Graded Lie algebras and regular prehomogeneous vector spaces with one-dimensional scalar multiplication

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Abstract: The aim of this paper is to study relations between regular reductive prehomogeneous vector spaces (PVs) with one-dimensional scalar multiplication and the structure of graded Lie algebras. We will show that the regularity of such PVs is described by an $\mathfrak{sl}_2$-triplet of a graded Lie algebra.

Key words: Prehomogeneous vector spaces; graded Lie algebras; standard pentads.

Introduction. A prehomogeneous vector space (abbrev. PV) is a triplet (G, ρ, V) consisting of a connected algebraic group G and its finite-dimensional rational representation (ρ, V) with a Zariski-dense orbit. Some particular cases of PVs are obtained from a graded finite-dimensional semisimple Lie algebra $\mathfrak{l} = \bigoplus_{n\in\mathbf{Z}} \mathfrak{l}_n$ as $(\mathfrak{l}_0, \mathfrak{l}_1)$. Such spaces are named PVs of parabolic type by H. Rubenthaler (see, for example, [2]) and studied by him. Then, today, it is known that PVs of parabolic type have rich structures related to the structure of graded Lie algebras. For example, the regularity of irreducible PVs of parabolic type $(\mathfrak{l}_0, \mathfrak{l}_1)$ is closely related to a subalgebra of $\mathfrak{l}$ which is isomorphic to $\mathfrak{sl}_2$ ([2, Corollaire II.2.15]).

In [2], H. Rubenthaler classified PVs of parabolic type using Dynkin diagrams of finite-dimensional semisimple Lie algebras. On the other hand, it is known that there exist infinitely many PVs which are not of parabolic type.

Recently, the author and H. Rubenthaler independently showed that any finite-dimensional reductive Lie algebra and its finite-dimensional representation can be embedded into some (finite or infinite-dimensional) graded Lie algebra ([5, the author], [3, H. Rubenthaler]). Hence, a PV with a reductive group and its representation can be also embedded. Thus, it is expected that we can extend the theory of PVs of parabolic type to the general theory of PVs. The aim of this paper is to study relations between the regularity of (not necessarily of parabolic type) PVs with 1-dimensional scalar multiplication and the structure of graded Lie algebras.

Notation 0.1.

- For an arbitrary vector space W, we denote by $\mathrm{Hom}(W, \mathbf{C})$ the set of all linear maps from W to $\mathbf{C}$. Moreover, when W is finite-dimensional, we denote by $W^* = \mathrm{Hom}(W, \mathbf{C})$ for simplicity.
- We denote the zero-matrix of size $k \times l$ by $O_{k,l}$ or O_k when $k = l$, the unit matrix of size k by I_k. We denote the set of all matrices of size $k \times l$ by $M(k, l)$ or $M(k)$ when $k = l$.
- In this paper, all objects are defined over the complex number field $\mathbf{C}$.

1. Constructions of graded Lie algebras. First of all, we shall introduce the notion of standard pentads.

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra, π a representation of $\mathfrak{g}$ on U, $\mathcal{U}$ a $\mathfrak{g}$-submodule of $\mathrm{Hom}(U, \mathbf{C})$, B a non-degenerate invariant bilinear form on $\mathfrak{g}$. When a pentad $(\mathfrak{g}, \pi, U, \mathcal{U}, B)$ satisfies the following conditions, we say that the pentad $(\mathfrak{g}, \pi, U, \mathcal{U}, B)$ is a standard pentad:

(a) the restriction to $U \times \mathcal{U}$ of the canonical pairing $\langle\cdot,\cdot\rangle : U \times \mathrm{Hom}(U, \mathbf{C}) \to \mathbf{C}$ is non-degenerate,

(b) there exists a linear map $\Phi_\pi : U \otimes \mathcal{U} \to \mathfrak{g}$, called the Φ-map of the pentad, satisfying an equation

$$B(a, \Phi_\pi(v \otimes \phi)) = \langle \pi(a)v, \phi\rangle$$

for any $a \in \mathfrak{g}$, $v \in U$ and $\phi \in \mathcal{U}$

(see [5, Definitions 2.1, 2.2]).

