An explicit formula of the unramified Shintani functions for $(\mathbf{GSp_4}, \mathbf{GL_2} \times_{\mathbf{GL_1}} \mathbf{GL_2})$ and its application

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Abstract: Let F be a non-archimedean local field of arbitrary characteristic. In this paper, we announce an explicit formula of the unramified Shintani functions for $(\mathbf{GSp}_4(F), (\mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)(F))$. As an application, we compute a local zeta integral, which represents the spin L-factor of $\mathbf{GSp}_4$.

Key words: Shintani functions; automorphic L-functions; zeta integrals.

1. Introduction. Let F be a non-archimedean local field of arbitrary characteristic. We announce a result of [G], which is an explicit formula of the unramified Shintani functions, and its application to an unramified local zeta integral of Murase–Sugano type for $(\mathbf{GSp}_4(F), (\mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)(F))$. For a non-archimedean local field F_0 of characteristic 0, Murase–Sugano [MS] introduced a new kind of local zeta integral for the pair $(\mathbf{SO}_n(F_0), \mathbf{SO}_{n-1}(F_0))$ of special orthogonal groups and proved that it represents the standard L-factors of $\mathbf{SO}_n$ ([MS, Theorem 1.6]). Shintani functions, which are our main objects, appear in such a local zeta integral of Murase–Sugano type. Later Kato–Murase–Sugano [KMS] gave an explicit formula of the unramified (Whittaker–)Shintani functions for the pair $(\mathbf{SO}_n(F), \mathbf{SO}_{n-1}(F))$ of split special orthogonal groups except for the case where the characteristic of F is 2.

In this note, we announce an explicit formula of the unramified Shintani functions for $(\mathbf{GSp}_4(F), (\mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)(F))$, where F is of *arbitrary* characteristic. Also we extend the local zeta integral for the pair $(\mathbf{SO}_5(F_0), \mathbf{SO}_4(F_0))$ of split special orthogonal groups to that for $(\mathbf{GSp}_4(F), (\mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)(F))$ and prove that our local zeta integral represents the spin L-factor of $\mathbf{GSp}_4$. Here we note that there are two important points about our results. First, we allow F to be of characteristic 2. Our explicit formula in the case where F is of characteristic 2 is *not* reduced to the result in [KMS], although that in the other case is reduced to a special case of their results (see §3.3). Second, our computation for the local zeta integral is more direct, compared to [MS, Theorem 1.6]. Namely we compute the local zeta integral directly by using our explicit formula (see Remark 4.2.3 and Remark 4.3.3).

Until the end of this paper, F is a non-archimedean local field of *arbitrary* characteristic. We denote by $\mathfrak{o} = \mathfrak{o}_F$ the ring of integers of F, and we let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. Let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$. Once and for all, we fix a generator ϖ of $\mathfrak{p}$. We denote by 1_n the identity matrix of size n.

2. Preliminaries. In this section, we introduce basic notation and objects which will be used throughout this paper.

2.1. Basic objects. Let $\mathbf{G}$ be an affine algebraic group over F defined by

$$\begin{aligned}\mathbf{G} &= \mathbf{GSp}_4 \\ &= \{g \in \mathbf{GL}_4 \mid {}^t g J g = \nu(g) J, \exists \nu(g) \in \mathbf{GL}_1\}.\end{aligned}$$

Here

$$J = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}.$$

Let $\mathbf{P}$ be a minimal parabolic subgroup of $\mathbf{G}$ defined by

$$\mathbf{P} = \mathbf{TN} = \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ & * & * & * \\ \hline & & * & \\ & & * & * \end{array} \right) \in \mathbf{G} \right\},$$

where $\mathbf{T}$ is a maximal (split) torus of $\mathbf{G}$ defined by

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$$\mathbf{T} = \{t(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in \mathbf{GL}_1\},$$
$$t(t_1, t_2, t_3) := \mathrm{diag}(t_1, t_2, t_3 t_1^{-1}, t_3 t_2^{-1})$$

and $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$.

