A characterization of the L^∞-representation algebra $\mathfrak{R}(S)$ of a foundation semigroup and its application to BSE algebras

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Abstract: For a locally compact Hausdorff semigroup S, the L^∞-representation algebra $\mathfrak{R}(S)$ was extensively studied by Dunkl and Ramirez. In this paper we give a characterization of the Banach algebra $\mathfrak{R}(S)$ of a foundation semigroup S and as an application we determine some BSE semigroup algerbras.

Key words: Representation algebra; BSE algebra; foundation semigroup; reflexive semigroup.

1. Introduction. The notation of the L^∞-*representation* Banach algebra $\mathfrak{R}(S)$ of a commutative topological semigroup S was introduced and extensively studied by Dunkl and Ramirez in [4]. Recall that an L^∞-representation of S is a triple (Ω, μ, T) where μ is a complete probability measure on a set Ω, and $s \mapsto T_s$ is a homomorphism of S into the unit ball of $L^\infty(\Omega, \mu)$ (where $L^\infty(\Omega, \mu)$ has the pointwise multiplication) and is weak-$*$ continuous (i.e., $\sigma(L^\infty(\Omega, \mu), L^1(\Omega, \mu))$) (see [4]). The L^∞-representation algebra $\mathfrak{R}(S)$ is defined to be the set of all functions

$$s \mapsto \int_\Omega T_s g d\mu$$

of S into $\mathbf{C}$, where (Ω, μ, T) is an L^∞-representation of S and $g \in L^1(\Omega, \mu)$.

It is shown in [12] that for a foundation semigroup S with identity and for every function $f \in \mathfrak{R}(S)$, there exists a unique measure $\mu_f \in M(\widehat{S})$ such that

$$(1) \qquad f(s) = \int_{\widehat{S}} \gamma(s) d\mu_f(\gamma) \quad (s \in S).$$

If we define $\|.\|_{\mathfrak{R}}$ on $\mathfrak{R}(S)$ by $\|f\|_{\mathfrak{R}} := \|\mu_f\|$ $(f \in \mathfrak{R})$, then $(\mathfrak{R}(S), \|.\|_{\mathfrak{R}})$ with the pointwise multiplication becomes a commutative Banach algebra.

Let A be a commutative Banach algebra. Denote by $\Delta(A)$ and $\mathcal{M}(A)$ the Gelfand spectrum and the multiplier algebra of A, respectively. A bounded continuous function σ on $\Delta(A)$ is called a *BSE-function* if there exists a constant $C > 0$ such that for every finite number of $\varphi_1, \dots, \varphi_n$ in $\Delta(A)$ and complex numbers $c_1, \dots, c_n$, the inequality

$$\left| \sum_{j=1}^n c_j \sigma(\varphi_j) \right| \le C. \left\| \sum_{j=1}^n c_j \varphi_j \right\|_{A^*}$$

holds. The BSE-norm of σ ($\|\sigma\|_{BSE}$) is defined to be the infimum of all such C. The set of all BSE-functions is denoted by $C_{BSE}(\Delta(A))$. Takahasi and Hatori [16] showed that under the norm $\|.\|_{BSE}$, $C_{BSE}(\Delta(A))$ is a commutative semisimple Banach algebra.

A bounded linear operator on A is called a *multiplier* if it satisfies $xT(y) = T(xy)$ for all $x, y \in A$. The set $\mathcal{M}(A)$ of all multipliers of A is a unital commutative Banach algebra, called the *multiplier algebra* of A.

For each $T \in \mathcal{M}(A)$ there exists a unique continuous function $\widehat{T}$ on $\Delta(A)$ such that $\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\hat{a}(\varphi)$ for all $a \in A$ and $\varphi \in \Delta(A)$. See [11] for a proof.

Define

$$\widehat{\mathcal{M}(A)} = \{\widehat{T} : T \in \mathcal{M}(A)\}.$$

A commutative Banach algebra A is called without order if $aA = \{0\}$ implies $a = 0$ $(a \in A)$.

A commutative and without order Banach algebra A is called a BSE algebra (or has *BSE-property*) if it satisfies the condition

$$C_{BSE}(\Delta(A)) = \widehat{\mathcal{M}(A)}.$$

The abbreviation BSE stands for Bochner-

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Schoenberg-Eberlin and refers to a famous theorem, proved by Bochner and Schoenberg [3,14] for the additive group of real numbers and in general by Eberlein [6] for a locally compact abelian group G, saying that, in the above terminology, the group algebra $L^1(G)$ is a BSE algebra. See [13] for a proof.

