

A character of the Siegel modular group of level 2 from theta constants

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Abstract: Given a characteristic, we define a character of the Siegel modular group of level 2, the computations of their values are obtained. Using our theorems, some key theorems of Igusa [2] can be recovered.

Key words: Theta constants; the Siegel modular groups of level 2; characters.

1. Introduction. The theta function of a characteristic m of degree g is the series

$$\theta_m(\tau, z) := \sum_{p \in \mathbf{Z}^g} \exp \pi i \left[\left(p + \frac{m'}{2}, \tau \left(p + \frac{m'}{2} \right) \right) + 2 \left(p + \frac{m'}{2}, z + \frac{m''}{2} \right) \right],$$

where $\tau \in \mathcal{H}_g$, $z \in \mathbf{C}^g$, $m = \begin{pmatrix} m' \\ m'' \end{pmatrix} \in \mathbf{Z}^{2g}$, \mathcal{H}_g is the

Siegel upper half-plane, m' and m'' denote vectors in \mathbf{Z}^g determined by the first and last g coefficients of m . If we put $z = 0$, we get the theta constant $\theta_m(\tau) = \theta_m(\tau, 0)$. The study of theta functions and theta constants has a long history, and they are very important objects in arithmetic and geometry. They can be used to construct modular forms and to study geometric properties of abelian varieties. Farkas and Kra's book [1] contains very detailed descriptions for the case of degree one. In [3], [4], and [5], Matsuda gives new formulas and applications. It is Igusa in [2] who began to study the cases of higher degrees. He used $\theta_m(\tau)\theta_n(\tau)$ to determine the structure of the graded rings of modular forms belonging to the group $\Gamma_g(4, 8)$.

In this note, we will define a character of the group $\Gamma_g(2)$, the principal congruence group of degree g and of level 2. We obtained its computation formula. Using our results, Igusa's key Theorem 3 in [2] can be recovered.

2. The Siegel modular group of level 2.

The Siegel modular group $\text{Sp}(g, \mathbf{Z})$ of degree g is the group of $2g \times 2g$ integral matrices M satisfying

$$M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} {}^t M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

in which ${}^t M$ is the transposition of M , I_g is the identity of degree g . If we put $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the condition for M in $\text{Sp}(g, \mathbf{Z})$ is $a^t d - b^t c = I_g$, $a^t b$ and $c^t d$ are symmetric matrices. In fact, if M is in $\text{Sp}(g, \mathbf{Z})$, then ${}^t b d$ and ${}^t a c$ are also symmetric, see [6, p. 437]. In this paper, we discuss two special subgroups of the Siegel modular group. The first is the principal congruence subgroup $\Gamma_g(2)$ of degree g and of level 2 which is defined by $M \equiv I_{2g} \pmod{2}$. The second is the Igusa modular group $\Gamma_g(4, 8)$, which is defined by $M \equiv I_{2g} \pmod{4}$ and $(a^t b)_0 \equiv (c^t d)_0 \equiv 0 \pmod{8}$. If s is a square matrix, we arrange its diagonal coefficients in a natural order to form a vector $(s)_0$. The Siegel modular group $\text{Sp}(g, \mathbf{Z})$ acts on \mathcal{H}_g by the formula

$$M\tau = (a\tau + b)(c\tau + d)^{-1},$$

where $\tau \in \mathcal{H}_g$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(g, \mathbf{Z})$. An element $m \in \mathbf{Z}^{2g}$ is called a theta characteristic of degree g . If n is another characteristic, then we have

$$\theta_{m+2n}(\tau, z) = (-1)^{{}^t m' n''} \theta_m(\tau, z).$$

Since

$$\theta_m(\tau, -z) = (-1)^{{}^t m' m''} \theta_m(\tau, z),$$

m is called even or odd according as ${}^t m' m''$ is even or odd. Only for even m , theta constants are none zero. Given a characteristic m and an element M in the Siegel modular group, we define

$$M \circ m = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} + \begin{pmatrix} (c^t d)_0 \\ (a^t b)_0 \end{pmatrix}.$$