Theorem 1.2. *Let $(\mathfrak{g}, \pi, U, \mathcal{U}, B)$ be an arbitrary standard pentad. Then there exists a (finite or infinite dimensional) graded Lie algebra*

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$L(\mathfrak{g},\pi,U,\mathcal{U},B)=\bigoplus_{n\in\mathbf{Z}}U_n$ such that

$$U_0 \simeq \mathfrak{g} \ (\textit{as Lie algebras}),$$
$$U_1 \simeq U, \quad U_{-1} \simeq \mathcal{U} \ (\textit{as } U_0 \simeq \mathfrak{g}\textit{-modules})$$

and that the restricted bracket product $[\cdot,\cdot]: U_1 \times U_{-1} \to U_0$ is induced by the Φ-map of the standard pentad $(\mathfrak{g},\rho,U,\mathcal{U},B)$ (see [5, Theorem 2.15]).

In the sense of Theorem 1.2, we can obtain a graded Lie algebra such that a given representation of a reductive Lie algebra can be embedded into its local part. To prove Theorem 1.2, the author constructed graded components $U_0, U_{\pm1}, U_{\pm2}\dots$ inductively.

On the other hand, H. Rubenthaler has obtained similar result independently in [3]. In [3, Theorem 3.1.2], he constructed a local Lie algebra $\Gamma(\mathfrak{g}_0,B_0,\pi)$ from a fundamental triplet $(\mathfrak{g}_0,B_0,(\pi,U))$, which consists of a quadratic Lie algebra $(\mathfrak{g}_0,B_0)$ and its finite-dimensional representation (π,U), and constructed a graded Lie algebra $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0,B_0,\pi))$ using [1, Proposition 4] by V. G. Kac. Although the constructions of him and of the author are based on different theories, their goals coincide.

Theorem 1.3. *Let $(\mathfrak{g},\pi,U,U^*,B)$ be a standard pentad with finite-dimensional objects and a symmetric bilinear form B. Then the corresponding Lie algebra $L(\mathfrak{g},\pi,U,U^*,B)$ is isomorphic to the Lie algebra $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g},B,\pi))$ as graded Lie algebras.*

Proof. To prove our claim, it suffices to show that $L(\mathfrak{g},\pi,U,\mathcal{U},B)$ is a minimal Lie algebra (for details on minimal Lie algebras, see [1, Definition 6]). We use a similar argument to the argument in [1, p. 1278, Proposition 5]. Suppose that $L(\mathfrak{g},\pi,U,U^*,B)=\bigoplus_{n\in\mathbf{Z}}U_n$ is not minimal. Then there exists a non-zero graded ideal $J=\bigoplus_{n\in\mathbf{Z}}(J\cap U_n)$ such that $J\cap(U_{-1}\oplus U_0\oplus U_1)=J\cap(U^*\oplus\mathfrak{g}\oplus U)=\{0\}$. Take an integer k such that $J\cap U_k\neq\{0\}$ and $J\cap U_n=\{0\}$ for any $|n|<|k|$. If $k>0$, there exists a non-zero element $v\in J\cap U_n$ such that $[v,U_{-1}]=\{0\}$. It contradicts the construction by the author that $U_k\subset \mathrm{Hom}(U_{-1},U_{k-1})$ (see [5, Definition 2.9]). The case where $k<0$ is similar. □

Thus, the theories of graded Lie algebras by the author and by H. Rubenthaler are essentially same. For example, using [6, Theorem 3.2] by the author, we can find the structure of a graded Lie algebra $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0,B_0,\pi))$, where $\mathfrak{g}_0$ is reductive and π is completely reducible, constructed by H. Rubenthaler. On the other hand, H. Rubenthaler obtained important results on relative invariant in [3, section 4]. In the remaining part of this paper, we shall use notion and notations based on the author's works unless noted otherwise. Here, we need to import some notions by H. Rubenthaler to the theory standard pentads.

Definition 1.4. Let $(\mathfrak{g},\pi,U,\mathcal{U},B)$ be a standard pentad. If $H_0\in\mathfrak{g}\subset L(\mathfrak{g},\pi,U,\mathcal{U},B)=\bigoplus_{n\in\mathbf{Z}}U_n$ satisfies the following conditions:

$$[H_0,A]=0,\quad [H_0,X]=2X,\quad [H_0,Y]=-2Y$$

for any $A\in\mathfrak{g}\simeq U_0$, $X\in U\simeq U_1$, $Y\in\mathcal{U}\simeq U_{-1}$, we say that H is a grading element of the pentad $(\mathfrak{g},\pi,U,\mathcal{U},B)$ or the graded Lie algebra $L(\mathfrak{g},\pi,U,\mathcal{U},B)$ or its local part $\mathcal{U}\oplus\mathfrak{g}\oplus U$ (cf. [3, Remark 3.4.4]).

