Termination of extremal rays of divisorial type for the power of étale endomorphisms

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Abstract: Let $f: X \to X$ be a non-isomorphic étale endomorphism of a smooth projective variety X. Suppose that there exists a K_X-negative extremal ray $R' \subset \overline{\mathrm{NE}}(X)$ of fiber type. Then we give a sufficient condition for a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$ of divisorial type to terminate under a suitable power f^k of $k > 0$.

Key words: Endomorphism; extremal ray; termination; divisorial contraction.

1. Introduction. The purpose of this note is to give a partial answer to the following question concerning termination of extremal rays of divisorial type on a smooth projective variety for the iteration of non-isomorphic étale endomorphisms.

Question. Let $f: X \to X$ be a non-isomorphic étale endomorphism of a smooth projective variety X. Suppose that there exists an extremal ray $R' \subset \overline{\mathrm{NE}}(X)$ such that $K_X R' < 0$ and the associated contraction morphism $\varphi := \mathrm{Cont}_{R'}: X \to Y$ is a fibration to a lower dimensional variety Y. Then is it true that for any K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$ of divisorial type, there exists a positive integer k such that $(f^k)_*(R) = R$ for the automorphism $(f^k)_*: \mathrm{N}_1(\mathrm{X}) \simeq \mathrm{N}_1(\mathrm{X})$ induced from the k-th power $f^k = f \circ \cdots \circ f$?

If we apply the minimal model program (MMP, for short, cf. [6]) to the study of non-isomorphic surjective endomorphisms of projective varieties, we sometimes encounter serious troubles: Let $f: X \to X$ be a non-isomorphic étale endomorphism. Then a K_X-negative extremal ray $R(\subset \overline{\mathrm{NE}}(X))$ is not necessarily preserved under a suitable power $(f^k)_*$ $(k > 0)$ of the push-foward mapping f_*: $N_1(X) \simeq N_1(X)$ (cf. [3]). Hence it is not at all clear that we can apply the MMP working compatibly with étale endomorphisms. Thus it is an interesting problem to give a sufficient condition for a K_X-negative extremal ray R to be preserved under a suitable power of f. A special case of the **Question** was studied in [3, Theorems 1.4, 8.6 and 9.6], which gives an affirmative answer to the **Question** in the case where $\dim X = 3$ and $\dim Y = 2$. In this note, we shall give some generalization.

Theorem 1.1. *The* **Question** *has an affirmative answer under the assumption that* $\rho(Y) = 2$.

If we drop the assumption that $\rho(Y) = 2$, then the **Question** does not necessarily have an affirmative answer. In Section 4, we shall give an easy counterexample using a rational elliptic surface with infinitely many (-1)-curves.

2. Notations and preliminaries. In this paper, we work over the complex number field $\mathbf{C}$. A projective variety is a complex variety embedded in a projective space. By an endomorphism $f: X \to X$, we mean a morphism from a projective variety X to itself.

The following symbols are used for a variety X.

K_X: the canonical divisor of X.

$\mathrm{Aut}(X)$: the algebraic group of automorphisms of X.

$N_1(X) := (\{1\text{-cycles on } X\}/\equiv) \otimes_{\mathbf{Z}} \mathbf{R}$, where $\equiv$ means a numerical equivalence.

$N^1(X) := (\{\text{Cartier divisors on } X\}/\equiv) \otimes_{\mathbf{Z}} \mathbf{R}$, where $\equiv$ means a numerical equivalence.

$\mathrm{NE}(X)$: the smallest convex cone in $N_1(X)$ containing all effective 1-cycles.

$\overline{\mathrm{NE}}(X)$: the Kleiman-Mori cone of X, i.e., the closure of $\mathrm{NE}(X)$ in $N_1(X)$ for the metric topology.

$\rho(X) := \dim_{\mathbf{R}} N_1(X)$, the Picard number of X.

$[C]$: the numerical equivalence class of a 1-cycle C.

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cl(D): the numerical equivalence class of a Cartier divisor D.

$\sim_{\mathbf{Q}}$: the $\mathbf{Q}$-linear equivalence of $\mathbf{Q}$-divisors of X.

For an endomorphism $f: X \to X$ and an integer $k > 0$, f^k stands for the k-times composite $f \circ \cdots \circ f$ of f.

