Semi-discrete finite difference schemes for the nonlinear Cauchy problems of the normal form

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Abstract: We consider the Cauchy problems of nonlinear partial differential equations of the normal form in the class of analytic functions. We apply semi-discrete finite difference approximation which discretizes the problems only with respect to the time variable, and we give a proof for its convergence. The result implies that there are cases of convergence of finite difference schemes applied to ill-posed Cauchy problems.

Key words: Finite difference method; nonlinear partial differential equations; Cauchy problem; ill-posed problems.

1. Introduction. Let us consider finite difference approximation of the Cauchy problems of nonlinear partial differential equations (PDE's) of the normal form, and we show here its convergence independently of stability of the Cauchy problems. We know, in the class of analytic functions, unique existence of solutions to the Cauchy problems of the normal form as the Kowalevskaya theorem [4] or the Cauchy-Kowalevskaya theorem ([2], [8] etc.), and our results correspond to a discretization of the Kowalevskaya theorem.

Within the standard framework of the theory of finite difference methods, convergence of finite difference approximation of the Cauchy problems is proved under assumption of their well-posedness. For the linear Cauchy problems, P. D. Lax and R. D. Richtmyer [5] show that stability and convergence of finite difference approximation of the well-posed Cauchy problems are equivalent to each other, and we are afraid that independence between the concepts of convergence and stability may be sometimes misunderstood. We remark the independence in the present paper and show convergence of finite difference approximation in the class of analytic functions.

We follow the approach of L. Nirenberg [6] and T. Nishida [7] to the nonlinear Cauchy problems of the normal form. Whereas the classical Kowalevskaya theorem requires analyticity of functions with respect to all the variables that appear, they radically weaken assumptions so that they do not require analyticity with respect to t. Their approach is to reduce the Cauchy problems of PDE's to that of abstract ordinary differential equations on a scale of Banach spaces. In the present work we discretize the reduced Cauchy problem by forward difference approximation. There is already a similar result for the linear cases by Y. Iso [3], and we generalize it in the present work.

In the final section we show some numerical results for an ill-posed Cauchy problem. Without stability of finite difference approximation, it is difficult to observe convergence property on the standard double precision arithmetic environments of computers, but it is possible to do it on a multiple-precision environment (e.g. *exflib* [1]).

2. Abstract Cauchy problems on Banach scale. Following L. Nirenberg [6] and T. Nishida [7], we introduce the Cauchy problems of abstract ordinary differential equations on a scale of Banach spaces. The abstract Cauchy problems contain those for systems of nonlinear PDE's of the normal form (see [6]). Let $\mathcal{B} = \{B_\rho\}_{0<\rho\le\rho_0}$ be a scale of Banach spaces: $\mathcal{B}$ is a family of Banach spaces B_ρ satisfying

$$(1) \qquad B_{\rho'} \subset B_\rho \text{ and } \|u\|_\rho \le \|u\|_{\rho'} \quad \text{for } 0 < \rho < \rho' \le \rho_0 \text{ and } u \in B_{\rho'},$$

where $\|\cdot\|_\rho$ is the norm on B_ρ. Let B_0 be a linear space which contains all B_ρ $(0 < \rho \le \rho_0)$. Let $F(t, u)$

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be a mapping from a subset of $\mathbf{R} \times B_0$ to B_0, and we consider the Cauchy problem on $\mathcal{B}$ of the form

(2) $$du/dt = F(t, u(t)), \quad u(t)|_{t=0} = 0.$$

Suppose the following conditions for some positive numbers η, R, C, and K:

(3) F is a continuous mapping of
$$\{t \in \mathbf{R};\ |t| < \eta\} \times \{u \in B_{\rho'};\ \|u\|_{\rho'} < R\}$$
to B_ρ for $0 < \rho < \rho' \leq \rho_0$;

(4) F satisfies
$$\|F(t,u) - F(t,v)\|_\rho \leq C\|u - v\|_{\rho'}/(\rho' - \rho)$$
for $\rho < \rho' \leq \rho_0$, $|t| < \eta$, $\|u\|_{\rho'} < R$, $\|v\|_{\rho'} < R$;

(5) F satisfies
$$\|F(t,0)\|_\rho \leq K/(\rho_0 - \rho)$$
for $0 < \rho < \rho_0$, $|t| < \eta$, $\|u\|_{\rho_0} < R$.