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This operation modulo 2 is a group action, i.e. $M_1 \circ (M_2 \circ m) \equiv (M_1 M_2) \circ m \pmod{2}$. Next we explain the transformation formulas for theta constants: for any $m \in \mathbf{Z}^{2g}$ and $M \in \text{Sp}(g, \mathbf{Z})$, we put

$$\begin{aligned} \Phi_m(M) := & -\frac{1}{8}({}^t m'^t b d m' + {}^t m''^t a c m'' \\ & - 2{}^t m'^t b c m'' - 2{}^t(a^t b)_0(dm' - c m'')), \end{aligned}$$

then we have

$$\theta_{M \circ m}(M\tau) = a(M)e(\Phi_m(M)) \det(c\tau + d)^{\frac{1}{2}} \theta_m(\tau),$$

in which $a(M)$ is an eighth root of unity depending only on M and the choice of square root sign for $\det(c\tau + d)^{\frac{1}{2}}$, and $e(\Phi_m(M)) = e^{2\pi i \Phi_m(M)}$. From now on, we always discuss the group $\Gamma_g(2)$, unless specified. Hence, we can write

$$\Phi_m(M) = -\frac{1}{8}({}^t m'^t b d m' + {}^t m''^t a c m'' - 2{}^t(a^t b)_0 d m').$$

3. Main theorems and proofs. The character mentioned in the abstract is as follows:

Definition. Let $m \in \mathbf{Z}^{2g}$, $M \in \Gamma_g(2)$, we define $\chi_m(M)$ by

$$\theta_m(M\tau) = a(M)\chi_m(M) \det(c\tau + d)^{\frac{1}{2}} \theta_m(\tau),$$

where $a(M)$ comes from the transformation formulas of theta constants.

Theorem 3.1. For a fixed m , $\chi_m(M)$ is a character of $\Gamma_g(2)$.

The proof of Theorem 3.1 needs two lemmas, in which $\Gamma_g(2)$ is essential.

Lemma 3.2. If $m, n \in \mathbf{Z}^{2g}$ and $m \equiv n \pmod{2}$, then $\Phi_m(M) \equiv \Phi_n(M) \pmod{1}$.

Proof. Let $m = n + 2\Delta$, then

$$\begin{aligned} & {}^t m'^t b d m' \\ &= {}^t(n' + 2\Delta')^t b d(n' + 2\Delta') \\ &= {}^t n'^t b d n' + 4{}^t \Delta'^t b d \Delta' + 2{}^t n'^t b d \Delta' + 2{}^t \Delta'^t b d n' \\ &\equiv {}^t n'^t b d n' \pmod{8}, \end{aligned}$$

since $b \equiv 0 \pmod{2}$ and ${}^t b d$ is symmetric, the last two terms are equal. Similarly, ${}^t a c$ is symmetric which implies that ${}^t m''^t a c m'' \equiv {}^t n''^t a c n'' \pmod{8}$. Moreover, $2{}^t(a^t b)_0 d m' \equiv 2{}^t(a^t b)_0 d n' \pmod{8}$ is trivial. By the definition of $\Phi_m(M)$, Lemma 3.2 is true. \square

Lemma 3.3. For $M, M' \in \Gamma_g(2)$, we have $\Phi_{M' \circ m}(M) \equiv \Phi_m(M) \pmod{1}$.

Proof. This lemma can be proved from the definition of $M' \circ m$ with $M' \in \Gamma_g(2)$ and Lemma

3.2. \square

Proof of Theorem 3.1. We firstly give a formula for $\chi_m(M)$. By the definition of the operation \circ , we can find a unique $n \in \mathbf{Z}^{2g}$ with $M \circ n = m$. Define $\Delta \in \mathbf{Z}^{2g}$ by $m + 2\Delta = n$, then by Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} & \theta_m(M\tau) \\ &= \theta_{M \circ n}(M\tau) \\ &= a(M)e(\Phi_n(M)) \det(c\tau + d)^{\frac{1}{2}} \theta_n(\tau) \\ &= a(M)e(\Phi_m(M)) \det(c\tau + d)^{\frac{1}{2}} \theta_n(\tau) \\ &= a(M)e(\Phi_m(M)) \det(c\tau + d)^{\frac{1}{2}} \theta_{m+2\Delta}(\tau) \\ &= a(M)e(\Phi_m(M)) \det(c\tau + d)^{\frac{1}{2}} (-1)^{{}^t m' \Delta''} \theta_m(\tau). \end{aligned}$$

