Hypertranscendence of the multiple sine function for a complex period

By Masaki KATO

Department of Mathematics, Graduate School of Science, Kobe University,
1-1 Rokkodai, Nada-ku, Kobe 657-8501, Japan

(Communicated by Masaki KASHIWARA, M.J.A., Jan. 15, 2019)

Abstract: It is known that the multiple sine function for a "rational" period satisfies an algebraic differential equation. However, for a non-"rational" period, the differential algebraicity of the multiple sine function is obscure. In this paper, we prove that, if there exists a non-real element in the set $\{\omega_j/\omega_i | 1 \leq i < j \leq r\}$, the multiple sine function $\mathrm{Sin}_r(x, (\omega_1, \cdots, \omega_r))$ does not satisfy any algebraic differential equation.

Key words: Multiple sine function; algebraic differential equations; hypertranscendence.

Let $\omega_1, \cdots, \omega_r \in \mathbf{C}$ all lie on the same side of some straight line through the origin. We put $\boldsymbol{\omega} := (\omega_1, \cdots, \omega_r)$. We define the multiple Hurwitz zeta, gamma and sine functions by

$$\zeta_r(s, x, \boldsymbol{\omega}) = \sum_{n_1,\ldots,n_2=0}^{\infty} (x + n_1\omega_1 + \cdots + n_r\omega_r)^{-s},$$

$$\Gamma_r(x, \boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, x, \boldsymbol{\omega})\Big|_{s=0}\right),$$

$$\mathrm{Sin}_r(x, \boldsymbol{\omega}) = \Gamma_r(x, \boldsymbol{\omega})^{-1}\Gamma_r(\omega_1 + \cdots + \omega_r - x, \boldsymbol{\omega})^{(-1)^r}.$$

The multiple gamma and sine functions were introduced by Barnes [2–4] and Kurokawa [9], respectively. When $r = 1$, the function Sin_1 is the usual sine function:

$$\mathrm{Sin}_1(x, \omega) = 2\sin\left(\frac{\pi x}{\omega}\right).$$

It is known that the multiple sine function has interesting applications: the Kronecker limit formula for real quadratic fields ([13]), expressions of special values of the Riemann zeta and Dirichlet L-functions ([9]), the calculation of the gamma factors of Selberg zeta functions ([9]), expression of solutions to the quantum Knizhnik-Zamolodchikov equation ([7]) and so on. Concerning basic properties of the multiple sine functions, we refer to [9].

The multiple sine function has similar properties to the usual sine function. Kurokawa and Wakayama [11] showed that, when the period $\boldsymbol{\omega}$ is "rational", that is, there exists a positive number c satisfying $\boldsymbol{\omega} \in c \cdot \mathbf{Q}^r$, $\mathrm{Sin}_r(x, \boldsymbol{\omega})$ satisfies the algebraic differential equation

$$F(x, y, y', \cdots, y^{(n)}) = 0$$
$$(n \in \mathbf{Z}_{\geq 0}, F(x, Y_0, Y_1, \cdots, Y_n) \in \mathbf{C}(x)[Y_0, Y_1, \cdots, Y_n]).$$

In particular, when $\boldsymbol{\omega} = (1, \cdots, 1)$, $y = \mathrm{Sin}_r(x, \boldsymbol{\omega})$ satisfies the algebraic differential equation

$$y'' + (\pi Q_r(x)^{-1} - 1)(y')^2 y^{-1} - Q_r'(x)Q_r(x)^{-1}y' + \pi Q_r(x)y = 0$$

with $Q_r(x) = (-1)^{r-1}\pi\binom{x-1}{r-1}$. (See [10, Theorem 2.2(d)].) However, for a general period $\boldsymbol{\omega}$, the differential algebraicity of the multiple sine function is still obscure.

On the other hand, in [8], we showed that the double cotangent function $\mathrm{Cot}_2(x, (1, \tau))$ (the logarithmic derivative of the double sine function) degenerates to the digamma function (the logarithmic derivative of the gamma function) as τ tends to infinity. This proposition enables us to regard the double cotangent function as a generalization of the digamma function. Thus it is natural to ask whether properties of the digamma and gamma functions can be extended to the double cotangent and sine functions.

One of the important properties of the gamma function is its hypertranscendence: It does not satisfy any algebraic differential equation. This theorem was proved by Hölder [6].