The notion of BSE algebras was introduced and studied by Takahasi and Hatori [16] and later by Kaniuth and Ülger [10]. There are several other papers on BSE algebras such as [7], [8] and [9].

It is worth noting that the semigroup algebra $l^1(\mathbf{Z}^+)$ (where $\mathbf{Z}^+$ is the additive semigroup of non-negative integers) is a BSE algebra [17], but for $k \geq 1$, $l^1(\mathbf{N}_k)$ $(\mathbf{N}_k = \{k, k+1, k+2, \ldots\})$ is not a BSE algebra.

In [9], we established affirmatively a question raised by Takahasi and Hatori [16] that whether $L^1(\mathbf{R}^+)$ is a BSE algebra.

Let S be a locally compact topological semigroup and $M(S)$ be the space of all bounded complex Borel measures on S. Then $M(S) = C_0(S)^*$ and $M(S)$ with convolution

$$\mu \star \nu(\psi) = \iint \psi(xy) d\mu(x) d\nu(y) \quad (\mu, \nu \in M(S), \psi \in C_0(S)),$$

is a Banach algebra. The subalgebra $M_a(S)$ of $M(S)$ consists of all measures μ in $M(S)$ for which the translations $x \to |\mu| \star \delta_x$ and $x \to \delta_x \star |\mu|$ from S into $M(S)$ are weakly continuous. A topological semigroup S is called a *foundation semigroup* if S coincides with the closure of $\cup\{\mathrm{supp}(\mu) : \mu \in M_a(S)\}$. This class of semigroups is very extensive for which discrete semigroups and topological groups are elementary examples. For more examples see [5] and [15].

In the present paper we first give a characterization of the L^∞-representation algebra $\mathfrak{R}(S)$ of a foundation semigroup S with identity and then we apply this characterization in order to prove that $M_a(S)$, for a reflexive foundation semigroup S, is a BSE algebra. We present examples which show that the assumption of reflexivity cannot be dropped.

We also prove that for a compact foundation semigroup S, the semigroup algebra $M_a(S)$ is BSE if and only if it has a Δ-weak bounded approximate identity.

2. Preliminaries. In this paper, the term *semigroup* will describe a set S endowed with an associative, binary operation mapping $S \times S$ into S. A commutative semigroup is a semigroup with a commutative operation. If S is also a Hausdorff topological space and the binary operation is continuous for the product topology of $S \times S$, then S is said to be a *topological semigroup*. If in addition S contains a unit with respect to the operation, we say S has an identity.

An *inverse semigroup* S is a semigroup in which every element x in S has a unique inverse x^{-1} in S in the sense that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$.

A Clifford semigroup is an inverse semigroup with $xx^{-1} = x^{-1}x$. Examples of Clifford semigroups are groups and commutative inverse semigroups.

In this paper all semigroups are considered to be commutative, so the term "inverse semigroup" is the same as "Clifford semigroup".

A *semicharacter* γ on a topological semigroup S is a bounded, continuous, complex-valued function on S, not identically zero, such that $\gamma(xy) = \gamma(x)\gamma(y)$ whenever x and y are in S. The set consisting of all semicharacters on S is denoted by $\widehat{S}$. If S has a unit, $\widehat{S}$ forms a semigroup under the pointwise multiplication. We endow $\widehat{S}$ with the topology of uniform convergence on compact subsets of S. Under this topology $\widehat{S}$ is a topological semigroup. A topological semigroup S is said to be *reflexive* if $S \cong \widehat{\widehat{S}}$ under the map $x \to \tilde{x}$ where $\tilde{x}(\gamma) = \gamma(x)$ for each γ in $\widehat{S}$ and x in S.

Remark 2.1. In [1] Austin proved that if S is discrete then S is topologically isomorphic to $\widehat{\widehat{S}}$ if and only if S is an inverse semigroup with identity. Also, he showed that if S is compact and $\widehat{S}$ separates the points of S and S is an inverse semigroup, then S is topologically isomorphic to $\widehat{\widehat{S}}$. A. C. Baker and J. W. Baker [2] showed that if S is topologically isomorphic to $\widehat{\widehat{S}}$ then S must be an inverse semigroup.

A bounded net $(e_\alpha)_\alpha$ in a Banach algebra A is called a bounded approximate identity for A if it satisfies $\|e_\alpha a - a\| \to 0$ for all $a \in A$.

Note that if S is a foundation semigroup with identity, then $M_a(S)$ has a bounded approximate identity [5].