The following theorem, called the Nirenberg-Nishida theorem, holds under the assumptions above, and it contains the Kowalevskaya theorem.

Theorem 2.1 ([7]). *Suppose* (1) *and* (3)–(5)*. Then there exists a positive number $a < \eta/\rho_0$ such that there exists a unique function $u(t)$ which solves the Cauchy problem* (2) *in the sense that*

> *for every $\rho < \rho_0$, the function $u(t)$ is C^1 with values in B_ρ on the interval $|t| < a(\rho_0 - \rho)$ and satisfies $\|u(t)\|_\rho < R$ and* (2)*.*

Remark 2.2. The choice of the value a given in [7] implies $\|u(t)\|_\rho \leq R/2$ for $0 < \rho < \rho_0 - |t|/a$, and we use this choice in Theorem 3.1.

In the present research, we apply finite difference method to approximate the solution to the Cauchy problem (2) whose unique existence is established by Theorem 2.1. We approximate d/dt by forward difference but we do not approximate the operator F, and therefore we call our scheme a semi-discrete finite difference scheme.

3. Main result. For a number $T > 0$ and an integer $N \geq 1$, we set $\Delta t := T/N$ and $t_k := k\Delta t$, and consider the following finite difference scheme:

$$(u^{k+1} - u^k)/\Delta t = F(t_k, u^k), \quad 0 \leq k \leq N-1,$$
$$u^0 = 0.$$

This is equivalent to

(6) $$u^k = \Delta t \sum_{j=0}^{k-1} F(t_j, u^j), \quad 0 \leq k \leq N,$$

where the summation for $k = 0$ equals zero. Our aim is to show that, for suitable T and large N, (6) determines $u^0, \ldots, u^N \in B_0$ such that (t_k, u^k) belongs to the domain of F, and u^k approximates $u(t_k)$. We suppose additional hypotheses as follows:

(7) F is a C^1 mapping of
$$\{t \in \mathbf{R};\ |t| < \eta\} \times \{u \in B_{\rho'};\ \|u\|_{\rho'} < R\}$$
to B_ρ for $0 < \rho < \rho' \leq \rho_0$;

(8) The partial derivative of F with respect to u is a bounded operator from $B_{\rho'}$ to B_ρ, and the operator norm satisfies
$$\|F_u(t,u)\| \leq C/(\rho' - \rho)$$
for $0 < \rho < \rho' \leq \rho_0$, $|t| < \eta$, $\|u\|_{\rho'} < R$;

(9) There is a number $L > 0$ such that the partial derivative of F with respect to t satisfies
$$\|F_t(t,u)\|_\rho \leq L/(\rho' - \rho)^2$$
for $0 < \rho < \rho' \leq \rho_0$, $|t| < \eta$, $\|u\|_{\rho'} < R$.

Note that conditions (1), (5), (7), (8) assumed in Theorem 3.1 imply unique existence of the solution to (2) because (7) implies (3), (8) implies (4), and Theorem 2.1 can be applied.

Theorem 3.1. *Suppose* (1), (5), *and* (7)–(9)*. Let $u(t)$ be the unique solution to* (2) *as asserted in Theorem* 3.1*, and suppose that $u(t)$ satisfies*

(10) $$\|u(t)\|_\rho \leq R/2, \quad 0 < \rho < \rho_0 - |t|/a.$$

Let c, ρ_1, and T be positive numbers satisfying

(11) $$c < \min\{a, 1/4C\}, \quad \rho_1 < \rho_0, \quad T < c\rho_1.$$

Then there exists a positive number S such that, if $S\Delta t < R/2$, equation (6) *determines $u^k \in B_{\rho_1 - t_k/c}$ for $0 \leq k \leq N$, and*

(12) $$\sup_{0 \leq k \leq N;\, t_k \leq c(\rho_1 - \rho)} \|u^k - u(t_k)\|_\rho \leq S\Delta t$$

for every positive $\rho < \rho_1$.

