On ergodic measures with negative Lyapunov exponents

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Abstract: We prove for $n \geq 3$ that every nonatomic ergodic measure of an n-dimensional flow whose Lyapunov exponents off the flow direction are all negative is supported on an attracting periodic orbit.

Key words: Ergodic measure; Lyapunov exponent; flow.

1. Introduction. It is well known that every ergodic measure whose Lyapunov exponents are all negative of a $C^{1+\alpha}$ diffeomorphism is supported on an attracting periodic orbit (cf. Corollary S.5.2 in [6]). This result was extended to C^1 diffeomorphisms by Araujo [1]. In the case of flows, Campanino [3] proved that every nonatomic ergodic measure whose Lyapunov exponents off the flow direction are all negative of a $C^{1+\alpha}$ flow is supported on an attracting periodic orbit. The methods in [10] imply that this is also true in the C^1 class but for *star flows*, i.e., flows which cannot be C^1 approximated by ones with nonhyperbolic critical elements.

In this paper we will extend Campanino's result to general C^1 n-dimensional flows with $n \geq 3$. More precisely, we will prove for all such flows that every nonatomic ergodic measure whose Lyapunov exponents off the flow direction are all negative is supported on an attracting periodic orbit. Let us state our result in a precise way.

Hereafter the term *n-dimensional flow* will mean a C^1 vector field X defined on a compact connected boundaryless Riemannian manifold M of dimension $n \in \mathbf{N}^+$. The one-parameter group of diffeomorphisms generated by X will be denoted by X_t, $t \in \mathbf{R}$. We say that $x \in M$ is a *periodic point* of X if there is a minimal positive number $\pi(x)$ (called *period*) such that $X_{\pi(x)}(x) = x$. Notice that if x is a periodic point, then 1 is an eigenvalue of the derivative $DX_{\pi(x)}(x)$ with eigenvector $X(x)$. The remainders eigenvalues of $DX_{\pi(x)}(x)$ will be referred to as the eigenvalues of x. We say that a periodic orbit $O(x) = \{X_t(x) : t \in \mathbf{R}\}$ is *attracting* if every eigenvalue of x has a modulus less than 1.

Let μ be a Borel probability measure of M. We say that μ is *nonatomic* if it has no points with positive mass. We say that μ is supported on $H \subset M$ if $\mathrm{supp}(\mu) \subset H$, where $\mathrm{supp}(\mu)$ denotes the support of μ. We say that μ is *invariant* if $\mu(X_t(A)) = \mu(A)$ for every Borelian A and every $t \in \mathbf{R}$. An invariant Borel probability measure is *ergodic* if every measurable invariant set has measure 0 or 1.

Oseledets's Theorem [11] ensures that every ergodic measure μ is equipped with a full measure set R, a positive integer k and real numbers $\chi_1 < \chi_2 < \cdots < \chi_k$ such that for every $x \in R$ there is a measurable splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^k$ such that $DX_t(x)(E_x^i) = E^i_{X_t(x)}$ ($\forall t \in \mathbf{R}$) and

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|DX_t(x)e^i\| = \chi_i,$$

for every $x \in R, e^i \in E_x^i \setminus \{0\}, 1 \leq i \leq k$. Such numbers are so-called the *Lyapunov exponents* of μ. Similar definitions and results hold for C^1 diffeomorphisms.

With these definitions we can state our result.

Theorem 1. *Let μ be a nonatomic ergodic measure of an n-dimensional flow with $n \geq 3$. If the Lyapunov exponents of μ off the flow direction are all negative, then μ is supported on an attracting periodic orbit.*

2. Proof. We divide the proof of Theorem 1 into three parts according to the following subsections.

2.1. Linear Poincaré flow. Given a flow X we denote by $\mathrm{Sing}(X)$ the set of *singularities* (i.e. zeroes) of X. Define $M^* = M \setminus \mathrm{Sing}(X)$ as the set of *regular* (i.e. nonsingular) points of X. To any $x \in M^*$ we define $N_x \subset T_xM$ as the set of tangent

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vectors which are orthogonal to $X(x)$. Denote by $N = \bigcup_{x \in M^*} N_x$ the vector bundle so induced and, correspondingly, denote by $\pi : T_{M^*}M \to N$ the orthogonal projection. Define the *linear Poincaré flow* as $P_t : N \to N$ by $P_t(x) = \pi_{X_t(x)} \circ DX_t(x)$.