A bounded net $(e_\alpha)_\alpha$ in a Banach algebra A is called a Δ-weak bounded approximate identity for A if $\varphi(e_\alpha) \to 1$ (equivalently, $\varphi(e_\alpha a) \to \varphi(a)$ for every $a \in A$) for all $\varphi \in \Delta(A)$. As is shown in [16], A

has a Δ-weak bounded approximate identity if and only if $\widehat{\mathcal{M}(A)} \subseteq C_{BSE}(\Delta(A))$.

3. A characterization of the Banach algebra $\mathfrak{R}(S)$. In this section we first give a characterization of the L^∞-representation algebra $\mathfrak{R}(S)$ of a foundation commutative semigroup S with identity.

Recall that if S is a topological semigroup, then $\widehat{S} \cup \{0\}$ is closed under pointwise multiplication and complex conjugation, the closure A of the linear span of $\widehat{S}$ in supremum norm is a C^*-subalgebra of bounded continuous functions on S. The semigroup $\widehat{S}$ contains the identity so that A is unital. It follows that the spectrum $\bar{S}$ of A is a compact Hausdorff space. Furthermore, since points of S determine complex homomorphisms of A, there is a continuous map $\alpha : S \to \bar{S}$, with dense image, such that $f \mapsto fo\alpha : C(\bar{S}) \to A$ is an isometric isomorphism of $C(\bar{S})$ onto A, when α is injective.

The map α is injective if and only if $\widehat{S}$ separates the points of S. We shall call $\bar{S}$ (together with the map α) the Bohr compactification of S. For more details the interested reader can refer to [18].

We start this section with the following Theorem which characterizes the L^∞-representation $\mathfrak{R}(S)$ of a foundation semigroup S.

Theorem 3.1. *Let S be a commutative foundation semigroup with identity. Then the following statements about a continuous function φ defined on S, are equivalent*:

(a) $\varphi \in \mathfrak{R}(S)$ *and* $\|\varphi\|_{\mathfrak{R}} \le \beta$.

(b) *For every function f on $\widehat{S}$ of the form*

$$f(\gamma) = \sum_{i=1}^n c_i\gamma(x_i) \quad (\gamma \in \widehat{S}),$$

where $c_1, \dots c_n$ are complex numbers and $x_1, \dots, x_n \in S$, we have

$$\left|\sum_{i=1}^n c_i\varphi(x_i)\right| \le \beta\|f\|_\infty. \tag{2}$$

Proof. Suppose that $\varphi \in \mathfrak{R}(S)$. Then, by equality (1), there exists a measure $\mu_f \in M(\widehat{S})$ such that $\|\mu_\varphi\| = \|\varphi\|_{\mathfrak{R}} \le \beta$ and

$$\varphi(x) = \int_{\widehat{S}} \gamma(x) d\mu_\varphi(\gamma).$$

With $f(\gamma) = \sum_{i=1}^n c_i\gamma(x_i)$ we have

$$\begin{aligned}\left|\sum_{i=1}^n c_i\varphi(x_i)\right| &= \left|\sum_{i=1}^n c_i \int_{\widehat{S}} \gamma(x_i) d\mu_\varphi(\gamma)\right| \\ &= \left|\int_{\widehat{S}} \sum_{i=1}^n c_i\gamma(x_i) d\mu_\varphi(\gamma)\right| \\ &= \left|\int_{\widehat{S}} f(\gamma) d\mu_\varphi(\gamma)\right| \\ &\le \|f\|_\infty\|\mu_\varphi\| \le \beta\|f\|_\infty.\end{aligned}$$

That is (a) which implies (b).

To prove the reverse implication, we consider the Bohr compactification semigroup $\overline{\widehat{S}}$ of the semigroup $\widehat{S}$. We then extend each f of the form

$$f(\gamma) = \sum_{i=1}^n c_i\gamma(x_i) \quad (x_i \in S, \quad \gamma \in \widehat{S})$$

to $\overline{\widehat{S}}$ by

$$f(\bar{\gamma}) = \sum_{i=1}^n c_i\bar{\gamma}(x_i) \quad (x_i \in S, \quad \bar{\gamma} \in \overline{\widehat{S}}).$$

Since $\widehat{S}$ is dense in $\overline{\widehat{S}}$, the norm $\|f\|_\infty$ is not altered by this extension. Let

$$\mathcal{A} = \left\{ f : \overline{\widehat{S}} \longrightarrow \mathbf{C} | \exists x_1, \dots, x_n \in S, \exists c_1, \dots, c_n \in \mathbf{C} : f(\bar{\gamma}) = \sum_{i=1}^n c_i\bar{\gamma}(x_i) \right\}.$$

Then $\mathcal{A}$ is a linear manifold in $C(\overline{\widehat{S}})$. Now define the linear functional $\mathcal{F}$ on $\mathcal{A}$ by

$$\mathcal{F}f = \sum_{i=1}^n c_i\varphi(x_i),$$

where $f(\bar{\gamma}) = \sum_{i=1}^n c_i\bar{\gamma}(x_i)$.