4. Proof of Theorem 3.1. First we estimate the second derivative of $u(t)$.

Proposition 4.1. *Suppose* (1), (5), *and* (7)–(10)*. Then the solution $u(t)$ to the Cauchy problem* (2) *is a C^2 function from $\{t \in \mathbf{R};\ |t| < a(\rho_0 - \rho)\}$ to B_ρ. Moreover, there is a number $V_2 > 0$ such that*

$$\|u''(t)\|_\rho \leq V_2/(\rho_0 - \rho - |t|/a)^2$$

for ρ and t satisfying $0 < \rho < \rho_0 - |t|/a$.

Proof. For ρ and t with $0 < \rho < \rho_0 - |t|/a$, let $\rho(t) := (\rho + \rho_0 - |t|/a)/2$. Then

$$0 < \rho < \rho(t) < \rho_0 - |t|/a$$

holds and $\|u(t)\|_{\rho(t)} \le R/2$ follows from (10). Using (4) with $\rho' = \rho(t)$ and (5), we have

$$\|u'(t)\|_\rho \le \|F(t,u(t)) - F(t,0)\|_\rho + \|F(t,0)\|_\rho \le C\|u(t)\|_{\rho(t)}/[\rho(t)-\rho] + K/(\rho_0-\rho).$$

Thus we obtain, with $V_1 = CR + K$,

$$\|u'(t)\|_\rho \le V_1/(\rho_0 - \rho - |t|/a) \tag{13}$$

for $0 < \rho < \rho_0 - |t|/a$. We apply (13) with $\rho = \rho(t)$ to get

$$\|u'(t)\|_{\rho(t)} \le V_1/(\rho_0 - \rho(t) - |t|/a).$$

Since F is C^1 by (7) and $u(t)$ is a C^1 function satisfying $u' = F(t,u)$, u is C^2 and

$$u''(t) = F_u(t,u(t))u'(t) + F_t(t,u(t)).$$

Using (8) and (9) with $\rho' = \rho(t)$, we obtain

$$\|u''(t)\|_\rho \le \|F_u(t,u(t))\|\|u'(t)\|_{\rho(t)} + \|F_t(t,u(t))\|_\rho \le V_2/(\rho_0 - \rho - |t|/a)^2$$

for $0 < \rho < \rho_0 - |t|/a$, with $V_2 = 4(CV_1 + L)$. □

To reduce our proof to Claim 4.2 below, we define truncation errors w_k $(0 \le k \le N-1)$ by

$$[u(t_{k+1}) - u(t_k)]/\Delta t = F(t_k, u(t_k)) + w_k. \tag{14}$$

Taylor's theorem and Proposition 4.1 show that there are $v_0, \dots, v_{N-1} \in B_0$ such that

$$u(t_{k+1}) - u(t_k) = F(t_k, u(t_k))\Delta t + v_k \Delta t^2, \tag{15}$$

$$\|v_k\|_\rho \le V_2/(\rho_0 - \rho - t_{k+1}/a)^2 \quad \text{for } 0 < \rho < \rho_0 - t_{k+1}/a, \tag{16}$$

for $0 \le k \le N-1$. From (14) and (15), we get

$$w_k = v_k \Delta t. \tag{17}$$

Claim 4.2. Under the hypotheses of Theorem 3.1, there exists a positive number S such that, if $S\Delta t < R/2$, elements $e_0, \dots, e_N \in B_0$ are defined by

$$e_k = \Delta t \sum_{j=0}^{k-1} [F(t_j, u(t_j) + e_j) - F(t_j, u(t_j)) - w_j] \tag{18}$$

for $0 \le k \le N$, and they satisfy

$$\|e_k\|_\rho \le S\Delta t, \quad 0 < \rho \le \rho_1 - t_k/c \tag{19}$$

for $0 \le k \le N$.

We assume Claim 4.2 for the moment. Suppose $S\Delta t < R/2$. For $0 \le k \le N$ and $0 < \rho \le \rho_1 - t_k/c$, (18) determines e_k, and $\|e_k\|_\rho < R/2$ by (19). Since $0 < \rho < \rho_0 - t_k/a$, we get $\|u(t_k)\|_\rho \le R/2$ from (10) hence $\|u(t_k) + e_k\|_\rho < R$.

