Some remarks on log surfaces

By Haidong Liu

Department of Mathematics, Graduate School of Science, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

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Abstract: Fujino and Tanaka established the minimal model theory for **Q**-factorial log surfaces in characteristic 0 and p, respectively. We prove that every intermediate surface has only log terminal singularities if we run the minimal model program starting with a pair consisting of a smooth surface and a boundary **R**-divisor. We further show that such a property does not hold if the initial surface is singular.

Key words: ε-log terminal; minimal model program on log surfaces.

1. Introduction. We work over an algebraically closed field of arbitrary characteristic throughout this paper. We will also follow the language and notational conventions of the book [KM98] unless stated otherwise.

Let (X, Δ) be a log surface. Remember that a pair (X, Δ) is called *log surface* if X is a normal algebraic surface and Δ is a boundary **R**-divisor on X such that $K_X + \Delta$ is **R**-Cartier (See [Fjn12, Definition 3.1]). To complete Fujita's results [Fjt84] on the semi-ampleness of semi-positive parts of Zariski decompositions of log canonical divisors and the finite generation of log canonical rings for smooth projective log surfaces, Fujino [Fjn12] developed the log minimal model program for projective log surfaces in characteristic 0. It is generalized to characteristic $p > 0$ by Tanaka in his paper [Tnk14]. One of their main results is the following

Theorem 1.1 ([Fjn12, Theorem 3.3], [Tnk14, Theorem 1.1]). *Let (X, Δ) be a log surface which is not necessarily log canonical, and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Assume that X is **Q**-factorial. Then we can run the log minimal model program over S with respect to $K_X + \Delta$ and get a sequence of at most $\rho(X/S) - 1$ contractions*

$$(X, \Delta) = (X_0, \Delta_0) \to (X_1, \Delta_1) \to \cdots \to (X_k, \Delta_k) = (X^*, \Delta^*)$$

over S such that one of the following holds:

(1) (*Minimal model*) *$K_{X^*} + \Delta^*$ is nef over S. In this case, (X^*, Δ^*) is called a minimal model of (X, Δ).*

(2) (*Mori fiber space*) *There is a morphism $g : X^* \to C$ over S such that $-(K_{X^*} + \Delta^*)$ is g-ample, $\dim C < 2$, C is projective over S and $\rho(X^*/C) = 1$. We sometimes call $g : (X^*, \Delta^*) \to C$ a Mori fiber space.*

*Note that X_i is **Q**-factorial for every i. Furthermore, if $K_X + \Delta$ is big, then on the minimal model (X^*, Δ^*), $K_{X^*} + \Delta^*$ is nef and big over S.*

First, we try to clarify that, given such a log surface (X, Δ) where X is smooth, what every intermediate surface X_i would look like after running this log minimal model program. Note that the final log surface (X^*, Δ^*) could be a minimal model or a Mori fiber space $g : (X^*, \Delta^*) \to C$. The following theorem is our main result in this paper to achieve this aim.

Theorem 1.2 (Theorem 3.1). *Notations are as in Theorem 1.1. Let ε be a real number such that $0 \le \varepsilon < 1$. If X is smooth and the coefficients of Δ are $\le 1 - \varepsilon$, then X_i is ε-log terminal for every i. In particular, X^* is ε-log terminal.*

Next, a natural question is that, given a log surface (X, Δ) where X is not smooth, what every intermediate surface X_i would look like after running log minimal model program (Theorem 1.1).

Proposition 1.3. *In Theorem 1.1, X_i is not necessarily log canonical even if X is log canonical.*

Moreover, we have:

Proposition 1.4. *In Theorem 1.1, X_i is not necessarily log canonical even if X is ε-log canonical and the coefficients of Δ are $\le 1 - \varepsilon$ for some $0 < \varepsilon < 1$.*

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In Section 4 we construct some examples to show that Propositions 1.3 and 1.4 are true. Furthermore, we show that X_i could not even be MR log canonical if X is not smooth. In fact this shows that Fujino and Tanaka's minimal model program on log surfaces is more general than Alexeev's minimal model program which is running mainly on MR log canonical surfaces in [Alex94, Section 10] (see Definition 2.2 for the definition of MR log canonical).