We will need the following lemmas about P_t. It is here where the hypothesis $n \geq 3$ is used in the proof of Theorem 1.

Lemma 2. *For every n-dimensional flow X with $n \geq 3$ and every $T \in \mathbf{R}$ there exists $K > 0$ such that $\|P_T(x)\| \geq K$ for every $x \in M^*$.*

Proof. Otherwise, there exists a sequence $x_k \in M^*$ such that $\|P_T(x_k)\| \to 0$ as $k \to \infty$. By compactness we can assume $x_k \to \sigma$ for some $\sigma \in M$. Since $\|P_T(x_k)\| \to 0$, σ is a singularity and so $X_T(x_k) \to \sigma$ too. Again by compactness we can assume that $E^X_{x_k} \to L$ and $E^X_{X_T(x_k)} \to L'$ for some one-dimensional subspaces L and L' of $T_\sigma M$, where E^X is the one-dimensional subbundle of $T_{M^*}M$ generated by X. Since $N_{x_k} \perp E^X_{x_k}$ for all k and $E^X_{x_k} \to L$, N_{x_k} converges to the orthogonal complement N of L in $T_\sigma M$. Similarly, $N_{X_T(x_k)}$ converges to the orthogonal complement N' of L' in $T_\sigma M$. It follows that $DX_T(x_k)|_{N_{x_k}} \to DX_T(\sigma)|_N$ and $\pi_{X_T(x_k)} \to \pi_{N'}$ where $\pi_{N'} : T_\sigma M \to N'$ is the orthogonal projection. Since $P_T(x_k) = \pi_{X_T(x_k)} \circ DX_T(x_k)|_{N_{x_k}}$, we conclude that $P_T(x_k) \to \pi_{N'} \circ DX_T(\sigma)|_N$ as $k \to \infty$. Since $\|P_T(x_k)\| \to 0$ as $k \to \infty$, we obtain $\pi_{N'} \circ DX_T(\sigma)|_N = 0$ which is equivalent to $DX_T(\sigma)N \subset L'$. However, $n \geq 3$ and $\dim(L) = 1$, so $\dim(N) \geq 2$, thus $\dim(DX_T(\sigma)N) \geq 2$. Since $\dim(L') = 1$ and $DX_T(\sigma)N \subset L'$, we obtain a contradiction. This ends the proof. $\square$

Lemma 3. *Let μ be a nonatomic ergodic measure of a flow X. If the Lyapunov exponents of μ off the flow direction are all negative, then there is $T_0 > 0$ such that μ is an ergodic measure of X_{T_0} and $\int \log \|P_{T_0}\| d\mu < 0$.*

Proof. First we prove $\mu(\mathrm{Sing}(X)) = 0$. Otherwise, $\mu(\mathrm{Sing}(X)) = 1$ since μ is ergodic and $\mathrm{Sing}(X)$ is closed invariant. Since every Lyapunov exponent of μ off the flow direction is negative, every Lyapunov exponent of μ is therefore negative (and so different from zero). In such a case, the results in p. 632 of [4] imply that μ is supported on a singularity. However, this contradicts that μ is nonatomic. Hence, $\mu(\mathrm{Sing}(X)) = 0$.

Let us continue with the proof. Since μ is ergodic and $\mu(\mathrm{Sing}(X)) = 0$, Oseledets's Theorem for the linear Poincaré flow (cf. Theorem 2.2 in [2]) implies that there exist a full measure set $R \subset M^*$, a P_t-invariant splitting $N_R = N^1 \oplus \cdots \oplus N^p$ and real numbers $\bar{\chi}_1 < \cdots < \bar{\chi}_p$ such that

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|P_t(x)v^i\| = \bar{\chi}_i,$$

for every $x \in R, v^i \in N^i_x, 1 \leq i \leq p$. Again by Oseledets's which is now applied to the flow X, we also have an invariant measurable splitting $E^1 \oplus \cdots \oplus E^k$ over R with Lyapunov exponents $\chi_1 < \cdots < \chi_k$ of μ as an ergodic measure of X. Since every Lyapunov exponent of μ off the flow direction is negative, $\chi_k = 0$ and so $\chi_{k-1} < 0$.