Extremal rays. For a smooth projective variety X, an extremal ray R means a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$, i.e., a 1-dimensional face of $\overline{\mathrm{NE}}(X)$ with $K_X R < 0$. An extremal ray R defines a proper surjective morphism with connected fibers $\mathrm{Cont}_R: X \to Y$ such that, for an irreducible curve $C \subset X$, $\mathrm{Cont}_R(C)$ is a point if and only if $[C] \in R$. This is called the contraction morphism associated to R.

We recall the following fundamental result.

Lemma 2.1 (cf. [1, Propositions 4.2 and 4.12])**.** *Let $f: Y \to X$ be a finite surjective morphism between smooth projective n-folds with $\rho(X) = \rho(Y)$. Then the following hold.*

(1) *The push-forward map $f_*: N_1(Y) \to N_1(X)$ is an isomorphism and $f_*\overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(X)$.*

(2) *Let $f_*: N^1(Y) \to N^1(X)$ be the map induced from the push-forward map $D \mapsto f_*D$ of divisors. Then the dual map $f^*: N_1(X) \to N_1(Y)$ is an isomorphism and $f^*\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(Y)$.*

(3) *If f is étale and K_X is not nef, then f^* and f_* above give a one-to-one correspondence between the set of extremal rays of X and Y.*

(4) *Under the same assumption as in (3), for an extremal ray $R(\subset \overline{\mathrm{NE}}(Y))$, and for the contraction morphisms $\mathrm{Cont}_R: Y \to Y'$ and $\mathrm{Cont}_{f_*R}: X \to X'$, there exists a finite surjective morphism $f': Y' \to X'$ such that $f' \circ \mathrm{Cont}_R = \mathrm{Cont}_{f_*R} \circ f$.*

The following theorem and some arguments used in its proof play another key role for proving our main Theorem 1.1.

Theorem 2.2 (cf. [7, Lemma 6.2])**.** *Let X be a normal $\mathbf{Q}$-factorial projective variety with at most log-canonical singularities and $f: X \to X$ a surjective endomorphism. Let $R(\subset \overline{\mathrm{NE}}(X))$ be an extremal ray and $\pi := \mathrm{Cont}_R: X \to Y$ the contraction morphism associated to R. Suppose that $E(\subset X)$ be a subvariety such that $\dim(\pi(E)) < \dim(E)$ and $f^{-1}(E) = E$. Then $(f^k)_*(R) = R$ for some positive integer k.*

Remark 2.3. In [7, Lemma 6.2], the assumption that 'X is $\mathbf{Q}$-factorial' is missing.

3. Proof of Theorem 1.1. Let $f: X \to X$ be a non-isomorphic étale endomorphism of a smooth projective variety X. Suppose that there exists an extremal ray $R' \subset \overline{\mathrm{NE}}(X)$ such that for the contraction morphism $\varphi := \mathrm{Cont}_{R'}: X \to Y$ associated to R' is a Mori fiber space (i.e., $\dim Y < \dim X$). Then by [5, Lemma 5-1-5], Y is $\mathbf{Q}$-factorial. **At first, we impose no restrictions on the Picard number $\boldsymbol{\rho(Y)}$ of $\boldsymbol{Y}$.** Then by [2], there exists a positive integer k such that $(f^k)_*R' = R'$. Hence replacing f by its power f^k, we may assume from the beginning that $f_*R' = R'$. Hence there exists a surjective endomorphism $g: Y \to Y$ such that $\varphi \circ f = g \circ \varphi$. Since φ is a Fano fibration, the general fiber of φ is simply-connected. Since f is étale, $X \xrightarrow{\varphi} Y \xrightarrow{g} Y$ gives a Stein factorization of $\varphi \circ f: X \to Y$ and g is also a non-isomorphic étale endomorphism with $\deg(g) = \deg(f)$. Thus we have the following Cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow{g} & Y. \end{array}$$

We take an arbitrary extremal ray $R(\subset \overline{\mathrm{NE}}(X))$ of *divisorial* type. We put $R_n := (f^n)_*R$ for $n > 0$ and $R_0 := R$. Then, by Lemma 2.1, $R_n(\subset \overline{\mathrm{NE}}(X))$ is also an extremal ray for any $n \geq 0$ and let $\psi_n := \mathrm{Cont}_{R_n}: X \to Z_n$ be a divisorial contraction associated to R_n. Then, by Lemma 2.1, there is induced a non-isomorphic finite morphism $g_n: Z_n \to Z_{n+1}$ which is étale in codimension one such that $\psi_{n+1} \circ f = g_n \circ \psi_n$ and $\deg(g_n) = \deg(f)$ for any $n \geq 0$. In summary, there exists another commutative diagram below:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\psi_n}\downarrow & & \downarrow{\scriptstyle\psi_{n+1}} \\ Z_n & \xrightarrow{g_n} & Z_{n+1}. \end{array}$$

