Weierstrass points on hyperelliptic modular curves

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Abstract: In this paper, we find all Weierstrass points on the hyperelliptic modular curves $X_0(N)$ whose hyperelliptic involutions are non-exceptional, i.e., induced by matrices in $\mathrm{GL}_2(\mathbf{R})$.

Key words: Weierstrass points; modular curve; hyperelliptic curve.

1. Introduction. Let X be a nonsingular algebraic curve of genus $g \geq 2$. A point P on X is called a *Weierstrass point* if there is a non-constant meromorphic function on X that has a pole of order less than or equal to g at P and is regular elsewhere. The set of Weierstrass points is an invariant of X which is useful in the study of arithmetic and geometric properties of X.

Let $\mathbf{H}$ be the complex upper half plane and Γ be a congruence subgroup of the full modular group $\mathrm{SL}_2(\mathbf{Z})$. We consider the modular curve $X(\Gamma)$ obtained from compactification of the quotient space $\Gamma\backslash\mathbf{H}$ by adding finitely many points called *cusps*. For any integer $N \geq 1$, we have subgroups $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ of $\mathrm{SL}_2(\mathbf{Z})$ defined by matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent modulo N to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, respectively. We let $X(N), X_1(N), X_0(N)$ be the modular curves defined over $\mathbf{Q}$ associated to $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$, respectively. The X's are compact Riemann surfaces.

The Weierstrass points of $X_1(N)$ and $X_0(N)$ have been investigated by Atkin [A], Choi [C], Kilger [K], Kohnen [K1,K2], Lehner and Newman [LN], Ogg [O2], Ono [On], Rohrlich [R], and Schöneberg [S].

In particular, the author [J] computed all Weierstrass points on the hyperelliptic curves $X_1(N)$. Note that a curve is said to be *hyperelliptic* if its genus is greater than or equal to 2 and it admits a map of degree 2 to $\mathbf{P}^1$. Mestre [M] determined that $X_1(N)$ is hyperelliptic if and only if $N = 13, 16, 18$ (See [Me]). Indeed, the Weierstrass points on a hyperelliptic curve X are the same as the fixed points on X by the hyperelliptic involution ν which is unique, and hence it is $\mathbf{Q}$-rational (See [J]).

Ogg [O1] classified all the hyperelliptic curves $X_0(N)$, and he determined all the hyperelliptic involutions which are the Atkin-Lehner involutions W_d except for $N = 37, 40, 48$. Note that for each divisor $d|N$ with $\gcd(d, N/d) = 1$, W_d is induced by the matrices of the form

$$\begin{pmatrix} dx & y \\ Nz & dw \end{pmatrix}$$

with $x, y, z, w \in \mathbf{Z}$ and determinant d. In particular, if $d = N$, then W_N is called the *full Atkin-Lehner involution*. We also denote by W_d a matrix of the above form. The hyperelliptic involutions, say μ and μ', on $X_0(40)$ and $X_0(48)$, respectively, are induced by the matrices

$$\mu = \begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}, \ \mu' = \begin{pmatrix} -6 & 1 \\ -48 & 6 \end{pmatrix},$$

respectively. However, the hyperelliptic involution on $X_0(37)$ is an exceptional automorphism which is not expressed by a matrix in $\mathrm{GL}_2(\mathbf{R})$.

In this paper, we find all Weierstrass points on the hyperelliptic modular curves $X_0(N)$, except for $N = 37$.

2. Quadratic forms. In this section we explain the result on quadratic forms by Gross, Kohnen and Zagier [GKZ] which will be used for computing the fixed points on $X_0(N)$ by W_d. For a negative integer D congruent to 0 or 1 modulo 4, we denote by $\mathcal{Q}_D$ the set of positive definite integral binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$. Then $\Gamma(1)$ acts on $\mathcal{Q}_D$ by

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$$Q \circ \gamma(x, y) = Q(px + qy, rx + sy)$$

where $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. A primitive positive definite form $[a, b, c]$ is said to be a *reduced form* if

$$|b| \le a \le c, \text{ and } b \ge 0 \text{ if either } |b| = a \text{ or } a = c.$$

Let $\mathcal{Q}_D^\circ \subset \mathcal{Q}_D$ be the subset of primitive forms. Then $\Gamma(1)$ also acts on $\mathcal{Q}_D^\circ$. As is well known, there is a 1-1 correspondence between the set of classes $\Gamma(1)\backslash\mathcal{Q}_D^\circ$ and the set of reduced primitive definite forms.