Definition 1.5. Let $(\mathfrak{g},\pi,U,\mathcal{U},B)$ be a standard pentad. When a triplet $(y,h,x)\in L(\mathfrak{g},\pi,U,\mathcal{U},B)^3$ satisfies the following conditions, we say that the pentad is an $\mathfrak{sl}_2$-triplet:

$$[h,x]=2x,\quad [h,y]=-2y,\quad [x,y]=h$$

(cf. [3, p. 53]).

We give the notion of prehomogeneity of standard pentads.

Definition 1.6. Let $(\mathfrak{g},\pi,U,\mathcal{U},B)$ be a standard pentad with Φ-map Φ_π. When the pentad satisfies the following condition, we say that the pentad $(\mathfrak{g},\pi,U,\mathcal{U},B)$ is a prehomogeneous pentad:

- there exists an element $X\in U$ such that a linear map $\Phi_\pi(X\otimes\cdot):\mathcal{U}\to\mathfrak{g}$ defined by $\phi\in\mathcal{U}\mapsto\Phi_\pi(X\otimes\phi)\in\mathfrak{g}$ is injective.

In other words, a pentad $(\mathfrak{g},\pi,U,\mathcal{U},B)$ is prehomogeneous if and only if its corresponding Lie algebra $L(\mathfrak{g},\pi,U,\mathcal{U},B)=\bigoplus_{n\in\mathbf{Z}}U_n$ has an element $X\in U_1$ such that the adjoint map $\mathrm{ad}X: U_{-1}\to U_0$ is injective. Moreover, we call such an element a generic point of the pentad (see [4, Definition 2.2]).

The terms "prehomogeneous" and "generic points" come from the theory of prehomogeneous vector spaces.

Definition 1.7. Let G be a connected linear algebraic group and (ρ,V) its finite-dimensional rational representation. We call a triplet (G,ρ,V) a prehomogeneous vector space (abbrev. PV) when there exists a Zariski-dense orbit $\rho(G)x$ in V. In particular, when a PV (G,ρ,V) has a reductive group G, we call it a reductive PV. An element $x'\in V$ is called a generic point when it belongs to the Zariski-dense orbit $\rho(G)x$. When a triplet is a PV, its Zariski-dense orbit is determined uniquely (see [7, p. 35, Definition 1]).

Theorem 1.8. *We let G be an arbitrary finite-dimensional reductive algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Let (ρ, V) be a finite-dimensional representation of G and $(d\rho, V)$ its infinitesimal representation of $\mathfrak{g}$. Then the following two conditions are equivalent*:

(a) *A triplet (G, ρ, V) is a PV,*

(b) *A pentad $(\mathfrak{g}, d\rho, V, V^*, B)$ is a prehomogeneous pentad for any non-degenerate invariant symmetric bilinear form B.*

Moreover, an element $x \in V$ is a generic point of (G, ρ, V) in the sense of PVs if and only if x is a generic point of $(\mathrm{Lie}(G), d\rho, V, V^*, B)$ *in the sense of prehomogeneous pentads* (*see* [4, Theorems 2.1, 2.4]).

The theory of PVs has the notion of the regularity.

Definition 1.9. Let (G, ρ, V) be a PV with a generic point x. Let $G_x = \{g \in G \mid \rho(g)x = x\}$ the isotropy subgroup of G at x. Let $\mathfrak{g}_1$ be a subalgebra of $\mathrm{Lie}(G)$ generated by $\mathrm{Lie}(G_x)$ and $[\mathrm{Lie}(G), \mathrm{Lie}(G)]$ and put $\overline{X}_1 = \{\omega \in \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbf{C}) \mid \omega\,|_{\mathfrak{g}_1} = 0\}$. Then the PV (G, ρ, V) is called quasi-regular if there exist $\omega \in \overline{X}_1$ and a rational map $\varphi_\omega : \rho(G)x \to V^*$ such that

$$\varphi_\omega(\rho(g)x') = \rho^*(g)\varphi_\omega(x'),$$
$$\langle d\rho(A)x', \varphi_\omega(x')\rangle = \omega(A)$$

for any $A \in \mathrm{Lie}(G)$, $g \in G$ and $x' \in \rho(G)x$ and that the image of φ_ω is Zariski-dense in V^*. In this case, ω is called non-degenerate. In particular, if there exists a character $\chi : G \to \mathbf{C}$ which corresponds to some relative invariants such that $\omega = d\chi$, then the PV (G, ρ, V) is called regular (see [8, p. 119]).

