Vertex operator algebras with central charge 1/2 and −68/7

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Abstract: In this article we study *simple vertex operator algebras* whose spaces of characters are 3-dimensional, and satisfy 3*rd order* (*modular*) *linear differential equations*. We classify such vertex operator algebras with central charge $1/2$ or $-68/7$. One of the main results is that these vertex operator algebras have *conformal weights* $\{0, 1/2, 1/16\}$ or $\{0, -2/7, -3/7\}$, respectively, and are isomorphic to the minimal models of central charge $c = c_{3,4} = 1/2$ and $c_{2,7} = -68/7$. Moreover, we give the explicit formulas representing characters of the minimal models with $c = 1/2$, $-68/7$ by using the classical Weber functions.

Key words: Vertex operator algebras; classification of vertex operator algebras.

1. Introduction. In this note we study simple vertex operator algebras (VOA for short) with central charges $1/2$ or $-68/7$ whose spaces of (formal) characters are 3-dimensional, and conformal weights and central charges satisfy a linear relation called *the non-zero Wronskian condition* (NZWC for short). We show that such VOAs are isomorphic to the minimal models.

It is known by Y. Zhu [12] that the space of characters $\mathcal{X}_V$ of a VOA V is invariant under the slash $|_0$-action of the group $\Gamma_1 = SL_2(\mathbf{Z})$. If V is C_2-cofinite then $\mathcal{X}_V$ is finite-dimensional. This fact leads the concept of *vector-valued modular forms* (VVMFs for short) with weight k introduced by G. Mason in [8], which are vector-valued functions that are invariant under the slash $|_k$-action of Γ_1. If the leading powers of all entries of a VVMF satisfy the NZWC the set of entries of a VVMF forms a fundamental system of solutions of a linear ordinary differential equation with regular singularities at $q = 0$ ($q = e^{2\pi\sqrt{-1}\tau}$, $\tau \in \mathbf{H}$). This differential equation has a significant property that the space of solutions is invariant under the slash $|_k$-action of Γ_1. This kind of equations is called **modular linear differential equations** (MLDE for short) by G. Mason [8]. Therefore the set of characters of a VOA with the non-zero Wronskian condition forms a fundamental system of solutions of an MLDE.

The well-known examples of VOAs with central charge $1/2$ and $-68/7$ whose spaces of characters are 3-dimensional are so-called the minimal models with the central charges $c_{3,4}$ and $c_{2,7}$ (by using the usual notation), respectively. The important point here is that they satisfy 3rd order MLDEs. Therefore, we classify 3rd order MLDEs which have coefficients involving central charge c and conformal weights as parameters (see (1)). The indices of MLDEs associated with VOAs are essentially $-c/24$, $h - c/24$ and $c/12 + 1/2 - h$, where h is one of non-zero conformal weights. We show that V with $c = 1/2$ and $c = -68/7$ are isomorphic to the minimal models $L(1/2, 0)$ and $L(-68/7, 0)$, respectively.

Finally we refer to relation between the current paper, [5] and [9]. The 2nd order MLDEs studied in the latter are included in the former in the sense that applying the Serre derivation to those MLDEs from the right we have 3rd order MLDEs discussed here. In fact, this relation is studied in [6]. This paper is closely related with [9] though the latter studied rational conformal field theories with 2 simple modules.

In §2 we review the concept of VVMFs and characters of VOAs. The §3 discusses the MLDEs which are related with the space of $\mathcal{X}_V$. In §4 we study simple VOAs with the central charge $1/2$ and NZWC, and show that V is isomorphic to the minimal model $L(1/2, 0)$. The §5 is devoted to study the simple VOAs with the central charge $-68/7$ and NZWC, and to prove isomorphism between V and $L(-68/7, 0)$.

2. 3rd order modular linear differential equations. Let $\rho : \Gamma_1 = SL_2(\mathbf{Z}) \to GL_n(\mathbf{C})$ be

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a representation of the group Γ_1. A (column) vector-valued function $\mathbf{F} = {}^t(f_1, f_2, \ldots, f_n)$ is called a size n **vector-valued modular form (VVMF) with weight k** if $\mathbf{F}|_k\gamma = {}^t(f_1|_k\gamma, f_2|_k\gamma, \ldots, f_n|_k\gamma) = \rho(\gamma)\mathbf{F}$ for any $\gamma \in \Gamma_1$ and entries of $\mathbf{F}$ are *linearly independent*. Let $\{\lambda_i\}$ be the set of exponents of leading terms of entries of a size n VVMF with weight k. Then we can assume, without loss of generality, that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, in which case the set $\{\lambda_i\}$ is called **normalized**. If the set of the normalized exponents satisfies the linear relation $n(n+k-1) - 12\sum_{i=1}^n \lambda_i = 0$, the VVMF is said to satisfy the **non-zero Wronskian condition** (This condition is equivalent to that the usual Wronskian with respect to $q(dq/dq)$ is not zero.). The set of entries of a VVMF with the non-zero Wronskian condition forms a fundamental system of solutions of an MLDE (see [6] and [8]).