Take $v \in N^i_x$ for some $x \in R$ and $1 \leq i \leq p$. Write $v = \sum_{j \in J} v_j$ for some $J \subset \{1, \cdots, k-1\}$ and $v_j \in E^j_x$ for all $j \in J$. Then,

$$\frac{1}{t} \log \|P_t(x)v\| \leq \frac{\log k}{t} + \max_{j \in J} \frac{1}{t} \log \|DX_t(x)v_j\|.$$

Letting $t \to \infty$ we get $\bar{\chi}_i \leq \max\{\chi_1, \cdots, \chi_{k-1}\} = \chi_{k-1} < 0$. Hence the numbers $\{\bar{\chi}_1, \cdots, \bar{\chi}_p\}$ are all negative too.

By [9] we can fix $T_1 > 0$ such that μ is *totally ergodic for X_{T_1}* (i.e. μ is ergodic for $X_{nT_1}, \forall n \in \mathbf{N}^+$). Additionally, it follows from the definitions that $\|P_{T_1}(x)\| \leq \|DX_{T_1}(x)\|$ for every $x \in M^*$. These, Lemma 2 and $\mu(M^*) = 1$ imply that there is $K > 0$ such that $\log K \leq \log \|P_{T_1}(x)\| \leq \log \|DX_{T_1}(x)\|$ for μ-a.e. $x \in X$. From this we obtain $\log \|P_{T_1}\| \in \mathcal{L}^1(M, \mu)$.

Now, let $\mathcal{A} = \{A_n : n \in \mathbf{N}^+\}$ be the sequence of linear maps $A_n : N \to N$ defined by

$$A_n(x) = P_{nT_1}(x) = P_{T_1}(X_{(n-1)T_1}(x)) \circ \cdots \circ P_{T_1}(x)$$

whenever $x \in M^*$. Since M^* is open and X of class C^1, we have that $\mathcal{A}$ is measurable. Since $\log \|P_{T_1}\| \in \mathcal{L}^1(M, \mu)$, the Furstenberg-Kesten Theorem implies that there is $\lambda \in \mathbf{R}$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \|P_{nT_1}(x)\| = \lambda, \quad \mu\text{-a.e. } x \in M. \tag{1}$$

Moreover, λ is the upper Lyapunov exponent of $\mathcal{A}$ (cf. p. 150 in [12]). Since the Lyapunov exponents $\{\bar{\chi}_1, \cdots, \bar{\chi}_p\}$ of $\mathcal{A}$ are all negative, we get $\lambda < 0$.

Next consider the sequence of functions $f_n : M \to \bar{\mathbf{R}}$ given by

$$f_n(x) = \begin{cases} \dfrac{1}{n} \log \|P_{nT_1}(x)\| & \text{if } x \notin \mathrm{Sing}(X) \\ -\infty & \text{otherwise.} \end{cases}$$

Since M^* is open and X of class C^1, f_n is a sequence of measurable functions. Moreover, (1) implies $|f_n| \le 2\lambda$ for large n. Since the constant map $x \mapsto 2\lambda$ is integrable (because $\mu(M) = 1$), (1) and the Dominated Convergence Theorem imply

$$\lim_{n\to\infty} \int \frac{1}{n} \log \|P_{nT_1}\| d\mu = \int \lambda d\mu = \lambda.$$

Pick $0 < \epsilon < -\lambda$ so $\lambda + \epsilon < 0$. The above limit implies that there is $n \in \mathbf{N}^+$ such that

$$\int \log \|P_{nT_1}\| d\mu < n(\lambda + \epsilon) < 0.$$

As μ is totally ergodic for X_{T_1}, μ is ergodic for X_{nT_1}. Then, we are done by taking $T_0 = nT_1$. $\square$

In what follows we will denote by $B(x, \delta)$ and $B[x, \delta]$ the open and closed balls of radius δ of M centered at $x \in M$ respectively.

Recall that the *support* of a Borel probability measure μ is the set $\mathrm{supp}(\mu)$ of points $x \in M$ such that $\mu(B(x, \delta)) > 0$ for every $\delta > 0$. For every flow X and $x \in M$ we define the omega-limit set as

$$\omega_X(x) = \Big\{ y \in M : y = \lim_{k\to\infty} X_{t_k}(x) \text{ for some sequences } t_k \to \infty \Big\}.$$

We say that $\Lambda \subset M$ is a *transitive set* of X if $\Lambda = \omega_X(x)$ for some $x \in \Lambda$.

Lemma 3 and a result by Liao [5] imply Theorem 1 when the involved measure has no singularities in its support. More precisely, we have the following result.