This diagram is Cartesian over a non-singular locus Z^0_{n+1} of Z_{n+1}. Let $E_n := \mathrm{Exc}(\psi_n)$ be the ψ_n-exceptional divisor. Then $f^{-1}(E_{n+1}) = E_n$ for any $n \geq 0$. Let $\gamma(\subset X)$ be an extremal rational curve whose numerical class $[\gamma]$ spans the extremal ray R'. Then γ is contracted to a point by φ. Since each E_n is effective, the restriction of E_n to a general fiber of φ is also effective. Hence $(E_n, \gamma) \geq 0$ for any $n \geq 0$.

Lemma 3.1. *We have two cases*;

(1) $(E_n, \gamma) = 0$ *for any* $n \geq 0$, *or*

(2) $(E_n,\gamma)>0$ *for any* $n\geq 0$.

Proof. Since $f_*R'=R'$, we have $f_*\gamma\equiv a\gamma$ for a positive rational number a. If we set $E:=E_0$, then $E=(f^n)^*E_n$. Hence by the projection formula, we see that $(E,\gamma)=(E_n,(f^n)_*\gamma)=a^n(E_n,\gamma)$ for any $n\geq 0$, which shows the claim. □

We set $k:=\dim Y$. Let ℓ_n be an extremal rational curve whose numerical class $[\ell_n]$ spans the extremal ray R_n, i.e., $R_n=\mathbf{R}_{\geq 0}[\ell_n]$.

Case (1). First, we consider the case that $(E_n,\gamma)=0$ for any $n\geq 0$. Then by [4], we see that for any $n\geq 0$, there exists a Cartier divisor D_n on Y such that $E_n\sim\varphi^*D_n$. Since $E_n=f^*E_{n+1}$ and $\varphi\circ f=g\circ\varphi$, we see that $E_n\sim\varphi^*g^*D_{n+1}$. Hence $D_n\sim g^*D_{n+1}$ for any $n\geq 0$. We have the following easy lemma.

Lemma 3.2. *D_n is not nef and $(D_n)^k=0$ for any $n\geq 0$.*

Proof. Since $R_n\neq R'$, $\varphi(\ell_n)$ is an irreducible curve on Y. Hence, by the projection formula, we see that $(D_n,\varphi_*\ell_n)=(\varphi^*D_n,\ell_n)=(E_n,\ell_n)<0$. Thus D_n is not nef. For the latter assertion, we may assume that $n=0$ without loss of generality. Since $D_n\sim g^*D_{n+1}$, we see that $(D_n)^k=\deg(g)(D_{n+1})^k$ for any $n\geq 0$. Hence $(D_0)^k=(\deg(g))^n(D_n)^k$ for any $n\geq 0$. Suppose that $(D_0)^k\neq 0$. Since $\deg(g)=\deg(f)\geq 2$, we see that $0<(D_n)^k<1$ for a sufficiently positive integer n. This contradicts the fact that $(D_n)^k\in\mathbf{Z}$. Thus $(D_0)^k=0$. □

Case (2). Next, we consider the case that $(E_n,\gamma)>0$ for any $n\geq 0$. We begin with an easy lemma.

Lemma 3.3. *We have $f_*\gamma\equiv\gamma$.*

Proof. Since $f_*R'=R'$, we have $f_*\gamma\equiv a\gamma$ for a positive rational number a. We set $E:=E_0$. Then by the projection formula, we have $((f^n)^*E,\gamma)=(E,(f^n)_*\gamma)=a^n(E,\gamma)>0$ for any $n>0$. Hence, if $0<a<1$, then $0<((f^n)^*E,\gamma)<1$ for a sufficiently positive integer n, which contradicts the fact that $((f^n)^*E,\gamma)\in\mathbf{Z}$. Suppose that $a>1$. Then by the same argument as in the proof of Lemma 3.1, we have $(E,\gamma)=a^n(E_n,\gamma)$ for any $n>0$. Thus $0<(E_n,\gamma)<1$ for a sufficiently positive integer n, which contradicts the fact that $(E_n,\gamma)\in\mathbf{Z}$. Hence $a=1$. □