Suppose that $V = \bigoplus_{n=0}^\infty V_n$ $(V_0 = \mathbf{C}\cdot\mathbf{1})$ is a C_2-cofinite VOA. Then it is known that central charges as well as conformal weights are rational numbers by G. Anderson, G. Moore [1], M. Miyamoto [10], and the space of characters of simple modules are invariant under the slash operator with weight 0, in other words, these define a VVMF with weight 0 (see [8] and [12]). Let M be a simple module of V. Then we have $M = \bigoplus_{n\in r+\mathbf{Z}_{\geq 0}} M_n$ and $\dim M_n < \infty$, where r is a rational number with $M_r \neq 0$ and L_0 is the grading operator of M, in particular, $r = 0$ for V. The rational number r is called a **conformal weight** of the simple V-module M. Let c be the central charge of V. Then the **formal character** of a simple V-module M is defined to be

$$\begin{aligned}\mathrm{ch}_M(\tau) &= \mathrm{tr}_M\, q^{L_0 - c/24}\\ &= \sum_{n=0}^\infty (\dim M_{r+n})\, q^{n+r-c/24}.\end{aligned}$$

3. 3-dimensional space of characters. Let V be simple VOA with $\dim \mathcal{X}_V = 3$. Suppose that $\{0, h_1, h_2\}$ is the set of conformal weights and the set of powers of leading terms of characters $\{-c/24, h_1 - c/24, h_2 - c/24\}$ satisfies the non-zero Wronskian condition. Then there is a rational number h such that $\{0, h_1, h_2\} = \{0, h, c/8 - h + 1/2\}$. Since 0 is a (trivial) conformal weight, it follows that the indicial equation of the corresponding MLDE has $-c/24$ as a root, and hence the set of its indices must be $\{-c/24, h - c/24, c/12 - h + 1/2\}$.

The general form of MLDEs of 3rd order (of weight 0) is defined to be $\vartheta^3(f) + P_2\vartheta(f) + P_3 f = 0$ where ϑ is the Serre derivation and P_k is a holomorphic modular form with weight $2k$ (see [8]). This MLDE is rewritten as $f''' - 1/2E_2 f'' + \{1/2E_2' + \alpha E_4\}f' + \beta E_6 f = 0$ with complex numbers α and β. Since an MLDE associated with an MLDE has $-c/24$, h and $c/8 - h + 1/2$ as indices, the MLDE of our interest is written as

$$(1)\quad \begin{aligned} f''' - \frac{1}{2}E_2 f'' + \Bigg\{&\frac{1}{2}E_2' \\ &- \left(h^2 - \frac{h}{2} - \frac{ch}{8} + \frac{c^2}{192} + \frac{c}{24}\right)E_4\Bigg\}f' \\ &+ \frac{c}{24}\left(\frac{1}{2} + \frac{c}{12} - h\right)\left(h - \frac{c}{24}\right)E_6 f = 0,\end{aligned}$$

where E_k is the normalized Eisenstein series of weight k on Γ_1, and the $'$ denotes $q(d/dq)$.

Suppose that (1) has a solution of the form $f = \sum_{n=0}^\infty a_n q^{\varepsilon+n}$ with $a_0 = 1$. (By definition it follows that $a_0 = 1$ of character of $\mathrm{ch}_V(\tau)$. Of course, a_0 of other characters may not be 1.) Then by Frobenius method we can determine a_n $(n > 0)$, recursively, by

$$(2)\quad \begin{aligned} a_n = &\left(n + \varepsilon + \frac{c}{24}\right)^{-1}\left(n + \varepsilon + \frac{c}{24} - h\right)^{-1} \\ &\times \left(n + \varepsilon - \frac{c}{12} - \frac{1}{2} + h\right)^{-1} \\ &\times \sum_{i=1}^n \Bigg\{(\varepsilon - i + n)12(2i - \varepsilon - n)\sigma_1(i) \\ &+ \frac{5}{4}(c^2 + 8c - 24hc - 96h + 192h^2)\sigma_3(i) \\ &- \frac{7}{96}c(c - 24h)(c - 12h + 6)\sigma_5(i)\Bigg\}a_{n-i},\end{aligned}$$

as far as the denominators are not 0, where $\sigma_{k-1}(i)$ is the sum of the $(k-1)$th powers of the divisors of i.