The purpose of this paper is, by generalizing this Hölder's proof, to show the hypertranscendence of the multiple sine function for a "complex" period:

Theorem 0.1. *Let $r \geq 2$. If there exists a non-real element in the set $\{\omega_j/\omega_i | 1 \leq i < j \leq r\}$,*

2010 Mathematics Subject Classification. Primary 33E30, 11J81, 11J91.

doi: 10.3792/pjaa.95.16

then the r-ple sine function $\mathrm{Sin}_r(x, \boldsymbol{\omega})$ *is hypertranscendental.*

When all elements in the set $\{\omega_i/\omega_j | 1 \leq i < j \leq r\}$ are positive real number and at least one element is irrational, it remains unclear whether or not the r-ple sine function $\mathrm{Sin}_r(x, \boldsymbol{\omega})$ is hypertranscendental. It may be possible that a totally different method from ours (for example, the Galois correspondence in differential Galois theory) provides a solution to this problem.

1. Hypertranscendence of a solution of a certain difference equation. In this section, by generalizing the argument of Hölder [6], we establish the following general result, which will be used in the proof of Theorem 0.1.

Proposition 1.1. *If a function $f(x)$ satisfies the difference equation*

$$f(x+\tau) = f(x)(2\sin(\pi x))^{-1} \tag{1.1}$$

for a non-real constant τ, then $f(x)$ is hypertranscendental over $\mathbf{C}(x, e^{\pi i x})$; *that is, $y = f(x)$ does not satisfy any algebraic differential equation over* $\mathbf{C}(x, e^{\pi i x})$, *which is given by*

$$\begin{aligned} &F(x, y, y', \cdots, y^{(n)}) = 0, \\ &n \in \mathbf{Z}_{\geq 0}, \\ &F(x, Y_0, Y_1, \cdots, Y_n) \in \mathbf{C}(x, e^{\pi i x})[Y_0, Y_1, \cdots, Y_n]. \end{aligned} \tag{1.2}$$

To prove Proposition 1.1, we use the hypertranscendence criteria, established in the differential Galois theory. We will briefly recall this criteria and then, with the aid of it, prove Proposition 1.1. (For details on the hypertranscendence criteria, we refer to [5] and the references therein.)

To describe the criteria, we introduce some definitions. A (ϕ, δ)-ring (R, ϕ, δ) is a ring R endowed with a ring automorphism ϕ and a derivation $\delta : R \to R$ (this means that δ is an additive map satisfying Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$) such that ϕ commutes with δ. If R is a field, then (R, ϕ, δ) is called a (ϕ, δ)-field.

Given a (ϕ, δ)-ring (R, ϕ, δ), a $(\widetilde{\phi}, \widetilde{\delta})$-ring $(\widetilde{R}, \widetilde{\phi}, \widetilde{\delta})$ is a (ϕ, δ)-algebras if $\widetilde{R}$ is a ring extension of R, $\widetilde{\phi}_{|R} = \phi$ and $\widetilde{\delta}_{|R} = \delta$; in this case, we will often denote $\widetilde{\phi}$ by ϕ and $\widetilde{\delta}$ by δ.

Let K be a (ϕ, δ)-field K. A δ-polynomial in the differential indeterminate y is a polynomial in the indeterminates $\{\delta^j y \mid j \in \mathbf{Z}_{\geq 0}\}$ with coefficients in K. Let R be a K-(ϕ, δ)-algebras and $a \in R$. If there exists a nonzero δ-polynomial $P(y)$ in the differential indeterminate y such that $P(a) = 0$, then we say that a is hyperalgebraic over K.

The hypertranscendency criteria is as follows:

Proposition 1.2 ([5], Proposition 2.6)**.** *Let K be a (ϕ, δ)-field with $k := \{f \in K \mid \phi(f) = f\}$ algebraically closed and let $a \in K^{\times}$. Let R be a K-(ϕ, δ)-algebra and let $v \in R \setminus \{0\}$. Assume that v is invertible in R. If $\phi(v) = av$ and if v is hyperalgebraic over K, then there exists a nonzero linear homogeneous δ-polynomial $L(y)$ and an element $f \in K$ such that*

$$L\left(\frac{\delta(a)}{a}\right) = \phi(f) - f.$$

The converse is also true if $R^{\phi} = k$.