Hence,

$$(1) \quad \chi_m(M) = e(\Phi_m(M))(-1)^{{}^t m' \Delta''}.$$

To prove Theorem 3.1 is equivalent to prove $\chi_m(M_1)\chi_m(M_2) = \chi_m(M_1 M_2)$ for any $M_1, M_2 \in \Gamma_g(2)$. Now fix M_1, M_2 and m , define Δ_1, Δ_2 by $m + 2\Delta_1 = n_1$, $M_1 \circ n_1 = m$; $m + 2\Delta_2 = n_2$, $M_2 \circ n_2 = m$. Write

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

by (1), we have

$$\chi_m(M_1) = e(\Phi_m(M_1))(-1)^{{}^t m' \Delta_1''}$$

and

$$\chi_m(M_2) = e(\Phi_m(M_2))(-1)^{{}^t m' \Delta_2''}.$$

In order to compute $\chi_m(M_1 M_2)$, we write $\tau' = M_2 \tau$, $M = M_1 M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and define Δ_3 by $m + 2\Delta_3 = \bar{n}$ with $M_1 \circ n_1 = m$, $M_2 \circ \bar{n} = n_1$. Then by Lemmas 3.2 and 3.3, we have

$$\begin{aligned} & \theta_m(M_1 M_2 \tau) \\ &= \theta_{M_1 \circ n_1}(M_1 \tau') \\ &= a(M_1)e(\Phi_{n_1}(M_1)) \det(c_1 \tau' + d_1)^{\frac{1}{2}} \theta_{n_1}(\tau') \\ &= a(M_1)e(\Phi_m(M_1)) \det(c_1 \tau' + d_1)^{\frac{1}{2}} \theta_{M_2 \circ \bar{n}}(M_2 \tau) \\ &= a(M_1)e(\Phi_m(M_1)) \det(c_1 \tau' + d_1)^{\frac{1}{2}} a(M_2)e(\Phi_{\bar{n}}(M_2)) \\ &\quad \times \det(c_2 \tau + d_2)^{\frac{1}{2}} \theta_{\bar{n}}(\tau) \\ &= a(M_1)a(M_2)e(\Phi_m(M_1))e(\Phi_{n_1}(M_2)) \det(c_1 \tau' + d_1)^{\frac{1}{2}} \\ &\quad \times \det(c_2 \tau + d_2)^{\frac{1}{2}} \theta_{\bar{n}}(\tau) \end{aligned}$$

$$\begin{aligned}
&= a(M_1)a(M_2)e(\Phi_m(M_1))e(\Phi_m(M_2)) \\
&\quad \times \det(c\tau + d)^{\frac{1}{2}}\theta_{m+2\Delta_3}(\tau) \\
&= a(M_1M_2)e(\Phi_m(M_1))e(\Phi_m(M_2)) \\
&\quad \times \det(c\tau + d)^{\frac{1}{2}}(-1)^{t_{m'}\Delta_3''}\theta_m(\tau),
\end{aligned}$$

in which, we use $c\tau + d = (c_1\tau' + d_1)(c_2\tau + d_2)$ and $a(M) = a(M_1)a(M_2)$, the latter is implied by Igusa's Theorem 2 in [2]. Hence,

$$\chi_m(M_1M_2) = e(\Phi_m(M_1))e(\Phi_m(M_2))(-1)^{t_{m'}\Delta_3''}.$$

Now we compute

$$(-1)^{t_{m'}\Delta_1''}, \quad (-1)^{t_{m'}\Delta_2''} \text{ and } (-1)^{t_{m'}\Delta_3''}.$$

From $M_1 \circ n_1 = m$, we get

$$(2)n_1 = \begin{pmatrix} {}^t a_1 m' + {}^t c_1 m'' - {}^t a_1 (c_1 {}^t d_1)_0 - {}^t c_1 (a_1 {}^t b_1)_0 \\ {}^t b_1 m' + {}^t d_1 m'' - {}^t b_1 (c_1 {}^t d_1)_0 - {}^t d_1 (a_1 {}^t b_1)_0 \end{pmatrix}.$$