2. Preliminaries. Let (X,Δ) be a log surface. If X is smooth, then it is $\mathbf{Q}$-factorial. Choose a set $I \subset [0, 1-\varepsilon]$ where $\varepsilon \in [0,1]$ is a fixed real number. Assume that the coefficients of Δ are in I. We say that I satisfies the *descending chain condition* or I is a DCC set for short, if it does not contain any infinite strictly decreasing sequence. Finally, recall that the *volume* of an $\mathbf{R}$-divisor D on a normal projective variety X of dimension n is defined as

$$\operatorname{vol}(D) = \limsup_{m\to\infty} \frac{h^0(X, \mathscr{O}_X(\lfloor mD \rfloor))}{m^n/n!}.$$

We recall some kinds of singularities and MR singularities following the same way of Alexeev.

Definition 2.1 ([Alex94, Definition 1.5])**.** Let (X,Δ) be a log surface. Let ε be a real number such that $0 \le \varepsilon < 1$. It is called:

1. ε-log canonical, if the total discrepancies $\ge -1+\varepsilon$,
2. ε-log terminal, if the total discrepancies $> -1+\varepsilon$

for every resolution $f : Y \to X$ in both cases. Simply, we call it ε-lc or ε-lt instead. Note that when ε is not zero, we can replace ε by a smaller positive ε', and assume that ε-log canonical is ε'-log terminal.

Definition 2.2 ([Alex94, Definition 1.7])**.** We call a log surface (X,Δ) MR log canonical, MR ε-log canonical, MR ε-log terminal etc. if we require the previous inequalities in Definition 2.1 to hold not for all resolutions $f : Y \to X$ but only for a minimal desingularization.

A strange but trivial example of MR log canonical log surface is the following

Example 2.3. Given a log surface (X,Δ), where X is smooth and Δ is a boundary. (X,Δ) is not necessarily log canonical in the usual sense. But $id : X \to X$ is the minimal desingularization, therefore (X,Δ) is MR log canonical.

3. Main results. Now we go to the proof of Theorem 1.2. Note that ε in this theorem could be zero:

Theorem 3.1. *Notations are as in Theorem 1.1. Let ε be a real number such that $0 \le \varepsilon < 1$. If X is smooth and the coefficients of Δ are $\le 1-\varepsilon$, then X_i is ε-log terminal for every i. In particular, X^* is ε-log terminal.*

Proof. **Step 1.** Run log minimal model program (Theorem 1.1) for $K_X + \Delta$ as in Theorem 1.1:

$$\begin{aligned}(X,\Delta) = (X_0,\Delta_0) \to (X_1,\Delta_1) \to \cdots \to (X_k,\Delta_k)\\ = (X^*,\Delta^*)\end{aligned}$$

where (X^*,Δ^*) is a minimal model or a Mori fiber space. In the following proof, we consider everything over X_j for a fixed j. Put $X^\dagger = X_j$ for this fixed j. Then take $X^\dagger$ as a base (that is, replace S by $X^\dagger$ and hence we are reduced to the case where $S = X^* = X_j$. Therefore, if needed, shrink $X^\dagger$ to be affine since ε-log terminal or not is a local property) and run $(K_X+\Delta)$-LMMP for the relative morphism $f : X \to X^\dagger$, which ends up again on $X^\dagger$ and $K_{X^\dagger} + \Delta^\dagger$ is nef over $X^\dagger$. In each step we have a relative morphism $X_i \to X^\dagger$ $(i \le j)$ and denote it by $X_i/X^\dagger$. We use f_i and h_i to denote the morphisms $(X_i,\Delta_i)/X^\dagger \to (X^\dagger,\Delta^\dagger)/X^\dagger$ and $(X,\Delta)/X^\dagger \to (X_i,\Delta_i)/X^\dagger$ with that $h_j = f_0 = f$. By [Fjn12, Section 3] and [Tnk14, Section 3],

$$K_{X_i} + \Delta_i = f_i^*(K_{X^\dagger} + \Delta^\dagger) + E_i$$

where E_i is all effective over $X^\dagger$ for every $0 \le i < j$. In particular, $h_{i*}(K_X+\Delta) = K_{X_i} + \Delta_i$, $h_{i*}(\Delta) = \Delta_i$. Furthermore, every curve in $\operatorname{Exc}(f) = \operatorname{Supp}(E_0)$ is a smooth rational curve by [Fjn12, Proposition 3.8] and [Tnk14, Theorem 3.19].