Remark 1.3. When

$$\begin{aligned} &K = \mathbf{C}(x), \phi(f(x)) = f(x+1), \delta = \frac{d}{dx}, \\ &R = \mathbf{C}(x, \Gamma(x), \Gamma^{(1)}(x), \Gamma^{(2)}(x), \cdots), \end{aligned}$$

the former part of Proposition 1.2 was proved by Hölder [6].

We will prove Proposition 1.1. Suppose that $y = f(x)$ satisfies the algebraic differential equation (1.2). Then, by applying Proposition 1.1 with

$$\begin{aligned} &K = \mathbf{C}(x, e^{\pi i x}), \phi(f(x)) = f(x+\tau), \delta = \frac{d}{dx}, \\ &R = \mathbf{C}(x, e^{\pi i x}, f, f^{(1)}, \cdots), v = f(x), \end{aligned}$$

we find that there exist an integer $n \geq 0$, $A_j \in \mathbf{C}$ not all zeros and $R \in \mathbf{C}(x, e^{\pi i x})$ such that

$$\sum_{j=0}^{n} A_j \frac{d^j}{dx^j} \cot(\pi x) = R(x+\tau) - R(x). \tag{1.3}$$

Since $x = 0$ is a pole of the left hand side of (1.3), at least one of $R(x+\tau)$ or $R(x)$ also must have a pole at $x = 0$.

We cosider the case where $R(x+\tau)$ has a pole at $x = 0$. Then $R(x)$ has a pole at $x = \tau$. Since $x = \tau$ is not a pole of the left hand side of (1.3), $R(x+\tau)$ must have a pole at $x = \tau$. Thus the function $R(x)$ have a pole at $x = 2\tau$. By repeating this process, it follows that the set of poles of $R(x)$ contains $\{\tau, 2\tau, 3\tau, \cdots\}$. This contradicts the fact that imaginary parts of zeros and poles of an arbitrary elements of $\mathbf{C}(x, e^{2\pi i x})$ are bounded.

Similarly, when the function $R(x)$ has a pole at $x = 0$, we find that the set of poles of $R(x)$ contains $\{0, -\tau, -2\tau, -3\tau, \cdots\}$, which also leads to a contradiction. Therefore we obtain the proposition.

2. Proof of Theorem 0.1. In this section, by applying Proposition 1.1, we prove Theorem 0.1. As a byproduct of Proposition 1.1, we also show that Appell's $\mathcal{O}$-functions, introduced by Appell [1], is hypertranscendental.

We will use the following result, which is due to Ostrowski [12]:

Proposition 2.1 (Ostrowski [12]).

Let Mer^{DA} be the set of the meromorphic functions over $\mathbf{C}$ which satisfy algebraic differential equations. Then we have the following;

a) *The set Mer^{DA} is a field.*

b) *For elements $f, g \in Mer^{DA}$, the composition $f \circ g$ belongs to Mer^{DA}.*

We also recall the quasiperiodicity of the multiple sine function:

Proposition 2.2 ([9], Theorem 2.1 (a)).

The multiple sine function satisfies the difference equation

$$\mathrm{Sin}_r(x+\omega_i, \boldsymbol{\omega}) = \mathrm{Sin}_r(x, \boldsymbol{\omega}) \mathrm{Sin}_{r-1}(x, \boldsymbol{\omega}(i))^{-1},$$

where we put $\boldsymbol{\omega}(i) = (\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_r)$ and $\mathrm{Sin}_0(x, \cdot) = -1$.

Proof of Theorem 0.1. We prove the theorem by induction on r. By Proposition 2.2 and Proposition 1.1, the theorem is obviously true for $r = 2$.