Let $\mathbf{G}_0$ be an affine algebraic group over F defined by

$$\begin{aligned}\mathbf{G}_0 &= \mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2 \\ &= \{(g_1, g_2) \in \mathbf{GL}_2 \times \mathbf{GL}_2 \mid \det(g_1) = \det(g_2)\}.\end{aligned}$$

We often identify $\mathbf{G}_0$ with a subgroup of $\mathbf{G}$ via the embedding

$$\mathbf{G}_0 \ni \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \mapsto \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \end{array}\right) \in \mathbf{G}.$$

Then $\mathbf{P}_0 = \mathbf{P} \cap \mathbf{G}_0 = \mathbf{T}_0 \mathbf{N}_0$ is a minimal parabolic subgroup of $\mathbf{G}_0$, where $\mathbf{T}_0$ is a maximal (split) torus of $\mathbf{G}_0$ defined by

$$\mathbf{T}_0 = \{t(t'_1, t'_2, t'_3) \mid t'_1, t'_2, t'_3 \in \mathbf{GL}_1\} \ (= \mathbf{T})$$

and $\mathbf{N}_0$ is the unipotent radical of $\mathbf{P}_0$.

We set $G := \mathbf{G}(F)$ and $G_0 := \mathbf{G}_0(F)$. Then $K := G \cap \mathbf{GL}_4(\mathfrak{o})$ and $K_0 := G_0 \cap \mathbf{GL}_4(\mathfrak{o})$ are maximal compact subgroups of G and G_0, respectively. Let Z be the center of G and Z_0 the center of G_0. We note that $Z \subset Z_0 \simeq Z \times \{\pm 1\}$.

Let W and W_0 be the Weyl groups of (G, T) and (G_0, T_0), respectively. Then W has eight elements and W_0 has four elements.

2.2. Satake isomorphism. Throughout this subsection, we put $H = G$ or G_0 and

$$(T_H, N_H, P_H, K_H) = \begin{cases} (T, N, P, K) & \text{if } H = G; \\ (T_0, N_0, P_0, K_0) & \text{if } H = G_0. \end{cases}$$

Let n be a positive integer. A character of $(F^\times)^n$ is called *unramified* if it is trivial on $(\mathfrak{o}^\times)^n$. We denote by $X_{nr}((F^\times)^n)$ the group of unramified characters of $(F^\times)^n$. We note that $(F^\times)^3$ is identified with T_H via an isomorphism

$$(F^\times)^3 \to T_H, \quad (t_1, t_2, t_3) \mapsto t(t_1, t_2, t_3).$$

Then the modulus character δ_{P_H} of P_H is an element of $X_{nr}(T_H) := X_{nr}((F^\times)^3)$. We sometimes identify $\chi \in X_{nr}(T_H)$ with $(\chi_1, \chi_2, \chi_3) \in X_{nr}(F^\times)^3$ via

$$\chi(t(t_1, t_2, t_3)) = \chi_1(t_1)\chi_2(t_2)\chi_3(t_3).$$

Also, we often identify $X_{nr}(T_H)$ with $(\mathbf{C}^\times)^3$ via

$$X_{nr}(T_H) \to (\mathbf{C}^\times)^3, \chi \mapsto (\chi_1(\varpi), \chi_2(\varpi), \chi_3(\varpi)).$$

Let $\mathcal{H}(H, K_H)$ be the Hecke algebra of (H, K_H) over $\mathbf{C}$, that is, $\mathcal{H}(H, K_H)$ is a $\mathbf{C}$-algebra consisting of continuous functions $\varphi \in C_c(H)$ with compact support which satisfies

$$\varphi(k_1 x k_2) = \varphi(x) \quad ({}^\forall x \in H, {}^\forall k_1, k_2 \in K_H).$$

The multiplication of $\varphi_1, \varphi_2 \in \mathcal{H}(H, K_H)$ is given by

$$(\varphi_1 * \varphi_2)(x) = \int_H \varphi_1(xh^{-1})\varphi_2(h)dh \quad ({}^\forall x \in H),$$

where dh is the Haar measure of H with $\mathrm{vol}(K_H; dh) = 1$. We note that the identity element of $\mathcal{H}(H, K_H)$ is ch_{K_H}. Here ch_A is the characteristic function of a subset $A \subset H$.