Step 2. Now we may assume that there is no (-1)-curve in $\operatorname{Exc}(f)$. Indeed, if there is some (-1)-curve, say C, in $\operatorname{Exc}(f)$, then by Castelnuovo's theorem, contracting this (-1)-curve in $X/X^\dagger$ leads to a new smooth surface $X'/X^\dagger$. Therefore we can run another $(K_{X'}+\Delta')$-LMMP over $X^\dagger$ until reaching to a final log surface $(\widetilde{X},\widetilde{\Delta})/X^\dagger$, where Δ' is the image of Δ. Every assumption of (X,Δ) is obviously keeping if we replace (X,Δ) by (X',Δ') except that we need to prove $(\widetilde{X},\widetilde{\Delta}) \cong (X^\dagger,\Delta^\dagger)$. We have three morphisms over $X^\dagger$: $\pi : X \to X'$, $g : X' \to \widetilde{X}$ and $\rho : \widetilde{X} \to X^\dagger$ such that

$$\begin{aligned}K_X + \Delta &= \pi^*(K_{X'} + \Delta') + aC,\\ K_{X'} + \Delta' &= g^*(K_{\widetilde{X}} + \widetilde{\Delta}) + E_0',\end{aligned}$$

$$K_{\widetilde{X}} + \widetilde{\Delta} = \rho^*(K_{X^\dagger} + \Delta^\dagger) + D$$

where $\pi : X \to X'$ is the Castelnuovo's contraction, ρ is not necessarily the identity and $K_{\widetilde{X}} + \widetilde{\Delta}$ is nef over $X^\dagger$. Then by negativity lemma (see [KM98, Lemma 3.39 and Lemma 3.40]), we have that $-D \geq 0$, since $K_{\widetilde{X}} + \widetilde{\Delta} - D \sim_\rho 0$ and D is ρ-exceptional. Remember that $K_X + \Delta = f^*(K_{X^\dagger} + \Delta^\dagger) + E_0$, $f^* = \pi^* g^* \rho^*$. That is, $E_0 \sim_f \pi^* g^* D + \pi^* E_0' + aC$. By negativity lemma again, $D > 0$ since E_0 is effective and both sides have the same support. Therefore we get a contradiction unless ρ is the identity. That is, $(\widetilde{X}, \widetilde{\Delta}) \cong (X^\dagger, \Delta^\dagger)$. Then, by contracting (-1)-curves finitely many times, we may assume that $\mathrm{Exc}(f)$ contains no (-1)-curve from now on.

Step 3. Assume that C_i is the contracted curve in step i of the log minimal model program, then $(K_{X_i} + \Delta_i) \cdot C_i < 0$. Therefore

$$(K_X + \Delta) \cdot h_i^*(C_i) = (K_{X_i} + \Delta_i) \cdot C_i < 0.$$

Note that $(h_i^*(C_i))^2 = (C_i)^2 < 0$ by the negativity lemma. Then $K_X \cdot h_i^*(C_i) \geq 0$ since $h_i^*(C_i)$ is effective and its support contains no (-1)-curve. Indeed, if $K_X \cdot h_i^*(C_i) < 0$, there must be a curve, say E, in $\mathrm{Supp} h_i^*(C_i)$ such that $K_X \cdot E < 0$. But $E^2 < 0$ since E is in $\mathrm{Exc}(f)$. Thus it is a (-1)-curve which contradicts our assumption. Therefore $\Delta \cdot h_i^*(C_i) < 0$. Then

$$\Delta_i \cdot C_i = h_{i*}(\Delta) \cdot C_i = \Delta \cdot h_i^*(C_i) < 0.$$

That is, C_i is in $\mathrm{Supp}\Delta_i$, and its strict transform is in $\mathrm{Supp}\Delta$. Therefore all those curves in $\mathrm{Exc}(f)$ must be such a strict transform of C_i under the assumption of the above step.