Therefore, from $m + 2\Delta_1 = n_1$, we have

$$\begin{aligned}
\Delta_1'' &= \frac{n_1'' - m''}{2} \\
&= \frac{{}^t b_1 m'}{2} + \frac{({}^t d_1 - I_g)m''}{2} - \frac{{}^t b_1 (c_1 {}^t d_1)_0}{2} - \frac{{}^t d_1 (a_1 {}^t b_1)_0}{2} \\
&\equiv \frac{{}^t b_1 m'}{2} + \frac{({}^t d_1 - I_g)m''}{2} - \frac{({}^t b_1)_0}{2} \pmod{2},
\end{aligned}$$

by noting that $b_1 \equiv c_1 \equiv 0 \pmod{2}$ and $a_1 \equiv d_1 \equiv I_g \pmod{2}$. So

$$\begin{aligned}
&{}^t m' \Delta_1'' \\
&\equiv \frac{{}^t m' {}^t b_1 m'}{2} + \frac{{}^t m' ({}^t d_1 - I_g)m''}{2} \\
&\quad - \frac{{}^t m' ({}^t b_1)_0}{2} \pmod{2} \\
&\equiv \frac{{}^t m' ({}^t d_1 - I_g)m''}{2} \pmod{2},
\end{aligned}$$

because $\frac{{}^t b_1}{2} \pmod{2}$ is symmetric, which follows from the fact $a {}^t b_1$ is symmetric. Similarly,

$${}^t m' \Delta_2'' \equiv \frac{{}^t m' ({}^t d_2 - I_g)m''}{2} \pmod{2}.$$

The computation for Δ_3'' is more complicated. Recall that $M_1 \circ n_1 = m$, $M_2 \circ \bar{n} = n_1$, $m + 2\Delta_3 = \bar{n}$ and (2), we have

$$n_1 \equiv \begin{pmatrix} {}^t a_1 m' + {}^t c_1 m'' - {}^t a_1 (c_1 {}^t d_1)_0 \\ {}^t b_1 m' + {}^t d_1 m'' - {}^t d_1 (a_1 {}^t b_1)_0 \end{pmatrix} \pmod{4}.$$

From $M_2 \circ \bar{n} = n_1$, we get

$$\bar{n} \equiv \begin{pmatrix} {}^t a_2 n'_1 + {}^t c_2 n''_1 - {}^t a_2 (c_2 {}^t d_2)_0 \\ {}^t b_2 n'_1 + {}^t d_2 n''_1 - {}^t d_2 (a_2 {}^t b_2)_0 \end{pmatrix} \pmod{4}.$$

Therefore,

$$\begin{aligned}
\bar{n}'' &\equiv {}^t b_2 ({}^t a_1 m' + {}^t c_1 m'' - {}^t a_1 (c_1 {}^t d_1)_0) + {}^t d_2 ({}^t b_1 m' \\
&\quad + {}^t d_1 m'' - {}^t d_1 (a_1 {}^t b_1)_0) - {}^t d_2 (a_2 {}^t b_2)_0 \pmod{4} \\
&\equiv {}^t b_2 {}^t a_1 m' + {}^t d_2 {}^t b_1 m' + {}^t d_2 {}^t d_1 m'' \\
&\quad - {}^t d_2 {}^t d_1 (a_1 {}^t b_1)_0 - {}^t d_2 (a_2 {}^t b_2)_0 \pmod{4}.
\end{aligned}$$

By the definition of Δ_3 , we have

$$\begin{aligned}
\Delta_3'' &= \frac{\bar{n}'' - m''}{2} \\
&\equiv \frac{{}^t b_2 {}^t a_1 m'}{2} + \frac{{}^t d_2 {}^t b_1 m'}{2} + \frac{({}^t d_2 {}^t d_1 - I_g)m''}{2} \\
&\quad - \frac{{}^t d_2 {}^t d_1 (a_1 {}^t b_1)_0}{2} - \frac{{}^t d_2 (a_2 {}^t b_2)_0}{2} \pmod{2} \\
&\equiv \frac{{}^t b_2 m'}{2} + \frac{{}^t b_1 m'}{2} + \frac{({}^t d_2 {}^t d_1 - I_g)m''}{2} \\
&\quad - \frac{({}^t b_1)_0}{2} - \frac{({}^t b_2)_0}{2} \pmod{2},
\end{aligned}$$

and

$$\begin{aligned}
&{}^t m' \Delta_3'' \\
&\equiv \frac{{}^t m' {}^t b_2 m'}{2} + \frac{{}^t m' {}^t b_1 m'}{2} + \frac{{}^t m' ({}^t d_2 {}^t d_1 - I_g)m''}{2} \\
&\quad - \frac{{}^t m' ({}^t b_1)_0}{2} - \frac{{}^t m' ({}^t b_2)_0}{2} \pmod{2} \\
&\equiv \frac{{}^t m' ({}^t d_2 {}^t d_1 - I_g)m''}{2} \pmod{2},
\end{aligned}$$