We recall the Satake isomorphism using the above notation (see [C, 4.2], for example). Let $\mathbf{C}[T_H/T_H \cap K_H]$ be the group algebra of $T_H/T_H \cap K_H$. Then we have $\mathrm{Hom}_{\mathbf{C}\text{-alg}}(\mathbf{C}[T_H/T_H \cap K_H], \mathbf{C}) \simeq (\mathbf{C}^\times)^3 \simeq X_{nr}(T_H)$. The Weyl group W_H of (H, T_H) acts on T_H by

$$w \cdot t := wtw^{-1} \quad ({}^\forall w \in W_H, {}^\forall t \in T_H).$$

The action is extended linearly to an action of W_H on $\mathbf{C}[T_H/T_H \cap K_H]$. The Satake transform $\omega : \mathcal{H}(H, K_H) \to \mathbf{C}[T_H/T_H \cap K_H]$ is defined by

$$\omega(\varphi)(t) := \delta_{P_H}(t)^{1/2} \int_{N_H} \varphi(tn)dn \quad ({}^\forall t \in T_H)$$

for all $\varphi \in \mathcal{H}(H, K_H)$. Here dn is the Haar measure of N_H with $\mathrm{vol}(N_H \cap K_H; dn) = 1$.

Theorem 2.2.1.

i) *The Satake transform ω is an algebra isomorphism from $\mathcal{H}(H, K_H)$ onto the subalgebra $\mathbf{C}[T_H/T_H \cap K_H]^{W_H}$ of $\mathbf{C}[T_H/T_H \cap K_H]$ consisting of the invariants of the Weyl group W_H;*

ii) *Any unitary homomorphism from $\mathcal{H}(H, K_H)$ to $\mathbf{C}$ is of the form*

$$\omega_\chi(\varphi) := \int_{T_H} \omega(\varphi)(t)\chi(t)dt \quad ({}^\forall \varphi \in \mathcal{H}(H, K_H))$$

for some $\chi \in X_{nr}(T_H)$. Here dt is the Haar measure of T_H with $\mathrm{vol}(T_H \cap K_H; dt) = 1$. Moreover, we have $\omega_\chi = \omega_{\chi'}$ if and only if there exists $w \in W_H$ such that $\chi' = w\chi$. Here W_H acts on $X_{nr}(T_H)$ by $(w\chi)(t) := \chi(w^{-1} \cdot t)$. In particular, we have a bijection

$$X_{nr}(T_H)/W_H \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}\text{-alg}}(\mathcal{H}(H, K_H), \mathbf{C}),$$
$$\chi \mapsto \omega_\chi.$$

3. Shintani functions. In this section, we introduce the unramified Shintani functions for (G, G_0) and state their explicit formula.

3.1. The definition of Shintani functions. For any $\xi \in X_{nr}(T_0), \Xi \in X_{nr}(T)$, we define $\mathcal{S}(\xi, \Xi)$ to be the $\mathbf{C}$-vector space consisting of all continuous functions $S : G \to \mathbf{C}$ such that

$$\begin{aligned}[L(\phi)R(\Phi)S](x) &:= \int_{G_0} dg' \int_G dg\ \phi(g')S(g'^{-1}xg)\Phi(g) \\ &= \omega_\xi(\phi)\omega_\Xi(\Phi)S(x)\end{aligned}$$

for all $(\phi, \Phi) \in \mathcal{H}(G_0, K_0) \times \mathcal{H}(G, K)$. Here dg (resp. dg') is the Haar measure of G (resp. G_0) with $\mathrm{vol}(K; dg) = 1$ (resp. $\mathrm{vol}(K_0; dg') = 1$). We call an element of $\mathcal{S}(\xi, \Xi)$ an *unramified Shintani function of type* (ξ, Ξ), or simply a *Shintani function*. The following lemma immediately follows from the definition.

Lemma 3.1.1. *Any Shintani function* $S \in \mathcal{S}(\xi, \Xi)$ *has the following properties*:

i) $S(k'xk) = S(x)$ *for all* $k' \in K_0, x \in G, k \in K$;

ii) $S(z_0xz) = \xi(z_0)^{-1}\Xi(z)S(x)$ *for all* $z_0 \in Z_0, x \in G, z \in Z$. *In particular, we have* $\mathcal{S}(\xi, \Xi) = \{0\}$ *if* $(\xi\Xi)|_Z \not\equiv 1$.