In general, we need to distinguish the notions of regularity and quasi-regularity. However, under the assumption that a group in a triplet is reductive, we have the following theorem.

Theorem 1.10. *We let (G, ρ, V) be a PV and assume that G is reductive. Then (G, ρ, V) is regular if and only if it is quasi-regular* (*see* [8, Proposition 1.3]).

2. PVs and graded Lie algebras. In this section, we shall consider how to describe the regularity of PVs using the theory of graded Lie algebras. In [3], H. Rubenthaler defined the following condition $(P)_X$:

$$(P)_X\colon \quad X \notin [[\mathfrak{g}, \mathfrak{g}], X]$$

and proved that the condition $(P)_x$ is closely related to $\mathfrak{sl}_2$-triplets and relative invariants of a representation (see [3, pp. 53–58, section 4]) under the Assumption (H).

Definition 2.1. We say that a representation $(\mathfrak{g}, \pi, U)$ of a Lie algebra $\mathfrak{g}$ satisfies Assumption (H) when the followings hold:

(a) The Lie algebra $\mathfrak{g}$ is a reductive Lie algebra with one-dimensional center:

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}], \quad \dim Z(\mathfrak{g}) = 1,$$

(b) We suppose also that $Z(\mathfrak{g})$ acts by a non-trivial character (i.e. $\pi(Z(\mathfrak{g})) = \mathbf{C}\mathrm{Id}$)

(see [3, p. 53]).

If $(\mathrm{Lie}(G), d\rho, V)$ satisfies the Assumption (H), it means that (G, ρ, V) is a group representation of a reductive group G with 1-dimensional scalar multiplication.

In the remaining part of this paper, we shall define similar conditions and consider relations between these conditions and the regularity of PVs.

Definition 2.2. Let $(\mathfrak{g}, \pi, U, \mathcal{U}, B)$ be a standard pentad. When elements $H \in \mathfrak{g}$ and $X \in U$ (respectively, $H \in \mathfrak{g}$ and $Y \in U^*$) have an element $\eta \in U^*$ (respectively, $\xi \in U$) such that a triplet (η, H, X) (respectively, (Y, H, ξ)) is an $\mathfrak{sl}_2$-triplet, we denote that $(P)_{(\cdot,H,X)}$ (respectively, $(P)_{(Y,H,\cdot)}$). Moreover, if an element η (respectively, ξ) is determined from H and X (respectively, H and Y) uniquely, we denote that $(P)^!_{(\cdot,H,X)}$ (respectively, $(P)^!_{(Y,H,\cdot)}$).

Proposition 2.3. *Let $(\mathfrak{g}, \pi, U, \mathcal{U}, B)$ be a standard pentad. If there exist elements $H \in \mathfrak{g}$ and $X \in U$ satisfying $(P)^!_{(\cdot,H,X)}$, then the pentad $(\mathfrak{g}, \pi, U, \mathcal{U}, B)$ is prehomogeneous with generic point X.*

Proof. We take a unique element $Y = Y(H, X)$ such that (Y, H, X) is an $\mathfrak{sl}_2$-triplet. If we suppose that X is not a generic point, there exists $0 \neq \eta \in \mathcal{U}$ such that $[X, \eta] = 0$. Then we have two $\mathfrak{sl}_2$-triplets (Y, H, X) and $(Y + \eta, H, X)$, of course $Y \neq Y + \eta$. It contradicts the assumption $(P)^!_{(\cdot,H,X)}$. □

Corollary 2.4. *Let (G, ρ, V) be a triplet and assume that G is a reductive group. If a pentad* $(\mathrm{Lie}(G), d\rho, V, V^*, B)$ *has elements $H \in \mathrm{Lie}(G)$ and $X \in V$ satisfying $(P)^!_{(\cdot,H,X)}$, then the triplet (G, ρ, V) is a PV.*

Similarly, we have the following proposition.

Proposition 2.5. *Let $(\mathfrak{g}, \rho, V, V^*, B)$ be a prehomogeneous pentad and $X \in V$ be a generic point of it. If there exists an element $H \in \mathfrak{g}$ satisfying $(P)_{(\cdot,H,X)}$, then H and X satisfy $(P)^!_{(\cdot,H,X)}$.*

Under these notations, we have the main theorem of this paper.