Put $u^k := u(t_k) + e_k$. Then we have $\|u^k\|_\rho < R$ for $0 \le k \le N$ and $0 < \rho \le \rho_1 - t_k/c$. Formula (14) yields

$$u(t_k) = \Delta t \sum_{j=0}^{k-1} [F(t_j, u(t_j)) + w_j], \quad 0 \le k \le N.$$

Adding this to (18) shows that $u^0, \dots, u^N$ satisfy (6), and (19) yields (12). The proof is therefore completed if we prove Claim 4.2.

5. Proof of Claim 4.2. Let c, ρ_1, and T be as in (11). We regard (18) as a fixed-point equation for $(e_0, \dots, e_N)$ in $(B_0)^{N+1}$ and find a solution by successive approximation. We introduce a family of linear subspaces $\{Y_\alpha\}_{\alpha \ge c}$ of $(B_0)^{N+1}$ as follows:

$$Y_\alpha := \{f = (f_0, \dots, f_N) \in (B_0)^{N+1};\ [f]'_\alpha < +\infty\},$$

where

$$[f]'_\alpha := \inf\{M \ge 0;\ \|f_k\|_\rho \le Mt_k/(\rho_1 - \rho - t_k/\alpha) \text{ for } 0 \le k \le N,\ 0 < \rho < \rho_1 - t_k/\alpha\}.$$

We remark that (a) $f_0 = 0$ for $f \in Y_\alpha$; (b) $[\,\cdot\,]'_\alpha$ is a norm on Y_α which makes Y_α a Banach space; (c) if $f \in Y_\alpha$, we have

$$[f]'_\beta \le [f]'_\alpha \quad \text{for } c \le \beta < \alpha \tag{20}$$

and therefore $Y_\alpha \subset Y_\beta \subset Y_c$; and (d) if $f \in Y_\alpha$, we have

$$\|f_k\|_\rho \le [f]'_\alpha/(1/\beta - 1/\alpha) \tag{21}$$

for $c \le \beta < \alpha$, $0 \le k \le N$, $0 < \rho \le \rho_1 - t_k/\beta$.

We hope to define a sequence $\{e^{(m)}\}_{m \ge 0}$ of elements $e^{(m)} = (0, e_1{}^{(m)}, \dots, e_N{}^{(m)})$ in $(B_0)^{N+1}$ by the following rule: let $e^{(0)} = (0, 0, \dots, 0)$; if $e^{(m)}$ is defined for some integer $m \ge 0$, define the next term $e^{(m+1)} = (0, e_1{}^{(m+1)}, \dots, e_N{}^{(m+1)})$ by

$$e_k{}^{(m+1)} := \Delta t \sum_{j=0}^{k-1} [F(t_j, u(t_j) + e_j{}^{(m)}) - F(t_j, u(t_j)) - w_j] \tag{22}$$

for $1 \le k \le N$. The goal is to show that the above rule defines a sequence convergent in Y_c and the limit, say $e = (e_0, \dots, e_N)$, satisfies (18).

(Step 1) By putting $m = 0$ in (22), components of $e^{(1)}$ are given by

$$e_k{}^{(1)} = -\Delta t \sum_{j=0}^{k-1} w_j, \quad 0 \le k \le N. \tag{23}$$

We show $[e^{(1)}]'_a \le V\Delta t$ with $V = V_2/(\rho_0 - \rho_1)$.

For $1 \le k \le N$ and $0 < \rho < \rho_1 - t_k/a$, we have

$$\|e_k{}^{(1)}\|_\rho \le \Delta t \sum_{j=0}^{k-1} \|v_j\|_\rho \Delta t$$

by (17). For $0 \le j \le k-1$, it follows from (16) that

$$\begin{aligned}\|v_j\|_\rho &\le V_2/(\rho_0 - \rho - t_{j+1}/a)^2\\ &\le V_2/(\rho_0 - \rho_1)(\rho_1 - \rho - t_k/a).\end{aligned}$$

Thus we obtain, with $V = V_2/(\rho_0 - \rho_1)$,

$$\|e_k{}^{(1)}\|_\rho \le (V\Delta t)t_k/(\rho_1 - \rho - t_k/a)$$

for $1 \le k \le N$, $0 < \rho < \rho_1 - t_k/a$. Since $e_0{}^{(1)} = 0$, $[e^{(1)}]'_a \le V\Delta t$ follows.