3.2. A Cartan type decomposition. We set

$$\Lambda^+ := \{\mu = (\mu_1, \mu_2, \mu_3) \in \mathbf{Z}^3 |\ \mu_1 \geq \mu_2, 2\mu_2 \geq \mu_3\},$$
$$\Lambda_0^{++} := \{\mu' = (\mu'_1, \mu'_2, \mu'_1) \in \mathbf{Z}^3 | \mu'_1 \geq 0, 2\mu'_2 \geq \mu'_1\}$$

and

$$\eta := \left(\begin{array}{cc|cc} 1 & 1 & & 1 \\ & 1 & 1 & \\ \hline & & 1 & \\ & & -1 & 1 \end{array}\right) \in G.$$

For all $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Z}^3$, we set $t(\lambda) := t(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \varpi^{\lambda_3})$.

Theorem 3.2.1 (Cartan type decomposition)**.**

$$G = \bigsqcup_{\substack{\mu \in \Lambda^+ \\ \mu' \in \Lambda_0^{++}}} K_0 g(\mu', \mu) K, \quad g(\mu', \mu) = t(\mu')\eta t(\mu).$$

This theorem is proved in the same way as [KMS, Theorem 5.2]. See [G] for more details. From Lemma 3.1.1 (i) and Theorem 3.2.1, it follows that the Shintani functions are determined by the values on the set $\{g(\mu', \mu) \mid \mu' \in \Lambda_0^{++}, \mu \in \Lambda^+\}$.

3.3. A relation between Shintani functions on $\mathbf{GSp_4}(F)$ and $\mathbf{SO_5}(F)$. As mentioned in Introduction, if F is not of characteristic 2, the unramified Shintani functions on $\mathbf{GSp}_4(F)$ are related to those on $\mathbf{SO}_5(F)$. To explain the relationship, we first recall an accidental isomorphism between $\mathbf{PGSp}_4$ and $\mathbf{SO}_5$. Here the split special orthogonal group $\mathbf{SO}_n$ and an embedding $\mathbf{SO}_4(F) \hookrightarrow \mathbf{SO}_5(F)$ are defined exactly in the same way as [KMS, §3]. We consider an F-vector space

$$V = \{X \in M_4(F) |\ XJ - J^tX = 0, \mathrm{Tr}(X) = 0\}$$

with a quadratic form $Q(X) = \mathrm{Tr}(X^2)/4$. Then (V, Q) has the following basis $\{f_i\}_{i=1}^5$:

$$f_1 = \left(\begin{array}{cc|cc} & 1 & & \\ & & & \\ \hline & & & \\ & & 1 & \end{array}\right),\quad f_2 = \left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & & & \\ & & & \end{array}\right),$$

$$f_3 = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right),$$

$$f_4 = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & -1 & & \\ 1 & & & \end{array}\right),\quad f_5 = \left(\begin{array}{cc|cc} & & & \\ 1 & & & \\ \hline & & & 1 \\ & & & \end{array}\right).$$

We identify V with F^5 via $f_i \mapsto {}^t(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$. Let $O(V, Q) \subset GL_5(F)$ be the orthogonal group of the quadratic space (V, Q). Then we have a group homomorphism $\gamma_5 : G \to O(V, Q)$ defined by

$$\gamma_5(g)X := w_2 g w_2^{-1} X (w_2 g w_2^{-1})^{-1}\ ({}^\forall(g, X) \in G \times V),$$

where

$$w_2 := \left(\begin{array}{c|ccc} 1 & & & \\ \hline & & & 1 \\ & & 1 & \\ & -1 & & \end{array}\right).$$

The above homomorphism induces an isomorphism $\overline{\gamma_5} : PGSp_4(F) = G/Z \xrightarrow{\sim} \mathbf{SO}_5(F)$. In particular, we have $\overline{\gamma_5}(t(t_1, t_2, t_3)Z) = d_5(t_1t_2t_3^{-1}, t_1t_2^{-1}) \in T_5$ for any $t(t_1, t_2, t_3) \in T$, where T_n is the maximal torus of $\mathbf{SO}_n(F)$ defined in [KMS, 3.2] and $d_5(s_1, s_2) := \mathrm{diag}(s_1, s_2, 1, s_2^{-1}, s_1^{-1})$.