By inequality (2), we have

$$|\mathcal{F}f| \le \beta\|f\|_\infty.$$

Thus $\|\mathcal{F}\| \le \beta$ and $\mathcal{F}$ can be extended to a bounded linear functional $\widetilde{\mathcal{F}}$ on $C(\overline{\widehat{S}})$ of the norm not exceeding β.

By the Riesz representation theorem, there is a unique measure $\mu \in M(\overline{\widehat{S}})$ such that $\|\mu\| \le \beta$ and

$$\widetilde{\mathcal{F}}f = \sum_{i=1}^n c_i\varphi(x_i) = \int_{\overline{\widehat{S}}} f(\bar{\gamma}) d\mu(\bar{\gamma}) \quad (f \in C(\overline{\widehat{S}})).$$

In particular, for $f(\bar{\gamma}) = \bar{\gamma}(x)$ $(x \in S)$ we have

$$\varphi(x) = \int_{\bar{\widehat{S}}} \bar{\gamma}(x) d\mu(\bar{\gamma}).$$

Putting $\Omega = \bar{\widehat{S}}$, $Tx = \bar{\gamma}(x)$ and $g = 1$, we conclude that $\varphi \in \mathfrak{R}(S)$ and $\|\varphi\|_{\mathfrak{R}} = \|\mu\| \leq \beta$. $\square$

Remark 3.2. Note that in the previous Theorem, (b) implies (a) for an arbitrary commutative topological (not necessarily foundation) semigroup.

4. The BSE-property of semigroup algebras related to foundation semigroups. In this section, as an application of Theorem 3.1, we prove that for any reflexive foundation semigroup S, the Banach algebra $M_a(S)$ is a BSE algebra.

However, in the case where S is a compact foundation semigroup, without appealing the L^∞-representation algebra $\mathfrak{R}(S)$, we prove that for a compact foundation semigroup S, $M_a(S)$ is a BSE algebra if and only if it has a Δ-weak bounded approximate identity.

Theorem 4.1. *Suppose that S is a reflexive foundation semigroup, then $M_a(S)$ is a BSE algebra.*

Proof. From reflexivity of S, it follows that it has an identity. Therefore $M_a(S)$ has a bounded approximate identity and by Corollary 5 of [16],

$$\mathcal{M}(\widehat{M_a(S)}) \subseteq C_{BSE}(\Delta(M_a(S))).$$

Since S is a foundation semigroup, we infer that $\Delta(M_a(S))$ is topologically isomorphic to $\widehat{S}$ (see [5] for a proof). Let $\sigma \in C_{BSE}(\Delta(M_a(S))) = C_{BSE}(\widehat{S})$. There exists $\beta > 0$ such that for every finite number of $\gamma_1, \ldots, \gamma_n \in \widehat{S}$ and $c_1, \ldots, c_n \in \mathbf{C}$,

$$\left| \sum_{i=1}^n c_i \sigma(\gamma_i) \right| \leq \beta \left\| \sum_{i=1}^n c_i \gamma_i \right\|_\infty.$$

This implies that for every f of the form

$$f(x) = \sum_{i=1}^n c_i \gamma_i(x) \quad (x \in S \cong \widehat{\widehat{S}}),$$

we have

$$\left| \sum_{i=1}^n c_i \sigma(\gamma_i) \right| \leq \beta \|f\|_\infty.$$

Applying Theorem 3.1 to $\widehat{S}$ instead of S, we obtain $\sigma \in \mathfrak{R}(\widehat{S})$ and $\|\sigma\| \leq \beta$. Since S is reflexive, from Remark 2.1 it is an inverse semigroup. Defining γ^{-1} by $\gamma^{-1}(x) = \gamma(x^{-1})$ $(\gamma \in \widehat{S})$, $\widehat{S}$ also defines an inverse semigroup. By Theorem 4.3 of [1] and Remark 11.6 of [15] $\widehat{S}$ is a union of closed groups and thus $\widehat{S}$ is a foundation semigroup. Applying Theorem 3 of [12] to $\widehat{S}$, we conclude that there exists a unique measure $\mu \in M((\widehat{\widehat{S}})) = M(S)$ such that

$$\sigma(\gamma) = \int_S \gamma(x) d\mu(x) \quad (\gamma \in \widehat{S}).$$

Since $M_a(S)$ is a closed ideal of $M(S)$, it follows that $M(S)$ is a subalgebra of $\mathcal{M}(M_a(S))$.