(Step 2) Take b_0 and c_0 with

$$c < c_0 < b_0 < \min\{a, 1/4C\},$$

and put $\delta := 1 - c_0/b_0$. Then $b_0 = c_0/(1-\delta)$ and $0 < \delta < 1$. Take θ with $0 < \theta < 1$ and $4Cb_0 < \theta^2$, and put $b_m := c_0/(1 - \delta\theta^m)$ for all integers $m \ge 0$. It will be needed later that

$$1/b_{m+1} - 1/b_m = \theta^m(1-\theta)(b_0 - c_0)/b_0c_0.$$

We have $c < c_0 < \cdots < b_2 < b_1 < b_0 < a$, hence

$$\begin{aligned}&[\,\cdot\,]'_c \le [\,\cdot\,]'_{c_0} \le \cdots \le [\,\cdot\,]'_{b_2} \le [\,\cdot\,]'_{b_1} \le [\,\cdot\,]'_{b_0} \le [\,\cdot\,]'_a,\\ &Y_a \subset Y_{b_0} \subset Y_{b_1} \subset Y_{b_2} \subset \cdots \subset Y_{c_0} \subset Y_c,\end{aligned}$$

by (20). Therefore

$$e^{(1)} \in Y_{b_0}, \quad [e^{(1)}]'_{b_0} \le V\Delta t, \tag{24}$$

in view of Step 1. By (21), we also have

$$e_k{}^{(1)} \in B_\rho, \tag{25}$$

$$\begin{aligned}\|e_k{}^{(1)}\|_\rho &\le [e^{(1)}]'_{b_0}/(1/b_1 - 1/b_0)\\ &\le b_0c_0V\Delta t/(b_0 - c_0)(1-\theta),\end{aligned} \tag{26}$$

for $0 \le k \le N$ and $0 < \rho \le \rho_1 - t_k/b_1$.

(Step 3) We take a positive number S as

$$S = b_0c_0V/(b_0 - c_0)(1-\theta)^2,$$

and suppose that $S\Delta t < R/2$. Then the following assertions hold for all integers $m \ge 0$:

$(1)_m$ $e^{(0)}, \ldots, e^{(m+1)}$ are defined by (22) and belong to Y_{b_m};

$(2)_m$ $[e^{(m+1)} - e^{(m)}]'_{b_m} \le \theta^{2m}V\Delta t$;

$(3)_m$ $\|e_k{}^{(m+1)}\|_\rho \le (1 - \theta^{m+1})S\Delta t$ for $0 \le k \le N$, $0 < \rho \le \rho_1 - t_k/b_{m+1}$.

We prove them by induction on m. The assertions $(1)_0$, $(2)_0$, $(3)_0$ follow from (24) and (26). Next suppose $(1)_m$, $(2)_m$, $(3)_m$ for some integer $m \ge 0$. The inductive step consists of paragraphs (A)–(C).

(A) Here we write $\lambda_k := \rho_1 - t_k/b_{m+1}$. For each k, $1 \le k \le N$, we show that the formula

$$\begin{aligned}e_k{}^{(m+2)} := \Delta t \sum_{j=0}^{k-1} [&F(t_j, u(t_j) + e_j{}^{(m+1)})\\ &- F(t_j, u(t_j)) - w_j]\end{aligned}$$

defines an element of B_{λ_k}.

Since $\lambda_k \le \rho_1 - t_k/b_1$, (23) and (25) show that

$$-\Delta t \sum_{j=0}^{k-1} w_j \in B_{\lambda_k}.$$

For $0 \le j \le k-1$, it follows from $(3)_m$ that $\|e_j{}^{(m+1)}\|_{\lambda_j} < R/2$. Since $\lambda_j < \rho_0 - t_j/a$ by $\rho_1 < \rho_0$ and $b_{m+1} < a$, we have $\|u(t_j)\|_{\lambda_j} \le R/2$ from (10) and hence $\|u(t_k) + e_k{}^{(m+1)}\|_{\lambda_j} < R$. Since $\lambda_k < \lambda_j$ for $0 \le j \le k-1$, we have

$$F(t_j, u(t_j) + e_j{}^{(m+1)}) \in B_{\lambda_k}, \quad F(t_j, u(t_j)) \in B_{\lambda_k}$$

from (8). Thus $e_k{}^{(m+2)}$ belongs to B_{λ_k}.