Let $\xi = (\xi_1, \xi_2, \xi_3) \in X_{nr}(T_0), \Xi = (\Xi_1, \Xi_2, \Xi_3) \in X_{nr}(T)$. In this subsection, we assume that $(\xi\Xi)|_Z \equiv 1$ (see Lemma 3.1.1). We fix an unramified character $\chi \in X_{nr}(F^\times)$ such that $\chi(\varpi)^2 = \Xi(\varpi 1_4)$ and set $\widehat{\chi} := \chi \circ \nu : G \to \mathbf{C}^\times$. Then $\widehat{\chi}|_T$ is an unramified character of T. We note that $\widehat{\chi}$ is K-bi-invariant and $\widehat{\chi}^n = \widehat{\chi^n}$ for every $n \in \mathbf{Z}$. Since $\widehat{\chi}(t(t_1, t_2, t_3)) = \chi(t_3)$, $\widehat{\chi}$ is identified with $(1, 1, \chi) \in X_{nr}(F^\times)^3$. We set $\Xi_{\chi^{-1}} := \Xi\widehat{\chi}^{-1} = (\Xi_1, \Xi_2, \Xi_3\chi^{-1})$ and $\xi_\chi := \xi\widehat{\chi} = (\xi_1, \xi_2, \xi_3\chi)$. We note that any $S \in \mathcal{S}(\xi_\chi, \Xi_{\chi^{-1}})$ satisfies $S(zx) = S(x)$ for all $(z, x) \in Z \times G$. Hence we can regard $S \in \mathcal{S}(\xi_\chi, \Xi_{\chi^{-1}})$ as a Shintani function on $\mathbf{SO}_5(F)$ via the accidental isomorphism. Indeed, $\mathcal{S}(\xi_\chi, \Xi_{\chi^{-1}})$ is naturally identified with $\mathcal{S}(\theta_{\xi,\chi}, \Theta_{\Xi,\chi^{-1}})_{\mathbf{SO}_5}$, where $\mathcal{S}(\theta_{\xi,\chi}, \Theta_{\Xi,\chi^{-1}})_{\mathbf{SO}_5}$ is the space of unramified Shintani functions on $\mathbf{SO}_5(F)$ considered in [KMS] and $(\theta_{\xi,\chi}, \Theta_{\Xi,\chi^{-1}}) := ((\xi_1\xi_2\xi_3\chi, \xi_1\xi_3\chi), (\Xi_1\Xi_2\Xi_3\chi^{-1}, \Xi_1\Xi_3\chi^{-1}))$ is an unramified character of $T_4 \times T_5$. For a Shintani function $S \in \mathcal{S}(\xi_\chi, \Xi_{\chi^{-1}})$, we set $S^\natural := \widehat{\chi}S$. We note that $S(1_4) = 1$ if and only if $S^\natural(1_4) = 1$.

Proposition 3.3.1. *We have an isomorphism*

$$\mathcal{S}(\theta_{\xi,\chi}, \Theta_{\Xi,\chi^{-1}})_{\mathbf{SO}_5} = \mathcal{S}(\xi_\chi, \Xi_{\chi^{-1}}) \xrightarrow{\sim} \mathcal{S}(\xi, \Xi), \quad S \mapsto S^\natural.$$

Hence we can obtain an explicit formula of the unramified Shintani functions on $\mathbf{GSp}_4(F)$ from that on $\mathbf{SO}_5(F)$ if F is not of characteristic 2.

3.4. An explicit formula of Shintani functions. We define a rational function $c_S(\xi, \Xi)$ on $X_{nr}(T_0) \times X_{nr}(T) \simeq (\mathbf{C}^\times)^3 \times (\mathbf{C}^\times)^3$ by

$$c_S(\xi, \Xi) := \frac{\mathbf{b}(\xi, \Xi)}{\mathbf{d}'(\xi)\mathbf{d}(\Xi)},$$