Suppose that the theorem is true for r and that $\mathrm{Sin}_{r+1}(x, (\omega_1, \cdots, \omega_{r+1}))$ satisfies an algebraic differential equation. For simplicity, we put $\boldsymbol{\omega} := (\omega_1, \cdots, \omega_{r+1})$. By the condition of the theorem, without loss of generality, we can assume that ω_2/ω_1 is a non-real complex number. Proposition 2.2 gives

$$\begin{aligned}&\mathrm{Sin}_{r+1}(x+\omega_{r+1}, \boldsymbol{\omega}) \\ &\quad = \mathrm{Sin}_{r+1}(x, \boldsymbol{\omega}) \mathrm{Sin}_r(x, \boldsymbol{\omega}(r+1))^{-1},\end{aligned}$$

which means that, by applying Proposition 2.1, $\mathrm{Sin}_r(x, (\omega_1, \cdots, \omega_r))$ also satisfies an algebraic differential equation. This contradicts the induction hypothesis. Thus we obtain the theorem. □

Proposition 1.1 is also applicable to the proof of the hypertranscendence for Appell's $\mathcal{O}$-function (also called q-Pochhammer symbol or q-shifted factorial). Appell's $\mathcal{O}$-functions are defined as follows; Let $r \geq 1$ and let $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_r)$ be a r-tuple consisting of complex numbers with positive imaginary part. We put

$$q_i = e^{2\pi i \omega_i} \quad (i = 1, \cdots, r)$$

and define the functions $\mathcal{O}_r(x, \boldsymbol{\omega})$ by

$$\mathcal{O}_r(x, \boldsymbol{\omega}) := \prod_{m_1, \cdots, m_r = 0}^{\infty} (1 - e^{2\pi i x} q_1^{m_1} \cdots q_r^{m_r}) \qquad (r = 1, 2, \cdots).$$

Theorem 2.3. *The function $\mathcal{O}_r(x, \boldsymbol{\omega})$ is hypertranscendental.*

Proof. We put

$$f(x) = \mathcal{O}_1(x, \omega_1) \exp\left(\frac{\pi i}{2}\left(\frac{x^2}{\omega_1} - \left(1 + \frac{1}{\omega_1}\right)x\right)\right).$$

Then it is easy to see that $f(x)$ satisfies the difference equation (1.1). Thus, by Proposition 2.1, the theorem is true for $r = 1$.

The remaining part of the proof is similar to that of Theorem 0.1, by observing that Appell's $\mathcal{O}$-functions satisfy the following difference equations:

$$\begin{aligned}&\mathcal{O}_{r+1}(x+\omega_i, (\omega_1, \cdots, \omega_{r+1})) \\ &\quad = \frac{\mathcal{O}_{r+1}(x, (\omega_1, \cdots, \omega_{r+1}))}{\mathcal{O}_r(x, (\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_{r+1}))} \\ &\qquad (i = 1, \cdots, r+1).\end{aligned}$$

□

References

[1] P. Appell, Sur une classe de fonctions analogues aux fonctions Eulériennes, Math. Ann. **19** (1881), no. 1, 84–102.

[2] E. W. Barnes, The genesis of the double gamma functions, Proc. Lond. Math. Soc. **31** (1899), 358–381.

[3] E. W. Barnes, The theory of the double gamma function, Philos. Trans. Roy. Soc. (A) **196** (1901), 265–388.

[4] E. W. Barnes, On the theory of the multiple gamma function, Trans. Cambridge Phil. Soc. **19** (1904), 374–425.

[5] T. Dreyfus, C. Hardouin, and J. Roques, Hypertranscendence of solutions of mahler equations, J. Eur. Math. Soc. (JEMS) **20** (2018), 2209–2238.

[6] O. Hölder, Ueber die Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen, Math. Ann. **28** (1887), 1–13.

[7] M. Jimbo and T. Miwa, Quantum KZ equation with $|q| = 1$ and correlation functions of the *XXZ* model in the gapless regime, J. Phys. A **29** (1996), no. 12, 2923–2958.

[8] M. Kato, An addition type formula for the double cotangent function, Ph.D. Thesis, Kobe University (2017).

[9] N. Kurokawa and S. Koyama, Multiple sine functions, Forum Math. **15** (2003), no. 6, 839–876.

[10] S.-Y. Koyama and N. Kurokawa, Zeta functions

and normalized multiple sine functions, Kodai Math. J. **28** (2005), no. 3, 534–550.
[11] N. Kurokawa and M. Wakayama, Differential algebraicity of multiple sine functions, Lett. Math. Phys. **71** (2005), no. 1, 75–82.
[12] A. Ostrowski, Über Dirichletsche Reihen und algebraische Differentialgleichungen, Math. Z. **8** (1920), no. 3–4, 241–298.
[13] T. Shintani, On a Kronecker limit formula for real quadratic fields, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 1, 167–199.