(B) Let $e^{(m+2)} := (0, e_1{}^{(m+2)}, \ldots, e_N{}^{(m+2)})$. To prove $(1)_{m+1}$, it suffices to show that

$$e^{(m+2)} - e^{(m+1)} \in Y_{b_{m+1}}$$

since we already have $e^{(0)}, \ldots, e^{(m+1)} \in Y_{b_{m+1}}$ in view of $(1)_m$ and $Y_{b_m} \subset Y_{b_{m+1}}$. For $1 \le k \le N$ and $0 < \rho < \rho_1 - t_k/b_{m+1}$, put

$$\rho'_j := (\rho + \rho_1 - t_j/b_{m+1})/2$$

for $0 \le j \le k-1$. Note that $\rho < \rho'_j < \rho_1 - t_j/b_{m+1}$. Using (8) with $\rho' = \rho'_j$, we compute

$$\begin{aligned}&\|e_k{}^{(m+2)} - e_k{}^{(m+1)}\|_\rho\\ &\le \Delta t \sum_{j=0}^{k-1} \|F(t_j, u(t_j) + e_j{}^{(m+1)})\\ &\qquad - F(t_j, u(t_j) + e_j{}^{(m)})\|_\rho\\ &\le \Delta t \sum_{j=0}^{k-1} \frac{C}{\rho'_j - \rho} \|e_j{}^{(m+1)} - e_j{}^{(m)}\|_{\rho'_j},\end{aligned}$$

and using $e^{(m+1)} - e^{(m)} \in Y_{b_{m+1}}$ and $[\,\cdot\,]'_{b_{m+1}} \le [\,\cdot\,]'_{b_m}$,

$$\begin{aligned}&\le \Delta t \sum_{j=0}^{k-1} \frac{C}{\rho'_j - \rho} \frac{[e^{(m+1)} - e^{(m)}]'_{b_{m+1}} t_j}{\rho_1 - \rho'_j - t_j/b_{m+1}}\\ &= \Delta t \sum_{j=0}^{k-1} \frac{C\,[e^{(m+1)} - e^{(m)}]'_{b_{m+1}} t_j}{(\rho_1 - \rho - t_j/b_{m+1})^2/4}\\ &\le 4Ct_k[e^{(m+1)} - e^{(m)}]'_{b_m} \sum_{j=0}^{k-1} \frac{\Delta t}{(\rho_1 - \rho - t_j/b_{m+1})^2}\\ &\le \frac{4Cb_{m+1}[e^{(m+1)} - e^{(m)}]'_{b_m} t_k}{\rho_1 - \rho - t_k/b_{m+1}},\end{aligned}$$

where the last inequality is obtained by using

$$\sum_{j=0}^{k-1} \frac{\Delta t}{(A-t_j)^2} \le \sum_{j=0}^{k-1} \frac{t_{j+1}-t_j}{(A-t_j)(A-t_{j+1})} = \frac{t_k}{A(A-t_k)}$$

and $t_k < b_{m+1}(\rho_1 - \rho)$. Since $4Cb_{m+1} < \theta^2$, we get

$$\|e_k{}^{(m+2)} - e_k{}^{(m+1)}\|_\rho \le \frac{\theta^2 [e^{(m+1)} - e^{(m)}]'_{b_m} t_k}{\rho_1 - \rho - t_k/b_{m+1}}$$

for $1 \le k \le N$ and $0 < \rho < \rho_1 - t_k/b_{m+1}$. This inequality holds also for $k = 0$ and thus

$$e^{(m+2)} - e^{(m+1)} \in Y_{b_{m+1}},$$
$$[e^{(m+2)} - e^{(m+1)}]'_{b_{m+1}} \le \theta^2 [e^{(m+1)} - e^{(m)}]'_{b_m}.$$

The former proves $(1)_{m+1}$ as remarked above. Combining the latter with $(2)_m$ proves $(2)_{m+1}$.