where

$$\begin{aligned}
\mathbf{d}(\Xi) &:= (1 - \Xi_1\Xi_2)(1 - \Xi_1\Xi_2^{-1})(1 - \Xi_1)(1 - \Xi_2),\\
\mathbf{d}'(\xi) &:= (1 - \xi_1)(1 - \xi_2),\\
\mathbf{b}(\xi, \Xi) &:= (1 - q^{-1/2}\xi_1\xi_3\Xi_1\Xi_3)(1 - q^{-1/2}\xi_2\xi_3\Xi_1\Xi_3)\\
&\quad \times (1 - q^{-1/2}\xi_1\xi_2\xi_3\Xi_1\Xi_3)(1 - q^{-1/2}\xi_1\xi_2\xi_3\Xi_2\Xi_3)\\
&\quad \times (1 - q^{-1/2}\xi_1\xi_3\Xi_1\Xi_2\Xi_3)(1 - q^{-1/2}\xi_2\xi_3\Xi_1\Xi_2\Xi_3)\\
&\quad \times (1 - q^{-1/2}\xi_3\Xi_1\Xi_2\Xi_3)(1 - q^{-1/2}\xi_1\xi_2\xi_3\Xi_1\Xi_2\Xi_3).
\end{aligned}$$

Then the main result of [G] is as follows:

Theorem 3.4.1. *Let (ξ, Ξ) be any element of $X_{nr}(T_0) \times X_{nr}(T)$. Then we have*

$$\dim_{\mathbf{C}} \mathcal{S}(\xi, \Xi) = \begin{cases} 1 & (\text{if } (\xi\Xi)|_Z \equiv 1),\\ 0 & (\text{otherwise}). \end{cases}$$

If $(\xi\Xi)|_Z \equiv 1$, for any nonzero Shintani function $S \in \mathcal{S}(\xi, \Xi)$ we have $S(1_4) \neq 0$, and the Shintani function $W_{\xi,\Xi} \in \mathcal{S}(\xi, \Xi)$ with $W_{\xi,\Xi}(1_4) = 1$ is given by

$$\begin{aligned}
&W_{\xi,\Xi}(g(\lambda', \lambda))\\
&\quad = \frac{(\Xi_1\Xi_2\Xi_3^2)^{\lambda_3}}{(1 - q^{-2})^2} \sum_{\substack{w \in W\\ w' \in W_0}} c_S(w'\xi, w\Xi)\\
&\quad\quad \times ((w\Xi)^{-1}\delta^{1/2})(t(\lambda))((w'\xi)^{-1}\delta_0^{1/2})(t(\lambda'))
\end{aligned}$$

for all $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$. Here δ (resp. δ_0) is the modulus character of P (resp. P_0).

If F is not of characteristic 2, Theorem 3.4.1 is reduced to a special case of [KMS, Theorem 10.9] by Proposition 3.3.1. However, since Theorem 3.4.1 in the case where F is of characteristic 2 is not reduced to their results, we do their proof all over again. See [G] for a proof which contains the case where F is of characteristic 2.

Remark 3.4.2. There are several papers studying (Whittaker–) Shintani functions on $\mathbf{GSp}_4(F)$ or related groups other than [KMS]. For instance, Whittaker–Shintani functions for $(\mathbf{Sp}_{2n}(F), \text{Jacobi group})$ were studied by Murase [M] for $n = 2$. Later Murase's result was generalized to any n by Shen [S]. Also, Bump–Friedberg–Furusawa [BFF] studied Bessel functions on $\mathbf{GSp}_4(F)$ and Hironaka [H] studied Shintani functions for $(\mathbf{Sp}_4(F), \mathbf{SL}_2(F) \times \mathbf{SL}_2(F))$.

4. Local zeta integrals of Murase–Sugano type. In this section, we introduce a local zeta integral of Murase–Sugano type for (G, G_0) and prove that the local zeta integral represents the local spin L-factor of $\mathbf{GSp}_4$. Details will appear in a forthcoming paper.