by using the expansions of quadratic forms and the fact that $\frac{{}^t b_2}{2}, \frac{{}^t b_1}{2}$ are symmetric modulo 2. Finally, the verification of

$${}^t m' \Delta_3'' \equiv {}^t m' \Delta_1'' + {}^t m' \Delta_2'' \pmod{2}$$

is easy, which comes from the simple fact

$$({}^t d_2 - I_g)({}^t d_1 - I_g) \equiv 0 \pmod{4}.$$

This completes the proof of Theorem 3.1. \square

In [2], Igusa gave the generators A_{ij}, B_{ij}, C_{ij} of $\Gamma_g(2)$, where

- (a) $1 \leq i \neq j \leq g$, $A_{ij} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $d = {}^t a^{-1}$, a is obtained by replacing (i, j) -coefficient in I_g by 2;
- (b) $1 \leq i \leq g$, $A_{ii} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $d = {}^t a^{-1}$, a is obtained by replacing (i, i) -coefficient in I_g by -1 ;

- (c) $1 \leq i < j \leq g$, $B_{ij} = \begin{pmatrix} I_g & b \\ 0 & I_g \end{pmatrix}$, b is obtained by replacing (i, j) - and (j, i) -coefficients in 0 by 2;
- (d) $1 \leq i \leq g$, $B_{ii} = \begin{pmatrix} I_g & b \\ 0 & I_g \end{pmatrix}$, b is obtained by replacing (i, i) -coefficient in 0 by 2;
- (e) $1 \leq i \leq j \leq g$, $C_{ij} = {}^t B_{ij}$.

By noting the computation of Δ , which depends on m and M , we find $(-1)^{{}^t m' \Delta''} = 1$ for $M = B_{ij}$ or C_{ij} , because in these cases, $d = I_g$, hence ${}^t m' \Delta'' = 0$. If $M = A_{ij}$, it is easy to find that ${}^t m' \Delta'' = -m'_i m''_j \equiv m'_i m''_j \pmod{2}$. We can easily compute $\Phi_m(M)$ for $M = A_{ij}, B_{ij}$ and C_{ij} . Now the values of $\chi_m(M)$ for the generators are

$$\begin{aligned} \chi_m(A_{ij}) &= (-1)^{m'_i m''_j}, & \chi_m(B_{ij}) &= (-1)^{m'_i m''_j}, \\ \chi_m(B_{ii}) &= (-1)^{m'_i} e\left(\frac{-(m'_i'')^2}{4}\right), \\ \chi_m(C_{ii}) &= e\left(\frac{-(m''_i')^2}{4}\right), & \chi_m(C_{ij}) &= (-1)^{m'_i m''_j}. \end{aligned}$$

Using the definitions of $\Phi_m(M)$ and ${}^t m' \Delta''$, it is easy to prove $\chi_m(M) = 1$ for M in the Igusa modular group $\Gamma_g(4, 8)$. Hence, by the computations above, we get

Theorem 3.4. *Write M in the form*

$$M = \prod_{1 \leq i, j \leq g} A_{ij}^{p_{ij}} \prod_{1 \leq i \leq j \leq g} B_{ij}^{q_{ij}} \prod_{1 \leq i \leq j \leq g} C_{ij}^{r_{ij}} M'$$

with $p_{ij}, q_{ij}, r_{ij} \in \mathbf{Z}$ and M' is in the commutator subgroup of $\Gamma_g(2)$, which is in $\Gamma_g(4, 8)$, then

$$\chi_m(M) = (-1)^A e\left(-\frac{B}{4}\right)$$

with

$$\begin{aligned} A &= \sum_{1 \leq i, j \leq g} p_{ij} m'_i m''_j + \sum_{1 \leq i \leq j \leq g} q_{ij} m'_i m''_j \\ &\quad + \sum_{1 \leq i < j \leq g} r_{ij} m''_i m''_j, \\ B &= \sum_{1 \leq i \leq g} q_{ii} (m'_i')^2 + \sum_{1 \leq i \leq g} r_{ii} (m''_i'')^2. \end{aligned}$$

4. Applications. If we define $\psi(\tau) = \theta_m(\tau)\theta_n(\tau)$, then for $M \in \Gamma_g(2)$,

$$\psi(M\tau) = a^2(M)\chi_m(M)\chi_n(M)\det(c\tau + d)\theta_m(\tau)\theta_n(\tau),$$

we find $\chi_m(M)\chi_n(M)$ is exactly the character defined by Igusa in [2], hence our theorems can recover Igusa's Theorem 3 in [2]. Our character

$\chi_m(M)$ is more fundamental, moreover we can see the relations between $\chi_m(M)$ and $\Phi_m(M)$.