4. Central charge $c = 1/2$. Here we completely determine the simple VOAs with central charge 1/2 and the non-zero Wronskian condition (see Theorem 1).

Let $L(1/2, 0)$ be a **minimal model** (which is also called *Virasoro VOA*) with the central charge $c = 1/2$. Then it is known that the set of conformal weights of $L(1/2, 0)$ is $\{0, 1/2, 1/16\}$ and hence the set of indices of characters is $\{-1/48, 23/48, 1/24\}$

(cf. [4]). It is immediate that these indices satisfy the non-zero Wronskian condition and then the set of characters forms a fundamental system of solutions of the MLDE

$$(3)\qquad f''' - \frac{1}{2}E_2 f'' + \left(\frac{1}{2}E_2' + \frac{7}{768}E_4\right)f' + \frac{23}{55296}E_6 f = 0.$$

Since the minimal models are uniquely determined by the space of characters as we see later, it suffices to show that $h = 1/2$ or $h = 1/16$ when $c = 1/2$ if any solution of (1) is of *character type*. Our discussions use the facts: **a)** The coefficients of every character are non-negative integers, particularly, **b)** The leading coefficient of the character ch_V is 1, **c)** Any conformal weight is rational if V is C_2-cofinite.

It follows by (2) with $\varepsilon = -1/48$ (since $c = 1/2$) that the 2nd coefficients of the character $\mathrm{ch}_V(\tau)$ is given by

$$a_1 = \frac{31(32h^2 - 18h + 1)}{4(h-1)(16h+7)}.$$

We now necessarily require that $m = a_1$ is a non-negative integer. Then the quadratic equation

$$(4)\qquad 32(31-2m)h^2 - 18(31-2m)h + 28m + 31 = 0$$

with an indeterminate h must have an integral square discriminant d^2 for a $d \in \mathbf{Z}$. Therefore we have

$$81(31-2m)^2 - 32(31-2m)(28m+31) = d^2,$$

which is rewritten as

$$(5)\qquad (1058m - 23d - 8959) \times (1058m + 23d - 8959) = 55353600.$$

This shows that integers $1058m \pm 23d - 8959$ are divisors of 55353600, and hence the set of all solutions (m, d) of (5) is given by

$$(6)\qquad \{(-166, \pm 8019), (0, \pm 217), (23, \pm 585), (93, \pm 3875)\}.$$

Substituting m in (6) into (4), we obtain the set of h with an integer a_1 as

$$(7)\qquad \left\{-\frac{15}{16}, -\frac{1}{2}, -\frac{9}{22}, \frac{1}{16}, \frac{1}{2}, \frac{171}{176}, \frac{17}{16}, \frac{3}{2}\right\}.$$

It is obvious that two different values of h can give the same set of conformal weights, which are given in the Table I. Hence (7) is reduced to a set, for instance,

Table I. Values of h and the conformal weights

Values of h	Indices
171/176, −9/22	−1/48, 251/264, −227/528
1/2, 1/16	−1/48, 23/48, 1/24
−15/16, 3/2	−1/48, −23/24, 71/4
−1/2, 17/16	−1/48, −25/48, 25/24

$$(8)\qquad \left\{-\frac{15}{16}, -\frac{1}{2}, \frac{1}{2}, \frac{171}{176}\right\}.$$

Now, the first several terms of q-series of solutions f_h (which depend on h) of the index $-1/48$ are given by

$$f_{\frac{171}{176}} = \frac{1}{\sqrt[48]{q}} - 166q^{47/48} + \cdots,$$

$$f_{\frac{1}{2}} = \frac{1}{\sqrt[48]{q}} + q^{95/48} + q^{143/48} + \cdots,$$

$$f_{-\frac{15}{16}} = \frac{1}{\sqrt[48]{q}} + 23q^{47/48} + \cdots,$$

$$f_{-\frac{1}{2}} = \frac{1}{\sqrt[48]{q}} + 93q^{47/48} + \frac{12131}{5}q^{95/48} + \cdots.$$

Then $h \neq 171/176$ since a_1 is negative, and $h \neq -1/2$ due to the 3rd coefficients is not integers. Therefore possible values of h are $-15/16$ and $1/2$. However, this is impossible because the 6th coefficient of the solution of index $h = -15/16$ is -31299. Therefore we conclude that $h \neq -5/16$. The final possibility of $h = 1/2$, indeed, there is a VOA with central charge 1/2 and conformal weights $\{0, 1/2, 1/16\}$, for instance, Ising model satisfies these conditions.