4.1. Iwasawa decomposition of $\mathbf{GSpin_6}$. In order to define a local zeta integral of Murase–Sugano type, we consider the Iwasawa decomposition of the split general spin group $\mathbf{GSpin}_6$. But, for simplicity, in this paper we consider a group G_1 defined as follows instead of $\mathbf{GSpin}_6(F)$:

$$G_1 := \{g \in GL_4(F) |\ \det(g) \in (F^\times)^2\}.$$

Remark 4.1.1. The split general spin group $\mathbf{GSpin}_6(F)$ is realized as follows:

$$\mathbf{GSpin}_6(F) = \{(g, r) \in GL_4(F) \times F^\times |\ \det(g) = r^2\}.$$

Hence we have an isomorphism

$$\mathbf{GSpin}_6(F)/\{(1_4, \pm 1)\} \xrightarrow{\sim} G_1.$$

Let P_{22} be a maximal parabolic subgroup of G_1 given by

$$P_{22} = \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ & * & & * \\ \hline * & * & * & * \\ & * & & * \end{array} \right) \in G_1 \right\} = M_{22}N_{22}.$$

Here M_{22} is a Levi subgroup of P_{22} given by

$$M_{22} = \left\{ \mathbf{m}_1(a,b) \mid a, b \in GL_2(F), \det(ab) \in (F^\times)^2 \right\},$$

$$\mathbf{m}_1(a,b) := \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}^{-1},$$

and N_{22} is the unipotent radical of P_{22}. We note that every $\mathbf{m}_1(a,b) \in M_{22}$ has a factorization

$$\mathbf{m}_1(a,b) = \mathbf{m}_1(\alpha^{-1} \cdot a, b)\mathbf{m}_1(\alpha \cdot 1_2, 1_2), \ \alpha^2 = \frac{\det(a)}{\det(b)}.$$

Namely, for any $m_1 \in M_{22}$ we have a factorization

$$m_1 = \beta(m_1)\operatorname{diag}(\alpha(m_1), 1, \alpha(m_1), 1)$$

for some $(\beta(m_1), \alpha(m_1)) \in G_0 \times F^\times$. We note that such a factorization of m_1 is not unique. We set $K_1 := G_1 \cap GL_4(\mathfrak{o})$. Then every $g \in G_1$ has an Iwasawa decomposition

$$\begin{aligned} g &= m_1(g)n_1(g)k_1(g) \\ &= \beta(m_1(g))\operatorname{diag}(\alpha(m_1(g)), 1, \alpha(m_1(g)), 1)n_1(g)k_1(g) \end{aligned}$$

for some $(m_1(g), n_1(g), k_1(g)) \in M_{22} \times N_{22} \times K_1$. For all $g \in G_1$, we fix such a factorization of g and set $\beta(g) = \beta(m_1(g))$ and $\alpha(g) = \alpha(m_1(g))$. The following lemma is easily checked by direct calculation.

Lemma 4.1.2. *The subgroup $P_{22} \cap K_1$ of G_1 is equal to the intersection of K_1 and*

$$\begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} GL_2(\mathfrak{o}) & M_2(\mathfrak{o}) \\ & GL_2(\mathfrak{o}) \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}^{-1}.$$

4.2. Unramified local zeta integrals of Murase–Sugano type. Let $(\xi, \Xi) \in X_{nr}(T_0) \times X_{nr}(T)$ such that $(\xi\Xi)|_Z \equiv 1$. For any Shintani function $S \in \mathcal{S}(\xi, \Xi)$, we define a local zeta integral of Murase–Sugano type by

$$Z_{MS}(s; S) := \int_{G_0\backslash G} S(\beta(g)^{-1}g)|\alpha(g)|^s d\dot{g}$$

where $d\dot{g}$ is the right invariant measure of $G_0\backslash G$ and $|\cdot|$ is the $\mathfrak{p}$-adic absolute value normalized so that $|\varpi| = q^{-1}$.

Remark 4.2.1. The local zeta integral $Z_{MS}(s; S)$ is a local component of a certain global zeta integral (*cf.* [MS]). Details will appear in a forthcoming paper.

Since any Shintani function $S \in \mathcal{S}(\xi, \Xi)$ can be regarded as a function on $K_0\backslash G/K$, it follows from Lemma 4.1.2 that the value $S(\beta(g)^{-1}g)|\alpha(g)|^s$ is independent of a choice of the Iwasawa decomposition of $g \in G \subset G_1$. For any $\chi = (\chi_1, \chi_2, \chi_3) \in (\mathbf{C}^\times)^3$ and $s \in \mathbf{C}$, we set

$$\begin{aligned} L(\chi; s) := &(1 - \chi_3 q^{-s})^{-1}(1 - \chi_1\chi_3 q^{-s})^{-1} \\ &(1 - \chi_2\chi_3 q^{-s})^{-1}(1 - \chi_1\chi_2\chi_3 q^{-s})^{-1}. \end{aligned}$$

We prove the following theorem as an application of Theorem 3.4.1.