Lemma 3.4. *There exists a positive rational number b such that $(K_X+bE_n,\gamma)=0$ for any $n\geq 0$.*

Proof. Since $(K_X,\gamma)<0$ and $(E_n,\gamma)>0$, there exists a unique positive rational number b_n such that $(K_X+b_nE_n,\gamma)=0$ for any $n\geq 0$. Combining Lemma 3.3 with the fact that $E_{n-1}=f^*E_n$, the projection formula shows that $(E_{n-1},\gamma)=(E_n,f_*\gamma)=(E_n,\gamma)$. Since

$$b_n=\frac{(-K_X,\gamma)}{(E_n,\gamma)},$$

we see that $b_{n-1}=b_n$ for any $n>0$. Thus $b_n\equiv b$ is indedependent of n. □

Then by [4], for any $n\geq 0$, there exists a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D'_n on Y such that $K_X+bE_n\sim_{\mathbf{Q}}\varphi^*D'_n$.

Lemma 3.5. *Each D'_n is not nef and $D'_{n-1}\sim_{\mathbf{Q}}g^*D'_n$ for any $n>0$. In particular, D'_n is not numerically trivial.*

Proof. Since $(K_X,\ell_n)<0,(E_n,\ell_n)<0$ and $b>0$, we see by the projection formula that $(D'_n,\varphi_*\ell_n)=(\varphi^*D'_n,\ell_n)=(K_X,\ell_n)+b(E_n,\ell_n)<0$. Since $R_n\neq R'$, $\varphi(\ell_n)$ is an irreducible rational curve on Y and thus each D'_n is not nef. Since $K_X\sim f^*K_X$ and $f^*E_n=E_{n-1}$, we see that

$$\begin{aligned}K_X+bE_{n-1}&\sim_{\mathbf{Q}}f^*(K_X+bE_n)\sim_{\mathbf{Q}}f^*\varphi^*D'_n\\&=\varphi^*g^*D'_n.\end{aligned}$$

Since $K_X+bE_{n-1}\sim_{\mathbf{Q}}\varphi^*D'_{n-1}$, we have $D'_{n-1}\sim_{\mathbf{Q}}g^*D'_n$ for any $n>0$. □

Lemma 3.6. *For any $n\geq 0$, the self-intersection number of D'_n is zero, i.e., $(D'_n)^k=0$.*

Proof. By Lemma 3.4,

$$b\equiv\frac{(-K_X,\gamma)}{(E_n,\gamma)}$$

is independent of n. Hence a positive integer (E_n,γ) is also independent of n. We set $d:=(E_n,\gamma)$ and let $\Gamma_n:=d(K_X+bE_n)$ be a Cartier divisor on X. Since $(\Gamma_n,\gamma)=0$, by [4], the Cartier divisor Γ_n is linearly equivalent to the pullback of some Cartier divisor on Y by φ. Hence for any $n\geq 0$, $\Delta_n:=dD'_n$ is a Cartier divisor on Y and $\Gamma_n\sim\varphi^*\Delta_n$. Since $\Gamma_n\sim f^*\Gamma_{n+1}\sim f^*\varphi^*\Delta_{n+1}=\varphi^*g^*\Delta_{n+1}$, we have $\Delta_n\sim g^*\Delta_{n+1}$. Since $\deg(g)>1$, applying the same method as in the proof of Lemma 3.3 to the Cartier divisor Δ_n on Y, we see that $(\Delta_n)^k=0$. Hence $(D'_n)^k=0$ for any $n\geq 0$. □

Corollary 3.7. *We have* $\dim Y\geq 2$ *and* $\rho(Y)\geq 2$.

Proof. If $\dim(Y)\leq 1$, then by Lemma 3.6, D'_n is numerically trivial, which contradicts Lemma 3.5. If $\rho(Y)=1$, then by Lemma 3.5, $-D'_n$ is ample, which contradicts Lemma 3.6. □

Proof of Theorem 1.1. Hereafter, **we always assume that $\boldsymbol{\rho(Y)=2}$**. Let $N^1_{\mathbf{C}}(Y) := N^1(Y) \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of the 2-dimensional $\mathbf{R}$-vector space $N^1(Y)$. We set $H := \{D \in N^1_{\mathbf{C}}(Y) | D^k = 0\}$. Then H is an affine curve in $N^1_{\mathbf{C}}(Y) \simeq \mathbf{C}^2$.