Step 4. Next, we need to prove that, for the resolution $f : X \to X^\dagger$ where $K_X = f^* K_{X^\dagger} + \sum a_i F_i$, we have that $a_i > -1 + \varepsilon$. Note that $K_X + \Delta = f^*(K_{X^\dagger} + \Delta^\dagger) + E_0$ where E_0 is effective in $\mathrm{Exc}(f)$ and F_i is in $\mathrm{Supp}\Delta$ by the above steps. Furthermore, let $\Delta = \sum \delta_i F_i + \Delta'$ where $\sum F_i$ and Δ' have no common components. Therefore, $f_* \Delta' = \Delta^\dagger$. Then

$$\begin{aligned} K_X + \Delta &= f^* K_{X^\dagger} + \sum a_i F_i + \sum \delta_i F_i + \Delta' \\ &= f^* K_{X^\dagger} + f^* \Delta^\dagger + E_0. \end{aligned}$$

That is,

$$\sum (a_i + \delta_i) F_i = f^* \Delta^\dagger - \Delta' + E_0$$

in which both sides are supported in $\mathrm{Exc}(f)$ and the right hand side is effective since $f^* \Delta^\dagger - \Delta' = f^* f_* \Delta' - \Delta'$ and Δ' is effective. Note that $\mathrm{Supp} E_0 = \mathrm{Supp}(\mathrm{Exc}(f))$ by our log minimal model program (Theorem 1.1). Thus comparing both sides, $a_i + \delta_i > 0$. That is, $a_i > -\delta_i \geq -1 + \varepsilon$ since the coefficients of Δ are $\leq 1 - \varepsilon$.

Step 5. We claim that, the resolution $f : X \to X^\dagger$ is a log resolution. That is, the reduced $\sum F_i$ must be a simple normal crossing curve. Remember that F_i is all smooth extremal rational curves since $X^\dagger$ has rational singularities by [FT12, Theorem 6.2] for any characteristic. Furthermore, the dual graph of $\sum F_i$ must be a tree. This shows that the reduced $\sum F_i$ must be a simple normal crossing curve. We get what we want. □

Remark 3.2. It is pointed out by Tanaka that, our claim in Step 5 can be proved by [KM98, Theorem 4.7].

From the above theorem, we know that when X is smooth, those contracting curves in log minimal model program consist of some images of (-1)-curves and some components of $\mathrm{Supp}\Delta$. Several direct but important implications of Theorem 3.1 are the following. When $K_X + \Delta$ is big, $K_{X^*} + \Delta^*$ is nef and big on the minimal model. What we have done in the proof of Theorem 3.1 is in fact showing that $f : X' \to X^*$ is exactly the minimal desingularization and (X^*, Δ^*) is MR ε-log terminal. Then the following corollaries are just simple consequences of [Alex94, Theorem 7.6, Theorem 7.7, Theorem 8.2]. It is another way to see that Fujino and Tanaka's minimal model program on log surfaces cover Alexeev's minimal model program stated in [Alex94, Section 10].

Corollary 3.3. *Let (X, Δ) be a projective log surface where X is smooth and $K_X + \Delta$ is big. Fixing $\varepsilon > 0$, let $I \subset [0, 1 - \varepsilon]$ be a DCC set and the coefficients of Δ be in I. If there is a positive integer M such that $(K_{X^*} + \Delta^*)^2 \leq M$ where (X^*, Δ^*) is a minimal model of (X, Δ), then these $(X^*, \mathrm{Supp}\Delta^*)$ belong to a bounded family.*

Corollary 3.4. *Let (X, Δ) be a projective log surface where X is smooth and $K_X + \Delta$ is big. Fixing $\varepsilon \geq 0$, let $I \subset [0, 1 - \varepsilon]$ be a DCC set and the coefficients of Δ be in I. Then $(K_{X^*} + \Delta^*)^2$ is a DCC set. In particular the volume $\mathrm{vol}(K_X + \Delta)$ is bounded from below away from 0.*

Proof. Since $\mathrm{vol}(K_X + \Delta) = \mathrm{vol}(K_X^* + \Delta^*) = (K_{X^*} + \Delta^*)^2$ by Theorem 3.1, this corollary is a direct consequence of [Alex94, Theorem 8.2]. □

