$. For any $z \in \mathbf{B}_n$, let $w = z + tz$ where t is a scalar, then we have

$$|\mathcal{R}f(z)| \leq \lim_{w \to z} \frac{|z||f(z) - f(w)|}{|z-w|} \leq 2g(z).$$

It implies $(1-|z|)^2 \mathcal{R}f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$. Thus we can get $f \in A^p_{\alpha,\beta}(\mathbf{B}_n)$ by Lemma 2.5.

(a) ⇒ (b). We consider any two points z, w in the unit ball. For a fixed radius $r > 0$, we first assume that $z \in E_r(w)$ or $w \in E_r(z)$. By Lemma 2.2, there is $r' > 0$ such that $d_\psi(z,w) < r'$. We have

$$|f(z) - f(w)| \leq |z-w| C \sup\{|\nabla f(u)| : u \in B_\psi(z,r')\}$$

and the function

$$h(z) = C \sup\{|\nabla f(u)| : u \in B_\psi(z,r')\}$$

is in $L^p(\mathbf{B}_n, dV_{\alpha+2p,\beta})$ following the proof of Theorem 3.1.

Next we assume that $z \notin E_r(w)$ and $w \notin E_r(z)$, then we have

$$|f(z) - f(w)| \leq |z-w| \left(\frac{|f(z)|}{r(1-|z|)^2} + \frac{|f(w)|}{r(1-|w|)^2} \right)$$

with triangle inequality. The function $g(z) = h(z) + \frac{|f(z)|}{r(1-|z|)^2}$ is a desired function in $L^p(\mathbf{B}_n, dV_{\alpha+2p,\beta})$. □

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