For each $\beta \in \mathbf{Z}/2N\mathbf{Z}$, we define

$$\mathcal{Q}_{D,N,\beta} = \{[aN, b, c] \in \mathcal{Q}_D \mid \beta \equiv b \pmod{2N}\}.$$

Gross, Kohnen and Zagier [GKZ] obtained $\Gamma_0(N)$-invariant decomposition as follows:

Proposition 2.1 ([GKZ]). *For each $\beta \in \mathbf{Z}/2N\mathbf{Z}$, we define*

$$\mathcal{Q}_{D,N,\beta}^\circ = \{[aN, b, c] \in \mathcal{Q}_D \mid \beta \equiv b \pmod{2N}, \gcd(a, b, c) = 1\}.$$

Then we have the following

(i) *Define $m = \gcd(N, \beta, \frac{\beta^2 - D}{4N})$ and fix a decomposition $m = m_1 m_2$ with $m_1, m_2 > 0$ and $\gcd(m_1, m_2) = 1$. Let*

$$\mathcal{Q}_{D,N,\beta,m_1,m_2}^\circ = \{[aN, b, c] \in \mathcal{Q}_{D,N,\beta}^\circ \mid \gcd(N, b, a) = m_1, \gcd(N, b, c) = m_2\}.$$

Then $\Gamma_0(N)$ acts on $\mathcal{Q}_{D,N,\beta,m_1,m_2}^\circ$ and there is a 1-1 correspondence between

$$\mathcal{Q}_{D,N,\beta,m_1,m_2}^\circ/\Gamma_0(N) \to \mathcal{Q}_D^\circ/\Gamma(1)$$
$$[aN, b, c] \mapsto [aN_1, b, cN_2]$$

where N_1N_2 is any decomposition of N into coprime factors such that $\gcd(m_1, N_2) = \gcd(m_2, N_1) = 1$. Moreover we have a $\Gamma_0(N)$-invariant decomposition as follows:

$$\mathcal{Q}_{D,N,\beta}^\circ = \bigcup_{\substack{m = m_1 m_2 \\ m_1, m_2 > 0 \\ \gcd(m_1, m_2) = 1}} \mathcal{Q}_{D,N,\beta,m_1,m_2}^\circ. \tag{1}$$

(ii) *The inverse image $[aN_2, b, c/N_2]$ of any primitive form $[\bar{a}, \bar{b}, \bar{c}]$ of discriminant D under the 1-1 correspondence in (i) is obtained by solving the following equations*:

$$a = \bar{a}p^2 + \bar{b}pr + \bar{c}r^2$$
$$b = 2\bar{a}pq + \bar{b}(ps + qr) + 2\bar{c}rs$$
$$c = \bar{a}q^2 + \bar{b}qs + \bar{c}s^2$$

satisfying $a \equiv 0 \pmod{N_1}, b \equiv \beta \pmod{2N}$, $c \equiv 0 \pmod{N_2}$ *and* $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma(1)$.