We now determine the characters of VOAs with central charge is 1/2, whose solutions have indices corresponding to conformal weights 0, 1/2 and 1/16. Let $\phi_1(q) = \eta(q)^2/\eta(q^{1/2})\eta(q^2)$, $\phi_2(q) = \eta(q^{1/2})/\eta(q)$ and $\phi_3(q) = \sqrt{2}\,\eta(q^2)/\eta(q)$ be the (classical) *Weber functions* (cf. [11]), where η is the Dedekind eta function. It is not hard to verify that they are solutions of (3) by using relations $48\phi_1' = \phi_1\eta^4(\phi_3^8 - \phi_2^8)$, $48\phi_2' = -\phi_2\eta^4(\phi_1^8 + \phi_3^8)$, $48\phi_3' = \phi_3\eta^4(\phi_1^8 + \phi_2^8)$, $24\eta' = E_2\eta$, $12E_2' = E_2^2 - E_4$, $3E_4' = E_2E_4 - E_6$ and $2E_4 = \sum_{i=3}^{3}\phi_i^{16}$.

Let g_0, $g_{1/2}$ and $g_{1/16}$ be solutions of (3) with the conformal weights 0, 1/2 and 1/16, which are expressed as (cf. [7, Theorem 8. (8.5) and (8.6)])

$$g_0 = \frac{\phi_1(q)+\phi_2(q)}{2} = \eta(q^2)^{-1}\sum_{n\in\mathbf{Z}} q^{(2n+1/4)^2} = \frac{1}{q^{1/48}} + q^{95/48} + q^{143/48} + \cdots, \tag{9}$$

$$g_{1/2} = \frac{\phi_1(q)-\phi_2(q)}{2} = \eta(q^2)^{-1}\sum_{n\in\mathbf{Z}} q^{(2n+3/4)^2} = q^{23/48} + q^{71/48} + q^{119/48} + \cdots, \tag{10}$$

$$g_{1/16} = \phi_3(q)/\sqrt{2} = \eta(q^2)^{-1}\sum_{n\in\mathbf{Z}} q^{2(n+1/4)^2} = q^{1/24} + q^{25/24} + q^{49/24} + \cdots. \tag{11}$$

Since it is well known that Fourier coefficients of $\eta(q)^{-1}$ are positive integers, it follows that Fourier coefficients of g_0, $g_{1/2}$ and $g_{1/16}$ are non-negative integers. Since the set of characters of V and the minimal model $L(1/2,0)$ coincide comparing characters of the vertex operator subalgebra generated by the Virasoro element and $L(1/2,0)$, we conclude that V is isomorphic to $L(1/2,0)$ as a vertex operator algebra. (This is kindly pointed out by Y. Arike.)

Theorem 1. *Let V be a simple vertex operator algebra with central charge $1/2$ whose space of characters is 3-dimensional. Suppose that V satisfies the non-zero Wronskian condition and does not have an index $47/48$. Then V is isomorphic to the minimal model $L(1/2,0)$ with central charge $1/2$. The conformal weights are 0, $1/2$, and $1/16$ whose characters are given by (9), (10) and (11).*

Remark 2. The $h=-1/2$ was excluded since the 3rd coefficient of $f_{-1/2}=(\phi_1^{25}-\phi_2^{25})/50$ is not integer. But the other two solutions for $h=-1/2$ (or $17/16$) are given by $(\phi_1^{25}\pm\phi_2^{25})/2\in \mathbf{Q}_{\geq 0}[g_0,g_{1/2}]$ and $(\phi_3/\sqrt{2})^{25}(=g_{1/16}^{25})$. It follows that their coefficients are non-negative integers by (9)–(11) and $(\phi_1^{25}\pm\phi_2^{25})/2\in((\phi_1\pm\phi_2)/2)\cdot\mathbf{Z}[\phi_1,\phi_2]$. Their indices coincide with solutions with $c=25/2$ and conformal weights 0, $1/2$ and $25/16$. The affine VOA B_{12} level 1 has these properties.