This implies that $\sigma = \widehat{\mu} \in \widehat{M(S)} \subseteq \mathcal{M}(\widehat{M_a(S)})$.

So $C_{BSE}(\widehat{S}) \subseteq \mathcal{M}(\widehat{M_a(S)})$ and $M_a(S)$ is a BSE algebra. $\square$

Note that for $k \geq 1$, $\mathbf{N}_k = \{k, k+1, \ldots\}$ under addition operation is a foundation semigroup which is not reflexive and the semigroup algebra $l^1(\mathbf{N}_k)$ is not BSE [16]. So we can not drop the hypothesis "reflexive" in the statements of Theorem 4.1.

In the following we present examples of semigroups which satisfy the hypothesis of the above Theorem.

Example 4.2. (a) For any discrete inverse semigroup S with identity, $l^1(S)$ is a BSE algebra. For instance, if $S = (\mathbf{Z}^+, \max)$, where $\mathbf{Z}^+$ is the discrete semigroup of non-negative integers, then S is a reflexive semigroup and so $l^1(S)$ is a BSE algebra.

(b) Let

$$T = \left\{ -\frac{1}{2n} : n \in \mathbf{N} \right\} \cup \{0\} \cup \left\{ \frac{1}{2n+1} : n \in \mathbf{N} \right\}$$

with the operation

$$xy = yx = x \text{ if } |x| \geq |y| \quad (x, y \in T),$$

and the topology of T coincides with the restriction of the line topology on $T = \{-\frac{1}{2n} : n \in \mathbf{N}\} \cup \{0\}$ while its restriction on $\{\frac{1}{2n+1} : n \in \mathbf{N}\}$ is discrete. Then T defines a compact inverse foundation semigroup with identity (p. 65 of [5]). So by Remark 2.1 and Theorem 4.1, $M_a(T)$ is BSE.

If we set $S := G \times T$, where G is an abelian topological group, then S is a reflexive foundation semigroup and again by Theorem 4.1, $M_a(S)$ is BSE.

(c) Let $S := \{0\} \cup \{\frac{1}{n} : n \in \mathbf{N}\}$ with the relative topology of the line and multiplication given by $xy = \max\{x, y\}$. Then S is a compact foundation semigroup with identity 0 (p. 34 of [5]). For any abelian locally compact group G, $T = S \times G$ is a reflexive foundation semigroup and by Theorem 4.1, $M_a(T)$ is BSE.

Before we give a necessary and sufficient condition for $M_a(S)$ of a commutative compact foundation semigroup S to be a BSE algebra, we need to quote the following result from Kaniuth and Ülger [10].

Theorem 4.3. *Let A be a semisimple commutative Banach algebra which is an ideal in its second dual. Then the following statements are equivalent*:

(i) *A is a BSE algebra.*

(ii) *A has a Δ-weak bounded approximate identity.*

(iii) *A has a bounded approximate identity.*

Dzinonotyiweyi [5] showed that if S is a compact foundation semigroup, then $M_a(S)$ is an ideal in its second dual. So as an application of the above Theorem we give the following result.

Theorem 4.4. *Let S be a compact foundation semigroup. Then $M_a(S)$ is a BSE algebra if and only if $M_a(S)$ has a Δ-weak approximate identity.*

Proof. Suppose that $M_a(S)$ is a BSE algebra. Then by Corollary 5 of [16] it has a Δ-weak approximate identity.

Conversely, suppose that $M_a(S)$ has a Δ-weak approximate identity, since S is a compact and foundation semigroup, $M_a(S)$ is an ideal in its second dual and by Theorem 4.3, $M_a(S)$ is a BSE algebra. □

Example 4.5. (a) Consider the semigroup $S = [0,1]^n$, $n \in \mathbf{N}$ with ordinary multiplication and restriction topology of $\mathbf{R}^n$. Since $[0,1]^n$ is a compact semigroup and $L^1([0,1]^n)$ has a bounded approximate identity, then $L^1([0,1]^n)$ is a BSE algebra, for all $n \in \mathbf{N}$.

(b) Let T be as in part (b) and S be as in part (c) of Example 4.2. Then by Theorem 4.4, $M_a(T)$ and $M_a(S)$ are BSE algebras.

(c) $S = [0,1]$ with the restriction topology of $\mathbf{R}$ and multiplication defined by $xy := \min\{x+y, 1\}$. Then S is a compact foundation semigroup with identity (p. 48 of [5]) and $M_a(S)$ is BSE.

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