(C) For $0 \le k \le N$ and $0 < \rho \le \rho_1 - t_k/b_{m+2}$, it follows from (21) and $(2)_{m+1}$ that

$$\begin{aligned}\|e_k{}^{(m+2)} - e_k{}^{(m+1)}\|_\rho &\le [e^{(m+2)} - e^{(m+1)}]'_{b_{m+1}}/(1/b_{m+2} - 1/b_{m+1}) \\ &\le \theta^{m+1}(1-\theta) S \Delta t.\end{aligned}$$

Adding this with $(3)_m$ proves $(3)_{m+1}$. Thus we finish the inductive step, and the assertions $(1)_m$, $(2)_m$, $(3)_m$ hold for all integers $m \ge 0$.

(Step 4) Formulas $(2)_m$ of Step 3 imply that (22) defines a Cauchy sequence $\{e^{(m)}\}_{m\ge 0}$ in the Banach space Y_{c_0}. Let $e = (e_0, \dots, e_N)$ denote the limit: $[e^{(m)} - e]'_{c_0} \to 0$ as $m \to \infty$. The definition of $[\,\cdot\,]'_{c_0}$ shows that

$$(27) \qquad \|e_k{}^{(m)} - e_k\|_\rho \to 0 \text{ as } m \to \infty$$

for $0 \le k \le N$ and ρ with $0 < \rho < \rho_1 - t_k/c_0$. For such k and ρ, formulas $(3)_m$ hold for all m, and by letting $m \to \infty$ we obtain

$$\|e_k\|_\rho \le S \Delta t.$$

Thus (19) follows since $c < c_0$.

For $0 \le k \le N$ fixed, we put $\rho := \rho_1 - t_k/c$ and $\rho' := \rho_1 - t_k/c_0$. For $0 \le j \le k-1$, we have $\rho' < \rho_1 - t_j/c_0$ and therefore $\|e_j{}^{(m)} - e_j\|_{\rho'} \to 0$ as $m \to \infty$ by (27). Since $\rho < \rho'$, it follows from (8) that for $0 \le j \le k-1$,

$$\|F(t_j, u(t_j) + e_j{}^{(m)}) - F(t_j, u(t_j) + e_j)\|_\rho \to 0 \quad \text{as } m \to \infty.$$

Taking the limit of (22) shows that e satisfies (18) in B_ρ. Thus we finish the proof of Claim 4.2. □

6. Numerical experiment. We illustrate our main result by the Cauchy problem for a system of quasi-linear PDE's of the normal form for $u = u(t,x)$ and $v = v(t,x)$:

$$(28) \qquad \begin{aligned} u_t = {}& (u-v)u_x + (u^2+1)v_x \\ &+ 1 + (tu+1)(xu - v), \end{aligned}$$

$$(29) \quad v_t = -(v^2+1)u_x - (u-v)v_x + 1 + x(1-u^2),$$

$$(30) \qquad u|_{t=0} = 0, \quad v|_{t=0} = 0.$$

The Cauchy problem (28)–(30) has a unique solution

$$(31) \qquad u = t, \quad v = t(1+x).$$

The system of PDE's (28)–(29) has a characteristic polynomial

$$\begin{vmatrix} \tau - (u-v)\xi & -(u^2+1)\xi \\ (v^2+1)\xi & \tau + (u-v)\xi \end{vmatrix} = \tau^2 + (uv+1)^2 \xi.$$

Thus in the region $x > -1 - 1/t^2$ the system is elliptic along the solution (31) since $uv > -1$, and the problem (28)–(30) is ill-posed in Hadamard's sense.

Theorem 3.1 discusses discretization only in t, but we discretize the system (28)–(29) also in x for numerical computation. For positive numbers Δt and Δx, we replace u, u_t, u_x by

$$u^k(x), \quad \frac{u^{k+1}(x) - u^k(x)}{\Delta t}, \quad \frac{u^k(x+\Delta x) - u^k(x)}{\Delta x},$$

where $u^k(x)$ is a function of x that is expected to be an approximation to $u(t_k, x)$. We replace v_t, v_x, v in a similar way. Let us consider (28)–(30) in the complex domain, and let B_ρ be the Banach space of holomorphic functions in $\{x \in \mathbf{C}; |x| < \rho\}$ with sup-norm. Then Theorem 3.1 implies that if we were able to compute exact solutions to a scheme that discretizes only in t but not in x, they would converge to the exact solution to the Cauchy problem (28)–(30) with the maximum error of order $O(\Delta t)$ as $\Delta t \to 0$.