Remark 3.5. Note that in Corollary 3.3, the ε is smaller, the bounded family of $(X^*, \mathrm{Supp}\Delta^*)$ is bigger. When ε goes to 0, all those X^* may not be in a bounded family, so may not be $(X^*, \mathrm{Supp}\Delta^*)$. See [Lin03, Remark 1.5] for the example showing that X^* could be $\mathbf{Q}$-Fano and not in a bounded family. Note also that Corollary 3.4 is an answer of the question coming from the first version of Di Cerbo's paper which has been confirmed by his second version [Dic, Corollary 4.3].

4. Examples. By [Alex94, Section 10], we easily see that if the log surface (X, Δ) is MR ε-log canonical, then so is every (X_i, Δ_i) in the step of log minimal model program; by Grothendieck spectral sequence and [FT12, Theorem 6.2], it is also easy to see that if X has only rational singularities, then so has every X_i. Now it is natural to generalize Theorem 3.1 and ask that if X is ε-log canonical, is so every X_i or not. But unfortunately we have the following example:

Example 4.1. Let us first recall an example of log canonical surface which is rational but not log terminal. Blowing up at a point of $\mathbf{P}^2$, we get a (-1)-curve E_0; find three points at E_0 and blow up several times (at these three points and some points at the exceptional curves over them), we can easily get a smooth and projective surface Y and four smooth rational curves E_0, E_1, E_2, E_3 on it such that $n_0 = -E_0^2 \geq 3$, $n_1 = -E_1^2 = 2$, $n_2 = -E_2^2 = 3$, $n_3 = -E_3^2 = 6$ where by abusing of notations, we still use E_0 to denote its strict transform on Y. By construction, $E_i \cdot E_0 = 1$, $E_i \cdot E_j = 0$ for $1 \leq i < j \leq 3$. Let $E = E_0 + E_1 + E_2 + E_3$, then its dual graph is a triple fork. See also that its intersection matrix is negative definite. Therefore by Artin's criterion [Art62, Theorem 2.3], we can contract E and finally get a surface X with a singular point. Now we have $f : Y \to X$ with $K_Y = f^*K_X + \sum a_i E_i$. Using adjunction, we have that:

$$-2 + n_0 = -a_0 n_0 + a_1 + a_2 + a_3;$$

$$0 = -2 + n_1 = a_0 - a_1 n_1 = a_0 - 2a_1;$$

$$1 = -2 + n_2 = a_0 - a_2 n_2 = a_0 - 3a_2;$$

$$4 = -2 + n_3 = a_0 - a_3 n_3 = a_0 - 6a_3.$$

Solving these equations we have $a_0 = -1$, $a_1 = -\frac{1}{2}$, $a_2 = -\frac{2}{3}$, $a_3 = -\frac{5}{6}$. These show that the singularity of X is exactly log canonical but not log terminal. Keeping this example in mind, we construct an example as follows:

Similar to the above blowing-up method, we can easily construct a surface Y and five smooth rational curves D, E_0, E_1, E_2, E_3 on it such that $n = -D^2$ is as big as we want, $n_0 = -E_0^2 \geq 3$, $n_1 = -E_1^2 = 2$, $n_2 = -E_2^2 = 3$, $n_3 = -E_3^2 = 6$ and $E_i \cdot E_0 = D \cdot E_0 = 1$, $E_i \cdot E_j = D \cdot E_i = 0$ for $1 \leq i < j \leq 3$. Let $E = E_0 + E_1 + E_2 + E_3$ and $F = E + D$. Then E is a triple fork and F is a quadruple fork in dual graph. Note that both of the intersection matrices of E and F are negative definite which are exercises of diagonalization of matrix. By contracting E on Y we get a morphism f from Y to a log canonical surface X which is rational but not log terminal as above. Now consider the log surface (X, D') where D' is the image of the smooth rational curve D. Note that D and D' are isomorphic outside the point $E \cap D$ and its image. Note also that $(K_X + D') \cdot D' < 0$. Indeed, let $f^*D' = D + \sum c_i E_i$. Then by $E_i \cdot f^*D' = 0$,

$$0 = 1 - c_0 n_0 + c_1 + c_2 + c_3;$$
$$0 = c_0 - c_1 n_1;$$
$$0 = c_0 - c_2 n_2;$$
$$0 = c_0 - c_3 n_3.$$