Theorem 2.6. *Let (G, ρ, V) be a triplet and assume that G is a reductive group. Assume that $(\mathrm{Lie}(G), d\rho, V)$ satisfies the Assumption (H). Then the following conditions are equivalent:*

(a) *The triplet (G, ρ, V) is a regular PV,*

(b) *For an arbitrary non-degenerate invariant bilinear form B on $\mathrm{Lie}(G)$, a pentad $(\mathrm{Lie}(G), d\rho, V, V^*, B)$ has elements $X \in V$ and $Y \in V^*$ such that (Y, H_0, X) is an $\mathfrak{sl}_2$-triplet and that the conditions $(P)^!_{(\cdot,H_0,X)}$ and $(P)^!_{(Y,H_0,\cdot)}$ hold, where H_0 is a grading element of the pentad.*

Proof. ((a) implies (b))

We assume that the triplet (G, ρ, V) is a regular PV with a generic point $x \in V$. Then there exists a non-degenerate linear map $\omega : Z(\mathrm{Lie}(G)) \to \mathbf{C}$ and a rational map $\varphi_\omega : \rho(G)x \to V^*$ such that

$$\varphi_\omega(\rho(g)x') = \rho^*(g)\varphi_\omega(x'),$$
$$\langle d\rho(A)x', \varphi_\omega(x')\rangle = \omega(A)$$

for any $g \in G$, $A \in \mathrm{Lie}(G)$, $x' \in \rho(G)x$. Since B is non-degenerate, there exists $H \in \mathrm{Lie}(G)$ such that $\omega(A) = B(A, H)$ for any $A \in \mathrm{Lie}(G)$. Since $\omega |_{[\mathrm{Lie}(G),\mathrm{Lie}(G)]} = 0$, we have $H = cH_0 \in Z(\mathrm{Lie}(G))$ for some $c \in \mathbf{C}$. Here, from the assumption that ω is non-degenerate, $c \neq 0$. Thus, in $L(\mathrm{Lie}(G), d\rho, V, V^*, B)$, we have an equation

$$B(A, [x', \varphi_\omega(x')]) = \langle d\rho(A)x', \varphi_\omega(x')\rangle$$
$$= \omega(A) = B(A, H)$$

for any $A \in \mathrm{Lie}(G)$ and $x' \in \rho(G)x$. From this, we can deduce that

$$[x, \varphi_\omega(x)] = H = cH_0.$$

Then, $(Y, H_0, X) = ((1/c)\varphi_\omega(x), H_0, x)$ is an $\mathfrak{sl}_2$-triplet. Since X belongs to the Zariski-dense orbit, we have $(P)^!_{(\cdot,H_0,X)}$. Moreover, since the orbit $\rho^*(G)Y =$ (the image of φ_ω) is Zariski-dense in V^*, we have $(P)^!_{(Y,H_0,\cdot)}$. Thus, we have the condition (b).

((b) implies (a))

We suppose the condition (b) and take an arbitrary non-degenerate invariant bilinear form B. Then (G, ρ, V) is a PV with a Zariski-dense orbit $\rho(G)X \subset V$. We can define a map $\varphi : \rho(G)X \to V^*$ by

$$(x', H_0, \varphi(x')) \text{ is an } \mathfrak{sl}_2\text{-triplet for } x' \in \rho(G)X$$

satisfying

$$\rho^*(g)\varphi(x') = \varphi(\rho(g)x') \quad (g \in G, x' \in \rho(G)X)$$

(see [3, proof of Proposition 4.2.7]). That is, $\eta = \varphi(x')$ is a unique solution of a linear equation $\mathrm{ad}(x')\eta = H_0$. Thus, φ is a rational map. If we define $\omega : \mathrm{Lie}(G) \to \mathbf{C}$ by $\omega(A) = B(A, H_0)$, then we have an equation

$$\langle d\rho(A)x', \varphi(x')\rangle = B(A, H_0) = \omega(A).$$

In the notations of Definition 1.9, ω clearly belongs to $\overline{X}_1$. From the assumption $(P)^!_{(Y,H_0,\cdot)}$, we have that (the image of φ) $= \rho^*(G)\varphi(X) = \rho^*(G)Y$ is Zariski-dense in V^*. Thus, we have that (G, ρ, V) is quasi-regular, and thus, regular. □

Under the notation of Theorem 2.6, note that $(P)_X$ is equivalent to $(P)_{(\cdot,H_0,X)}$ (see [3, Theorem 4.1.2]).