Figures 1 and 2 are profiles of the numerical solution $v^k(x)$ with $\Delta t = \Delta x = 0.01$ over the region $0 \le t \le 0.5$, $0 \le x \le 10$. Figure 1 is obtained by the standard double precision arithmetic, and the oscillation results from instability of the scheme and effect of the growth of rounding errors. Figure 2 is obtained by 200 decimal digits arithmetic on *exflib* [1], and we see that the use of multiple precision arithmetic prevents the growth of rounding errors.

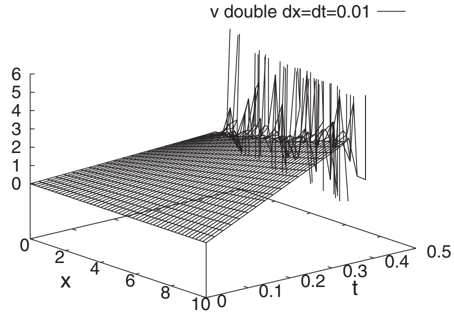

Fig. 1. Numerical solution $v^k(x)$ with $\Delta t = \Delta x = 0.01$ obtained by the standard double precision arithmetic.

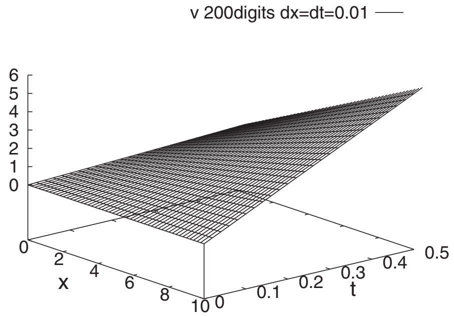

Fig. 2. Numerical solution $v^k(x)$ with $\Delta t = \Delta x = 0.01$ obtained by 200 decimal digits precision arithmetic.

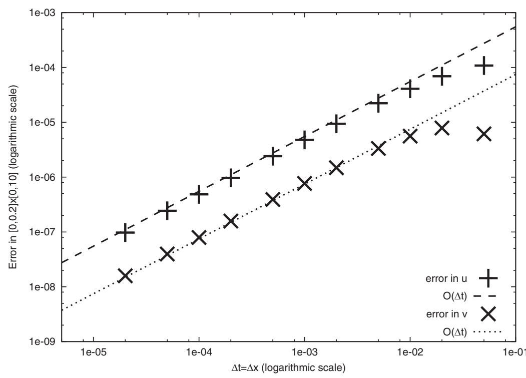

Fig. 3. The maximum error between the exact solution and the numerical solution over $0 < t < 0.2$, $0 < x < 10$, plotted versus $h = \Delta t = \Delta x$ and computed with 5000 decimal digits precision.

Figure 3 is the plot of the maximum errors

$$\sup_{0 \leq t_k \leq 0.2,\ 0 \leq x \leq 10} |u^k(x) - u(t_k, x)|,$$

$$\sup_{0 \leq t_k \leq 0.2,\ 0 \leq x \leq 10} |v^k(x) - v(t_k, x)|,$$

versus $h = \Delta t = \Delta x$. Both axes are in logarithmic scale and the numerical solutions are computed with 5000 decimal digits precision. The dashed line and the dotted one indicate the rate of convergence of order $O(\Delta t)$, and the numerical results are in good agreement with Theorem 3.1 for sufficiently small Δt.

The numerical experiment given above suggests that the order of magnitude of the errors coming from Δx does not exceed that coming from Δt as long as $\Delta x = \Delta t$. We consider that the experiment visualizes Theorem 3.1 and suggests a convergence result for fully discretized schemes under suitable restriction on the ratio $\Delta x / \Delta t$.

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