(iii) *We have the following $\Gamma_0(N)$-invariant decomposition*:

$$\mathcal{Q}_{D,N,\beta} = \bigcup_{\substack{\ell > 0 \\ \ell^2 \mid D}} \bigcup_{\substack{\lambda\,(2N) \\ \ell\lambda \equiv \beta\,(2N) \\ \lambda^2 \equiv D/\ell^2\,(4N)}} \ell \mathcal{Q}_{D/\ell^2,N,\lambda}^\circ. \tag{2}$$

3. Weierstrass points. In this section we explain a method to compute the fixed points on $X_0(N)$ of W_d for $d > 4$ by using Proposition 2.1. Using this method, we obtain the Weierstrass points on the hyperelliptic curves $X_0(N)$ except for $N = 37$. According to [O1, Theorem 2], if W_d on $X_0(N)$ is a hyperelliptic involution, then $d > 4$. Also, if $d \ne 4$, then W_d does not fix any cusps by [O1, Proposition 3]. Thus, it will suffice to find non-cuspidal fixed points of W_d on $X_0(N)$.

Delaunay [D] suggested an algorithm to find all the fixed points on $\mathbf{H}$ by some matrices W_d. They are infinitely many, and one should find inequivalent points modulo $\Gamma_0(N)$ among them. However, he did not explain how to explicitly choose such points. Later, the author, Kim, and Schweizer [JKS] developed a method to obtain inequivalent points $\Gamma_0(N)$ by using the quadratic forms. We explain their methods in more detail. Suppose $d > 4$. Since W_d has a non-cuspidal fixed point on $X_0(N)$, W_d can be represented by an elliptic element, i.e.

$$W_d = \begin{pmatrix} dx & y \\ Nz & -dx \end{pmatrix}.$$

Then one can easily check that

$$\tau = \frac{2dx + \sqrt{-4d}}{2Nz} \tag{3}$$

is a fixed point of W_d. Conversely, every fixed point has the form (3).

Note that each fixed point in (3) can be considered as the Heegner point of a quadratic form $[Nz, -2dx, -y]$. Thus, if we can find inequivalent quadratic forms $[Nz, -2dx, -y] \bmod \Gamma_0(N)$, then they produce inequivalent points modulo $\Gamma_0(N)$ among the fixed points in (3).

Now we explain how to use Proposition 2.1 to find inequivalent quadratic forms $[Nz, -2dx, -y] \bmod \Gamma_0(N)$. Since the discriminant of the quadratic form of $[Nz, -2dx, -y]$ is $-4d$, take $D = -4d$. Then we can follow the subsequent steps:

(i) We search $\beta \pmod{2N}$ such that

(4) $\beta^2 \equiv -4d \pmod{4N}$ with $\beta \equiv -2dx \pmod{2N}$

where $x \in \mathbf{Z}$.

(ii) We search ℓ and λ as described in Proposition 2.1-(iii).

(iii) For each ℓ and λ obtained in (2), we set the decomposition of $\mathcal{Q}^\circ_{D/\ell^2,N,\lambda}$ as in Proposition 2.1-(ii).

(iv) For each factor in the decomposition of (3), we find the quadratic form representations and taking the inverse image of reduced forms under the map which is described in Proposition 2.1-(ii).

(v) After multiplying ℓ by each quadratic form in (4), we find their Heegner points.

For example, consider $X_0(28)$. According to [O1, Theorem 2], W_7 is the hyperelliptic involution on $X_0(28)$. Since $X_0(28)$ is of genus 2, W_7 has 6 fixed points on $X_0(28)$. Let $D = -28$, then $\beta \equiv \pm 14 \pmod{56}$. First consider the case of $\beta = 14$. Then we have $\ell = 1$ or 2. If $\ell = 1$, then $\lambda = 14$, and we have the following decomposition:

$$\mathcal{Q}^\circ_{-28,28,14} = \mathcal{Q}^\circ_{-28,28,14,2,1} \bigcup \mathcal{Q}^\circ_{-28,28,14,1,2}.$$

If $\ell = 2$, then $\lambda = 35$, and we have the following decomposition:

$$\mathcal{Q}^\circ_{-7,28,35} = \mathcal{Q}^\circ_{-7,28,35,1,1}.$$

By using the BCMATH program by Matthews [M], we obtain the following reduced forms:

$$\mathcal{Q}^\circ_{-28}/\Gamma(1) = \{[1,0,7]\} \text{ and } \mathcal{Q}^\circ_{-7}/\Gamma(1) = \{[1,1,2]\}.$$