5. Central charge $c=-68/7$. We can apply almost the same method to the case of $c=-68/7$. The Eq. (1) has a solution of the form $f=\sum_{i=0}^{\infty} b_i q^{17/42+i}$ with $b_0=1$, and the 2nd coefficient b_1 is given by

$$b_1 = -\frac{2108(49h^2+35h+6)}{49(h-1)(7h+12)},$$

which is rewritten as

$$(103292-343m)h^2 + (73780-245m)h + 588m + 12648 = 0, \tag{12}$$

where $m=b_1\in\mathbf{Z}_{\geq 0}$. By the same discussions given in §4 it follows that (12) in the indeterminate h has rational solutions if and only if $(m,d\ )\in\mathbf{Z}^2$ satisfies

$$(2527m-19d-381548)(2527m+19d-381548) = 143974713600.$$

Therefore any such integer m must be one of -2848879, -593636, -13243, -8944, -8366, -1088, -868, -527, -446, -374, -319, 0, 10, 496, 1501, 2914 and 26877. The coefficient b_1 is a non-negative integer if and only if h is one of $132/133$, $-227/133$, $72/77$, $-127/77$, $179/203$, $-324/203$, $5/7$, $-10/7$, $-8/77$, $-47/77$, $-2/7$ and $-3/7$. If h is one of $-227/133$, $-127/77$, $-324/203$ and $-10/7$, then the 2nd coefficients of the q-series with the index $h+59/42$ are negative rational numbers. Similarly, for $h=132/133$, $72/77$, $179/203$ and $5/7$, the 2nd coefficients of q-series with the index $-13/42-h$ are negative. Finally h is neither $-8/77$ nor $-47/77$ since the 3rd coefficients of the q-series with the index $17/42$ are not integers. Therefore we have $h=-2/7$ or $-3/7$ and hence the conformal weights are 0, $-2/7$ and $-3/7$. The corresponding MLDE with indices $17/42$, $5/42$, and $-1/42$ is given by

$$f''' - \frac{1}{2}E_2 f'' + \left(\frac{1}{2}E_2' + \frac{1}{28}E_4\right)f' + \frac{85}{74088}E_6 f = 0.$$

The solutions of this equation are

$$x(q) = \frac{1}{\eta(q)}\sum_{n\in\mathbf{Z}}(-1)^n q^{(14n+5)^2/56} = q^{17/42}\prod_{\substack{n>0\\ n\not\equiv 0,\pm1 \bmod 7}}(1-q^n)^{-1},$$

$$y(q) = \frac{1}{\eta(q)}\sum_{n\in\mathbf{Z}}(-1)^n q^{(14n+3)^2/56} = q^{5/42}\prod_{\substack{n>0\\ n\not\equiv 0,\pm2 \bmod 7}}(1-q^n)^{-1},$$

$$z(q) = \frac{1}{\eta(q)}\sum_{n\in\mathbf{Z}}(-1)^n q^{(14n+1)^2/56} = q^{-1/42}\prod_{\substack{n>0\\ n\not\equiv 0,\pm3 \bmod 7}}(1-q^n)^{-1},$$

respectively, where it is used that the relations $x' = \eta^4\{x(x^7+5y^7+17z^7)-28x^2yz^2(xz^2+y^3)\}/42$ (derivatives y' and z' are obtained by the permutation $(x,y,z)\to(-y,z,-x)$) and the fact that E_4/η^8, E_6/η^{12} can be written as homogenous polynomials in x, y and z of degree 14 and 21, respectively (see [3, Section 2.3]).

The functions x, y and z are modular forms of weight $2/7$ on the principal congruence subgroup of level 7 if they are divided by $\eta(q)^{4/7}$ (cf. [3, Section 2.2]). Since the genus of this group is 3, there exist 3 cusp forms of weight 2 ([2, Section 4.1. (4.6)]). By comparing enough number of Fourier coefficients (up to 28 coefficients are surely enough), we can find that they are $x(q)\eta(q)^4$, $y(q)\eta(q)^4$ and $z(q)\eta(q)^4$, respectively. Hence, by [2, Section 4.1. (4.6)], we have other expressions of x, y and z, and we find that Fourier coefficients of x, y and z are non-negative integers. By the same discussion in the previous sections we have proved:

Theorem 3. *Let V be a simple vertex operator algebra with central charge $-68/7$, whose space of characters is 3-dimensional. Suppose that V satisfies the non-zero Wronskian condition and does not have an index $25/42$. Then V is isomorphic to the minimal model with the central charge $-68/7$. The conformal weights are 0, $-2/7$ and $-3/7$, and the corresponding characters are x, y and z.*

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