Case (1). First, we consider the case where $(E_n, \gamma) = 0$ for any $n \geq 0$. For each $n \geq 0$, let $L_n := \langle \mathrm{cl}(D_n)\rangle_{\mathbf{C}}$ be the 1-dimensional complex vector space spanned by $\mathrm{cl}(D_n)$. Then by Lemma 3.2, each affine line L_n is an irreducible component of H. The pull-back $g^*: N^1_{\mathbf{C}}(Y) \simeq N^1_{\mathbf{C}}(Y)$ induces an automorphism of H. Since $g^*D_n \sim D_{n-1}$, there is induced an isomorphism $g^*|_{L_n}: L_n \simeq L_{n-1}$ for any $n > 0$. We have the following commutative diagram:

$$\begin{array}{ccc} N^1_{\mathbf{C}}(Y) & \xrightarrow{g^*} & N^1_{\mathbf{C}}(Y) \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j} \\ H & \xrightarrow{g^*|_H} & H \\ {\scriptstyle i_n}\uparrow & & \uparrow{\scriptstyle i_{n-1}} \\ L_n & \xrightarrow{g^*|_{L_n}} & L_{n-1}, \end{array}$$

where $j: H \hookrightarrow N^1_{\mathbf{C}}(Y)$, $i_n: L_n \hookrightarrow H$ and $i_{n-1}: L_{n-1} \hookrightarrow H$ are all inclusions. The automorphism $g^*|_H: H \simeq H$ induces a permutation of a finite number of irreducible components of H. Hence, replacing g (resp. f) by its suitable power $g^k (k > 0)$ (resp. f^k), there exists some positive integer n such that $L_n = g^*L_n = L_{n-1}$. Pulling back by $\varphi: X \to Y$, we have $E_n \equiv \lambda E_{n-1}$ for some $\lambda \in \mathbf{Q}_{>0}$, since any E_n is a non-zero effective divisor. Since $(\ell_{n-1}, E_n) = \lambda(\ell_{n-1}, E_{n-1}) < 0$, we see that $\ell_{n-1} \subset E_n$. Since ℓ_{n-1} sweeps out E_{n-1}, we have $E_{n-1} \subset E_n$. Thus $E_{n-1} = E_n$, since E_n is irreducible. Hence $f^{-1}(E_n) = E_{n-1} = E_n$ and we can apply Theorem 2.2 to the ψ_n-exceptional divisor E_n. There exists a positive integer p such that $(f^p)_* R_n = R_n$. Since $f_*: N_1(X) \simeq N_1(X)$ is an automorphism, we have $(f^p)_* R = R$.

Case (2). Next, we consider the case where $(E_n, \gamma) > 0$ for any $n \geq 0$. For each $n \geq 0$, let $L'_n := \langle \mathrm{cl}(D'_n)\rangle_{\mathbf{C}}$ be the 1-dimensional complex vector space spanned by $\mathrm{cl}(D'_n)$. Then by Lemma 3.6, each affine line L'_n is an irreducible component of the affine curve H. Since $f^*(K_X + bE_n) \sim_{\mathbf{Q}} K_X + bE_{n-1}$, we see that $g^*(D'_n) \sim_{\mathbf{Q}} D'_{n-1}$. Thus the automorphism $g^* \in \mathrm{Aut}(N^1(Y)_{\mathbf{C}})$ induces an isomorphism $g^*|_{L'_n}: L'_n \simeq L'_{n-1}$ for each $n > 0$. Then we can apply the same argument as in the **Case (1)**. After replacing g (resp. f) by its suitable power g^k $(k > 0)$ (resp. f^k), there exists some positive integer n such that $L'_n = g^*L'_n = L'_{n-1}$. Pulling back by $\varphi: X \to Y$, we have $K_X + bE_{n-1} \equiv \lambda(K_X + bE_n)$ for some $\lambda \in \mathbf{Q}$. We have $\lambda \neq 0$, since L'_{n-1} is not numericall trivial by Lemma 3.5. If $\lambda < 0$, then we replace f by f^2. Thus we may assume from the beginning that $\lambda > 0$.