Definition 2.7. Define a bilinear form T_n on $\mathfrak{gl}_n$ by

$$T_n(X, Y) = \mathrm{Tr}(XY)$$

for any $X, Y \in \mathfrak{gl}_n$. Clearly, the bilinear form T_n is non-degenerate and invariant. Moreover, for a Lie subalgebra $l \subset \mathfrak{gl}_n$, we also denote the restriction $T_n |_{l\times l}$ by the same symbol T_n.

Example 2.8. An irreducible PV

$$(G, \rho, V)$$
$$= (GL_1 \times Sp_n \times SO_3, \square \otimes \Lambda_1 \otimes \Lambda_1, M(2n, 3))$$

$(2n \geq 3)$ is very important for us since it is an example of a non-regular PV which has a relative invariant (see [7, p. 105, Proposition 19]). Let us show this claim using a pentad

$$(\mathrm{Lie}(G), d\rho, V, V^*, B)$$
$$= (\mathfrak{gl}_1 \oplus \mathfrak{sp}_n \oplus \mathfrak{so}_3, \square \otimes \Lambda_1 \otimes \Lambda_1, M(2n, 3),$$
$$M(2n, 3), T_1 \oplus T_{2n} \oplus T_3),$$

where

$$\mathfrak{sp}_n = \{A \in M(2n) \mid A \cdot J_n + J_n \cdot {}^t A = O_{2n}\},$$
$$J_n = \left(\begin{array}{c|c} O_n & I_n \\ \hline -I_n & O_n \end{array}\right).$$

The representations $d\rho$ and its dual $d\rho^*$ are given by:

$$d\rho(a,A,B)v = av + Av - vB,$$
$$d\rho^*(a,A,B)u = -au + Au - uB$$

($a \in \mathfrak{gl}_1, A \in \mathfrak{sp}_n, B \in \mathfrak{so}_3$) via a bilinear form

$$\langle v,u\rangle = \mathrm{Tr}({}^t v \cdot J_n \cdot u) \quad (v,u \in M(2n,3)).$$

The Φ-map $\Phi_{d\rho}$ of this pentad is given by

$$\Phi_{d\rho}(v \otimes u) = \left(\mathrm{Tr}({}^t v \cdot J_n \cdot u), -\frac{1}{2}(v \cdot {}^t u + u \cdot {}^t v)J_n, \frac{1}{2}({}^t v \cdot J_n \cdot u + {}^t u \cdot J_n \cdot v)\right).$$

We can easily check that the pentad has a grading element H_0 and a generic point X

$$H_0 = (2, O_{2n}, O_3),$$
$$X = {}^t\left(\begin{array}{cc|c|c|c} 1 & 0 & & 0 & \\ 0 & 1 & O_{3,n-2} & 0 & O_{3,n-1} \\ 0 & 0 & & 1 & \end{array}\right)$$

and satisfies the Assumption (H). We can easily check that H_0 and X satisfy $(P)^!_{(\cdot,H_0,X)}$. In fact, if we put

$$\eta = {}^t\left(\begin{array}{c|c|c|c} 0 & & 1 & \\ 0 & O_{3,n-1} & 0 & O_{3,n-1} \\ -1 & & 0 & \end{array}\right),$$

then we can obtain an $\mathfrak{sl}_2$-triplet (η, H_0, X), and thus, $(P)_{(\cdot,H_0,X)}$ holds. Since X is a generic point, we have $(P)^!_{(\cdot,H_0,X)}$. From the result of H. Rubenthaler, [3, Theorem 4.2.3], we can deduce that there exists a non-trivial relative invariant on $V =$ (the Zariski closure of $\rho(G)X$). Here, (G,ρ,V) is not regular. If we suppose that (G,ρ,V) is regular, then we have an $\mathfrak{sl}_2$-triplet (Y', H_0, X') such that $(P)^!_{(\cdot,H_0,X')}$ and $(P)^!_{(Y',H_0,\cdot)}$ hold. Then, there exists $g \in G$ such that $X' = \rho(g)X$. Then, we have that $\eta = \rho^*(g^{-1})Y'$ belongs to the Zariski-dense orbit $\rho^*(G)Y'$ in V^*. However, since $\operatorname{rank}\eta = 2$, the orbit $\rho^*(G)\eta$ cannot be Zariski-dense in $M(2n,3)$. It is a contradiction.

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