By taking the inverse image of reduced forms under the map which is described in Proposition 2.1-(ii), we obtain the following forms:

$$\mathcal{Q}^\circ_{-28,28,14,2,1}/\Gamma_0(28) = \{[28,14,2]\},$$
$$\mathcal{Q}^\circ_{-28,28,14,1,2}/\Gamma_0(28) = \{[56,14,1]\},$$
$$\mathcal{Q}^\circ_{-7,28,35,1,1}/\Gamma_0(28) = \{[28,-21,4]\}.$$

Hence $2\mathcal{Q}^\circ_{-7,28,35,1,1}/\Gamma_0(28) = \{[56,-42,8]\}$. Next consider the case of $\beta = -14$. By the exact same method as in the case of $\beta = 14$, we have the following forms:

$$\mathcal{Q}^\circ_{-28,28,-14,2,1}/\Gamma_0(28) = \{[28,-14,2]\},$$
$$\mathcal{Q}^\circ_{-28,28,-14,1,2}/\Gamma_0(28) = \{[56,-14,1]\},$$
$$\mathcal{Q}^\circ_{-7,28,21,1,1}/\Gamma_0(28) = \{[28,21,4]\}.$$

Hence $2\mathcal{Q}^\circ_{-7,28,21,1,1}/\Gamma_0(28) = \{[56,42,8]\}$. From the above forms, we obtain their Heegner points as follows:

$$\pm\frac{1}{4} + \frac{\sqrt{-7}}{28}, \pm\frac{1}{8} + \frac{\sqrt{-7}}{56}, \pm\frac{3}{8} + \frac{\sqrt{-7}}{56}.$$

They are indeed all the Weierstrass points on $X_0(28)$. By the same method, we obtain all the Weierstrass points on the hyperelliptic curves $X_0(N)$, except for $N = 37, 40, 48$, which are listed in Table I.

Using this method, we have the following result:

Theorem 3.1. *Let N be a positive integer, and $d > 4$ a divisor of N with $\gcd(d, N/d) = 1$. Then the number of fixed points on $X_0(N)$ by W_d is equal to*

$$\sum_{\beta \text{ satisfis } (4)} \#\mathcal{Q}_{-4d,N,\beta}.$$

Now we consider the curve $X_0(40)$. As stated in the introduction, its hyperelliptic involution is

$$\mu = \begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}$$

which is not of Atkin-Lehner type. Consider the matrices

$$U = \begin{pmatrix} 10x & y \\ 40z & -10x \end{pmatrix}$$

with $\det(U) = 20$. Then $\det(U) = -100x^2 - 40yz = 20$, and we have $-5x^2 - 2yz = 1$. Thus, x should be odd. If either y or z is even, then $-5x^2 - 2yz \equiv -x^2 \equiv 1 \pmod 4$, i.e., $x^2 \equiv -1 \pmod 4$, which is a contradiction. Thus, y and z are odd, and

$$\mu^{-1}U = \begin{pmatrix} 5x - 2z & \frac{y+x}{2} \\ 20(3x - z) & 6y + 5x \end{pmatrix}$$

is contained in $\Gamma_0(40)$, hence U define the same involution on $X_0(40)$ as μ. Then one can easily check that

(5) $$\tau = \frac{20x + \sqrt{-80}}{80z}$$

is a fixed point of μ. Since $X_0(40)$ is of genus 3, it is sufficient to find 8 inequivalent points modulo $\Gamma_0(40)$ among the fixed points in (5). In this case, the fixed points in (5) can be viewed as Heegner points of the quadratic forms $[40z, -20x, -y]$. Similar to the fixed points of Atkin-Lehner involutions, by finding inequivalent quadratic forms

Table I. List of all Weierstrass points on the hyperelliptic curves $X_0(N)$ with $N \neq 37$