Lemma 3.8. *We have* $\lambda = 1$.

Proof. Suppose that $0 < \lambda < 1$. Since $\lambda b E_n \equiv (1-\lambda)K_X + bE_{n-1}$, we have

$$\lambda b(E_n, \ell_{n-1}) = (1-\lambda)(K_X, \ell_{n-1}) + b(E_{n-1}, \ell_{n-1}) < 0.$$

Hence $(E_n, \ell_{n-1}) < 0$ and ℓ_{n-1} is contained in E_n. Since ℓ_{n-1} sweeps out E_{n-1}, we have $E_{n-1} \subset E_n$. Thus $E_{n-1} = E_n$, since E_n is irreducible. Then $K_X + bE_n \equiv \lambda(K_X + bE_n)$, which shows that $K_X + bE_n \equiv 0$. This contradicts Lemma 3.5. Next, suppose that $\lambda > 1$. Since $bE_{n-1} \equiv (\lambda - 1)K_X + b\lambda E_n$, we have

$$b(E_{n-1}, \ell_n) = (\lambda - 1)(K_X, \ell_n) + b\lambda(E_n, \ell_n) < 0.$$

Hence $(E_{n-1}, \ell_n) < 0$. By the same argument as above, we have $E_{n-1} = E_n$ and $K_X + bE_n \equiv 0$, which again contradicts Lemma 3.5. □

By Lemma 3.8, we have $E_{n-1} \equiv E_n$. Then applying the same argument as in the **Case (1)**, we see that $E_{n-1} = E_n$. Thus $E_n = E_{n-1} = f^{-1}(E_n)$. Hence we can apply Theorem 2.2 to the ψ_n-exceptional divisor E_n. By the same argument as in the **Case (1)**, we have $(f^p)_* R = R$ for some integer $p > 0$. □

4. Counterexamples. The **Question** does not necessarily have an affirmative answer if we drop the assumption that $\rho(Y) = 2$. We shall construct such an example (cf. [2], Remark *A*.9). Let S be a rational elliptic surface with global sections whose Mordell-Weil rank is positive. It is obtained as 9-points blowing-up of $\mathbf{P}^2$. We regard S as an elliptic curve C_K defined over the function field K of the base curve C. Since S is relatively minimal, the translation mapping $C_K \to C_K$ given by the non-torsion section γ induces a relative automorphism $t: S \simeq S$ over C, which is of infinite order. Let $X := S \times E \times \mathbf{P}^1$ be the product variety of S, an elliptic curve E and $\mathbf{P}^1$. We take a point $o \in E$ and a point $0 \in \mathbf{P}^1$. Since γ is a (-1)-curve on S, the curve $\gamma \times \{o\} \times \{0\}$ on X spans the extremal ray $R(\subset \overline{\mathrm{NE}}(X))$ of divisorial type. If we denote by

$\mu_k: E \to E$ multiplication by $k > 1$, then the product mapping $f := t \times \mu_k \times \mathrm{id}_{\mathbf{P}^1}: X \to X$ gives a non-isomorphic étale endomorphism of X. We set $Y := S \times E$ and let $p_{1,2}: X \to Y$ and $p: Y \to S$ be natural projections. Then $p_{1,2}$ is a trivial $\mathbf{P}^1$-bundle, which is a Mori fiber space. Furthermore, the product mapping $g := t \times \mu_k: Y \to Y$ also gives a non-isomorphic étale endomorphism of Y.

We have the following Cartesian diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle p_{1,2}}\downarrow & & \downarrow{\scriptstyle p_{1,2}} \\ Y & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{t} & S. \end{array}$$

If $(f^n)_* R = R$ for some $n > 0$, then $t^n(\gamma) = \gamma$. This is impossible, since $t \in \mathrm{Aut}(S/C)$ is of infinite order. Hence, $(f^n)_* R \neq R$ for any $n > 0$. In this case, $\rho(S) = 10$ and $\rho(Y) \geq 10$. Let $\psi := \mathrm{Cont}_R: X \to Z$ be the extremal contraction associated to R and $\Delta := \mathrm{Exc}(\psi)$ the ψ-exceptional divisor. Then $\Delta \simeq \gamma \times E \times \mathbf{P}^1$ does not intersect with the general fiber of $p_{1,2}$. Thus our counterexample corresponds to the **Case (1)** in Section 3.

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