We can use our results to give a more transparent proof of the key part of Theorem 5 in Igusa's paper [2]. The key part of Theorem 5 in that paper is from the invariant condition that $\frac{\theta_m(M\tau)}{\theta_n(M\tau)} = \frac{\theta_m(\tau)}{\theta_n(\tau)}$ holds for all even m, n to infer M is in $\Gamma_g(4, 8)$, where M is in $\Gamma_g(2)$. By the definition of $\chi_m(M)$, this is equivalent to the congruence $\chi_m(M) = \chi_n(M)$ holds for all even m, n , i.e.

$$e(\Phi_m(M))(-1)^{\frac{{}^t m'({}^t d - I_g)m''}{2}}$$

is equal to

$$e(\Phi_n(M))(-1)^{\frac{{}^t n'({}^t d - I_g)n''}{2}}.$$

Let $n' = n'' = 0$, we get for any even m ,

$$\frac{{}^t m'({}^t d - I_g)m''}{2} \equiv 0 \pmod{2}$$

and

$${}^t m' {}^t b d m' + {}^t m'' {}^t a c m'' - 2^t (a^t b)_0 d m' \equiv 0 \pmod{8}.$$

The first congruence implies $d \equiv I_g \pmod{4}$, from $a^t d - b^t c = I_g$, and we get $a \equiv I_g \pmod{4}$. In the second congruence, let $m' = 0$, we get ${}^t m'' {}^t a c m'' \equiv 0 \pmod{8}$, this implies ${}^t a c \equiv 0 \pmod{4}$, hence $c \equiv 0 \pmod{4}$, and $({}^t a c)_0 \equiv 0 \pmod{8}$. If we write ${}^t a = I_g + 4\bar{a}$, then we have

$$({}^t a c)_0 = ((I_g + 4\bar{a})c)_0 \equiv (c)_0 \equiv 0 \pmod{8}.$$

Let $m'' = 0$, we get the congruence

$${}^t m' {}^t b d m' - 2^t (a^t b)_0 d m' \equiv 0 \pmod{8},$$

which is equivalent to the congruence

$$(3) \quad {}^t m' {}^t b d m' - 2^t (b)_0 m' \equiv 0 \pmod{8}.$$

Write ${}^t b = (2\bar{b}_{ij})$ and $d = I_g + 4\bar{d}$, then

$$\begin{aligned} & {}^t m' {}^t b d m' - 2^t (b)_0 m' \\ &= {}^t m' 2\bar{b}_{ij} (I_g + 4\bar{d}) m' - 2(2\bar{b}_{ij})_0 m' \\ &\equiv 2^t m' \bar{b}_{ij} m' - 4(\bar{b}_{ij})_0 m' \pmod{8} \\ &\equiv 2\bar{b}_{11} m_1'^2 - 4\bar{b}_{11} m_1' \pmod{8} \end{aligned}$$

by taking ${}^t m' = (m_1', 0, 0, \dots, 0)$. Hence (3) implies $\bar{b}_{11} \equiv 0 \pmod{4}$. Similarly, we can prove $\bar{b}_{ii} \equiv 0 \pmod{4}$ holds for each $1 \leq i \leq g$. Therefore, we have $(b)_0 \equiv 0 \pmod{8}$. Combining it with (3), we find the congruence ${}^t m' {}^t b d m' \equiv 0 \pmod{8}$ holds for any even m' , which implies ${}^t b d \equiv 0 \pmod{4}$, hence $b \equiv 0 \pmod{4}$. The analysis above shows that M is

in $\Gamma_g(4)$ and $(b)_0 \equiv (c)_0 \equiv 0 \pmod{8}$. The observation of Igusa in [2, p. 222, line 4-line 6] shows M is in $\Gamma_g(4, 8)$.

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