$X_0(N)$	genus	Weierstrass points
$X_0(22)$	2	$-\frac{1}{2}+\frac{\sqrt{-11}}{22}$, $\pm\frac{1}{4}+\frac{\sqrt{-11}}{44}$, $\pm\frac{1}{6}+\frac{\sqrt{-11}}{66}$, $-\frac{1}{12}+\frac{\sqrt{-11}}{132}$
$X_0(23)$	2	$\frac{\sqrt{-23}}{23}$, $\pm\frac{1}{6}+\frac{\sqrt{-23}}{138}$, $\pm\frac{1}{8}+\frac{\sqrt{-23}}{184}$, $-\frac{1}{12}+\frac{\sqrt{-23}}{276}$
$X_0(26)$	2	$\frac{\sqrt{-26}}{26}$, $\pm\frac{1}{3}+\frac{\sqrt{-26}}{78}$, $\pm\frac{2}{5}+\frac{\sqrt{-26}}{130}$, $-\frac{5}{31}+\frac{\sqrt{-26}}{806}$,
$X_0(28)$	2	$\pm\frac{1}{4}+\frac{\sqrt{-7}}{28}$, $\pm\frac{1}{8}+\frac{\sqrt{-7}}{56}$, $\pm\frac{3}{8}+\frac{\sqrt{-7}}{56}$
$X_0(29)$	2	$\frac{\sqrt{-29}}{29}$, $\pm\frac{1}{3}+\frac{\sqrt{-29}}{87}$, $\pm\frac{1}{5}+\frac{\sqrt{-29}}{145}$, $-\frac{1}{15}+\frac{\sqrt{-29}}{435}$
$X_0(30)$	3	$-\frac{1}{2}+\frac{\sqrt{-15}}{30}$, $\pm\frac{1}{4}+\frac{\sqrt{-15}}{60}$, $\pm\frac{1}{6}+\frac{\sqrt{-15}}{120}$, $-\frac{3}{8}+\frac{\sqrt{-15}}{120}$, $-\frac{1}{16}+\frac{\sqrt{-15}}{240}$, $-\frac{3}{34}+\frac{\sqrt{-15}}{510}$
$X_0(31)$	2	$\frac{\sqrt{-31}}{31}$, $\pm\frac{1}{4}+\frac{\sqrt{-31}}{124}$, $\pm\frac{2}{5}+\frac{\sqrt{-31}}{155}$, $-\frac{1}{16}+\frac{\sqrt{-31}}{496}$
$X_0(33)$	3	$\pm\frac{1}{3}+\frac{\sqrt{-11}}{33}$, $\pm\frac{1}{6}+\frac{\sqrt{-11}}{66}$, $\pm\frac{2}{9}+\frac{\sqrt{-11}}{99}$, $\pm\frac{1}{12}+\frac{\sqrt{-11}}{132}$
$X_0(35)$	3	$\frac{\sqrt{-35}}{35}$, $\pm\frac{1}{4}+\frac{\sqrt{-35}}{140}$, $-\frac{1}{6}+\frac{\sqrt{-35}}{210}$, $\pm\frac{1}{12}+\frac{\sqrt{-35}}{420}$, $-\frac{5}{12}+\frac{\sqrt{-35}}{420}$, $-\frac{1}{18}+\frac{\sqrt{-35}}{630}$
$X_0(39)$	3	$\frac{\sqrt{-39}}{39}$, $\pm\frac{1}{4}+\frac{\sqrt{-39}}{156}$, $\pm\frac{1}{8}+\frac{\sqrt{-39}}{312}$, $-\frac{3}{8}+\frac{\sqrt{-39}}{312}$, $-\frac{5}{16}+\frac{\sqrt{-39}}{624}$, $-\frac{1}{20}+\frac{\sqrt{-39}}{780}$
$X_0(40)$	3	$\pm\frac{1}{4}+\frac{\sqrt{-5}}{20}$, $\pm\frac{1}{12}+\frac{\sqrt{-5}}{60}$, $\pm\frac{5}{12}+\frac{\sqrt{-5}}{60}$, $\pm\frac{5}{36}+\frac{\sqrt{-5}}{180}$