Theorem 4.2.2. *Let $(\xi, \Xi) \in X_{nr}(T_0) \times X_{nr}(T)$ such that $(\xi\Xi)|_Z \equiv 1$. For the Shintani function $S \in \mathcal{S}(\xi, \Xi)$ with $S(1_4) = 1$, the local zeta integral $Z_{MS}(s; S)$ is absolutely convergent if $\operatorname{Re}(s) > s_\Xi := \max\{\log_q \|\Xi_3\|, \log_q \|\Xi_1\Xi_3\|, \log_q \|\Xi_2\Xi_3\|, \log_q \|\Xi_1\Xi_2\Xi_3\|\}$. Here $\|\cdot\|$ is the usual absolute value on $\mathbf{C}$. If $\operatorname{Re}(s) > s_\Xi$, the zeta integral $Z_{MS}(s; S)$ can be evaluated as*

$$Z_{MS}(s; S) = \frac{L(\Xi; s)}{L(\xi^{-1}; s + 1/2)}.$$

Remark 4.2.3. Theorem 4.2.2 is generalization of [MS, Theorem 1.6] for the pair $(\mathbf{SO}_5(F), \mathbf{SO}_4(F))$ of split special orthogonal groups. While they proved their result without using the explicit formula of Shintani functions for $(\mathbf{SO}_5(F), \mathbf{SO}_4(F))$, we compute the local zeta integral $Z_{MS}(s; S)$ using that for (G, G_0).

4.3. Evaluation of the unramified local zeta integrals. In this subsection, we evaluate the local zeta integral $Z_{MS}(s; S)$ by using the explicit formula of $S \in \mathcal{S}(\xi, \Xi)$. First we state the following theorem.

Theorem 4.3.1. *We have the decomposition*

$$G = \bigsqcup_{l \geq 0} G_0 a(l) K.$$

Here $a(l) := g((0,0,0),(l,l,l))$.

For any integrable function $F : G_0\backslash G \to \mathbf{C}$ which is right K-invariant, Theorem 4.3.1 yields

$$\int_{G_0\backslash G} F(g) d\dot{g} = \sum_{l=0}^{\infty} F(a(l))v_l,$$

where

$$v_l := \operatorname{vol}(G_0 \cap a(l)Ka(l)^{-1}; dg')^{-1}.$$

We note that the integrand $S(\beta(g)^{-1}g)|\alpha(g)|^s$ of the local zeta integral $Z_{MS}(s; S)$ is a function on $G_0\backslash G/K$. Hence we have

$$Z_{MS}(s;S)=\sum_{l=0}^{\infty}S(\beta(a(l))^{-1}a(l))|\alpha(a(l))|^{s}v_{l}$$
$$=\sum_{l=0}^{\infty}S(\beta(a(l))^{-1}a(l))v_{l}q^{-ls}.$$

Since $\beta(a(l))^{-1}a(l)\in K_0a(l)K$, it is enough to compute the volume v_l and the value

$$S(\beta(a(l))^{-1}a(l))=S(a(l)).$$

Proposition 4.3.2. *For $l\geq 0$, we have*

$$v_l=\begin{cases}1 & (\textit{if } l=0),\\ q^{3l}(1-q^{-2}) & (\textit{if } l>0).\end{cases}$$

In particular, the generating function for the sequence $\{v_l\}_{l\geq 0}$ is given by

$$\sum_{l=0}^{\infty}v_lt^l=\frac{1-qt}{1-q^3t}.$$

The domain of convergence of the above power series is $\|t\|<q^{-3}$.

From Theorem 3.4.1 and Proposition 4.3.2, we obtain Theorem 4.2.2.

Remark 4.3.3. Murase–Sugano obtained Proposition 4.3.2 as a corollary of computation of their local zeta integral (see [MS, Lemma 1.12]). However we can also prove Proposition 4.3.2 by directly computing the index $[K_0^{(l)}:K_0^{(l+1)}]$ for all $l\geq 0$. Here $K_0^{(l)}:=G_0\cap a(l)Ka(l)^{-1}$.

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