That is, $c_0 = \frac{1}{n_0 - 1}$, $c_1 = \frac{c_0}{2}$, $c_2 = \frac{c_0}{3}$, $c_3 = \frac{c_0}{6}$. Then

$$\begin{aligned}
(K_X + D') \cdot D' &= (K_Y + D) \cdot f^*D' \\
&= (K_Y + D) \cdot \left(D + \sum c_i E_i\right) \\
&= (K_Y + D) \cdot D + \sum c_i (K_Y \cdot E_i) + \sum c_i (D \cdot E_i) \\
&= -2 + c_0(-2 + n_0) + c_1(-2 + 2) + c_2(-2 + 3) \\
&\quad + c_3(-2 + 6) + c_0 \\
&= -2 + c_0(-2 + n_0) + \frac{c_0}{3} + \frac{2c_0}{3} + c_0 \\
&= -2 + 1 + c_0 = c_0 - 1 < 0
\end{aligned}$$

since $c_0 = \frac{1}{n_0 - 1} < 1$. Therefore, by [Tnk14, (1) of Theorem 3.19], D' is a smooth rational curve. Moreover, contracting D' on X is indeed a step of log minimal model program (Theorem 1.1). Finally, we get a log surface $(X^*, 0)$ where X^* is no longer log canonical since the dual graph of F is a quadruple fork which is not in the classification of dual graph of log canonical singularities in [KM98, Theorem 4.7]. Furthermore, it is not even MR log canonical by calculating the discrepancy of E_0 (which is $\frac{2 - nn_0}{nn_0 - n - 1} < -1$). But remember that X^* still has rational singularities by [FT12, Theorem 6.2]. This example proves Proposition 1.3.

Example 4.2. We just gave an example for Proposition 1.3 where $\varepsilon = 0$. In fact, by a similar construction as above, we can get some examples where $\varepsilon > 0$. A sketch of construction is the following. As Example 4.1, we can easily construct a smooth and projective surface Y and five smooth rational curves D, E_0, E_1, E_2, E_3 on it with $n = -D^2$, $n_i = -E_i^2$ such that $n = 3$, $n_0 = 5$, $n_1 = n_2 = n_3 = 2$, $E_i \cdot E_0 = D \cdot E_0 = 1$, $E_i \cdot E_j = D \cdot E_i = 0$ for $1 \leq i < j \leq 3$. Let $E = E_0 + E_1 + E_2 + E_3$ and $F = E + D$. Then E is a triple fork and F is a quadruple fork which is not in the classification of dual graph of log canonical singularities. Note that both of the intersection matrices of E and F are negative definite again by diagonalization of the intersection matrices. Choose an ε such that $0 < \varepsilon \leq \frac{1}{7}$. The same calculation as Example 4.1 shows that by contracting E on Y we get a morphism f from Y to an ε-log canonical surface X. Now consider the log surface (X, bD') where D' is the image of D and b is a non-negative real number. Note that D' is still a smooth rational curve by construction. Choose a proper real number b such that $(K_X + bD') \cdot D' < 0$. Indeed, by careful calculations as in Example 4.1, we can check that $(K_X + bD') \cdot D' < 0$ for $b > \frac{13}{19}$. Therefore, $(K_X + (1-\varepsilon)D') \cdot D' < 0$ for $0 < \varepsilon \leq \frac{1}{7}$. Now contracting D' on (X, bD') by log minimal model program, we get a log surface $(X^*, 0)$ where X^* is no longer log canonical. This gives an example to confirm Proposition 1.4.

Remark 4.3. The above two examples are based on one of the dual graphs of log canonical singularities in [KM98, Theorem 4.7]. In fact, we can construct similar examples based on the other dual graphs there and get a bunch of similar examples.

It will be interesting to ask the following question:

Question 4.4. In Theorem 1.1, if X is canonical, is X_i log canonical?

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