$X_0(41)$	3	$\frac{\sqrt{-41}}{41}$, $-\frac{1}{2}+\frac{\sqrt{-41}}{82}$, $\pm\frac{1}{3}+\frac{\sqrt{-41}}{123}$, $\pm\frac{1}{6}+\frac{\sqrt{-41}}{246}$, $\pm\frac{2}{5}+\frac{\sqrt{-41}}{205}$
$X_0(46)$	5	$-\frac{1}{2}+\frac{\sqrt{-23}}{46}$, $\pm\frac{1}{4}+\frac{\sqrt{-23}}{92}$, $\pm\frac{1}{6}+\frac{\sqrt{-23}}{138}$, $\pm\frac{1}{8}+\frac{\sqrt{-23}}{184}$, $\pm\frac{3}{8}+\frac{\sqrt{-23}}{184}$, $\pm\frac{1}{12}+\frac{\sqrt{-23}}{276}$, $-\frac{1}{24}+\frac{\sqrt{-23}}{552}$
$X_0(47)$	4	$\frac{\sqrt{-47}}{47}$, $\pm\frac{1}{3}+\frac{\sqrt{-47}}{141}$, $\pm\frac{1}{4}+\frac{\sqrt{-47}}{188}$, $\pm\frac{1}{6}+\frac{\sqrt{-47}}{282}$, $\pm\frac{2}{9}+\frac{\sqrt{-47}}{423}$, $-\frac{1}{24}+\frac{\sqrt{-47}}{1128}$
$X_0(48)$	3	$\pm\frac{1}{8}+\frac{\sqrt{-3}}{24}$, $\pm\frac{3}{8}+\frac{\sqrt{-3}}{24}$, $\pm\frac{3}{56}+\frac{\sqrt{-3}}{168}$, $\pm\frac{17}{56}+\frac{\sqrt{-3}}{168}$
$X_0(50)$	2	$\frac{\sqrt{-2}}{10}$, $\pm\frac{3}{11}+\frac{\sqrt{-2}}{110}$, $\pm\frac{1}{17}+\frac{\sqrt{-2}}{170}$, $-\frac{7}{43}+\frac{\sqrt{-2}}{430}$
$X_0(59)$	5	$\frac{\sqrt{-59}}{46}$, $-\frac{1}{2}+\frac{\sqrt{-59}}{118}$, $\pm\frac{1}{3}+\frac{\sqrt{-59}}{177}$, $\pm\frac{1}{4}+\frac{\sqrt{-59}}{236}$, $\pm\frac{1}{5}+\frac{\sqrt{-59}}{295}$, $\pm\frac{1}{6}+\frac{\sqrt{-59}}{354}$, $\pm\frac{3}{7}+\frac{\sqrt{-59}}{413}$
$X_0(71)$	6	$\frac{\sqrt{-71}}{71}$, $-\frac{1}{2}+\frac{\sqrt{-71}}{142}$, $\pm\frac{1}{3}+\frac{\sqrt{-71}}{213}$, $\pm\frac{1}{4}+\frac{\sqrt{-71}}{284}$, $\pm\frac{2}{5}+\frac{\sqrt{-71}}{355}$, $\pm\frac{1}{6}+\frac{\sqrt{-71}}{426}$, $\pm\frac{1}{8}+\frac{\sqrt{-71}}{568}$, $\pm\frac{3}{8}+\frac{\sqrt{-71}}{568}$

$[40z, -20x, -y]$ modulo $\Gamma_0(40)$, we can obtain inequivalent points modulo $\Gamma_0(40)$ among the fixed points in (5). We apply this method to obtain the Weierstrass points on $X_0(40)$ which are listed in Table I. Compared with the case of Atkin-Lehner involutions W_d, the discriminant is not $-4d$ but $-8d$ if we regard d as 10. By the same method as $X_0(40)$, we obtain the Weierstrass points on $X_0(48)$ which are listed in Table I as well.

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