Corollary 4. *Let μ be a nonatomic ergodic measure of a flow X with* $\mathrm{supp}(\mu) \cap \mathrm{Sing}(X) = \emptyset$. *If the Lyapunov exponents of μ off the flow direction are all negative, then μ is supported on an attracting periodic orbit.*

Proof. By Lemma 3 there exists $T_0 > 0$ such that $\int \log \|P_{T_0}\| d\mu < 0$. Then, by a result of Liao (Lemma 3.2 in [5]) we have that $\mathrm{supp}(\mu)$ contains an attracting periodic orbit of X. Since $\mathrm{supp}(\mu)$ is ergodic, $\mathrm{supp}(\mu)$ is transitive and so μ is supported on that periodic orbit. This completes the proof. $\square$

2.2. Scaled Poincaré flow and $(\eta, T_0)^*$-contractible orbits. Liao defined the *scaled linear Poincaré flow* by

$$P_t^*(x) = \frac{\|X(x)\|}{\|X(X_t(x))\|} P_t(x), \quad \forall x \in M^*.$$

By an *orbit* of X we mean $O = \{X_t(x) : t \in \mathbf{R}\}$. In such a case we say that O is the orbit *through* x. The orbit O is *regular* if $X(x) \neq 0$.

Given $\eta > 0$ and $T_0 > 0$ we call a regular orbit O *eventually* $(\eta, T_0)^*$*-contractible* if there are $x \in O$ and $n_x \in \mathbf{N}^+$ such that

$$(2) \qquad \frac{1}{nT_0} \sum_{i=0}^{n-1} \log \|P^*_{T_0}(X_{iT_0}(x))\| \le -\eta, \quad \forall n \ge n_x.$$

If above n_x can be chosen as 1, then O is called $(\eta, T_0)^*$*-contractible* [7]. Clearly every $(\eta, T_0)^*$-contractible orbit is eventually $(\eta, T_0)^*$-contractible. As in [7], in each case we call x *reference point* of O.

Given $x \in M$ we define $W^{sta}(x)$ as the set of points $y \in M$ for which there exists a continuous monotonic function $h : [0, \infty[\to [0, \infty[$ with $h(0) = 0$ such that

$$\lim_{t\to\infty} d(X_t(x), X_{h(t)}(y)) = 0.$$

By Proposition 6.1 of Liao [7] for every flow X and every pair of numbers $\eta, T_0 > 0$ there exists $\xi > 0$ such that if O is a $(\eta, T_0)^*$-contractible orbit with reference point x, then $B(x, \xi\|X(x)\|) \subset W^{sta}(x)$. The proof is based on the following statistical property (see (1.3) in p. 3 of [8]):

$$\limsup_{n\to\infty} \frac{1}{nT_0} \sum_{i=0}^{n-1} \log \|P^*_{T_0}(X_{iT_0}(x))\| < 0.$$

Since this statistical property is also true for eventually $(\eta, T_0)^*$-contractible orbits with reference point x (just take a limit superior in (2)), Proposition 6.1 in [7] is also true in the eventual case as well. Specifically, we have the following result.

Lemma 5. *For every flow X and every pair of numbers $\eta, T_0 > 0$ there exists $\xi > 0$ such that if O is an eventually $(\eta, T_0)^*$-contractible orbit with reference point x, then $B(x, \xi\|X(x)\|) \subset W^{sta}(x)$.*

We say that x is *recurrent* if $x \in \omega_X(x)$. Denote by $R(X)$ the set of recurrent points of X. A similar definition holds for diffeomorphisms. The following lemma is a consequence of Lemma 3.

Lemma 6. *Let μ be a nonatomic ergodic measure of a flow X. If the Lyapunov exponents of μ off the flow direction are all negative, then there are $\eta, T_0 > 0$ and an orbit O which is eventually $(\eta, T_0)^*$-contractible with reference point $x \in R(X) \cap \mathrm{supp}(\mu)$.*

Proof. Applying Lemma 3 there are $\eta_0, T_0 > 0$ such that μ is an ergodic measure of X_{T_0} and

$\int \log \|P_{T_0}\| d\mu < -\eta_0$. As $\log \|P_{T_0}\| \in \mathcal{L}^1(M,\mu)$, Birkhoff's ergodic theorem implies

$$\lim_{n\to\infty} \frac{1}{nT_0} \sum_{i=0}^{n-1} \log \|P_{T_0}(X_{iT_0}(x))\| < -\eta,$$

μ-a.e. $x \in M$, where $\eta = \frac{\eta_0}{T_0}$. However,

$$\lim_{n\to\infty} \frac{1}{nT_0} (\log \|X(X_{nT_0}(x))\| - \log \|X(x)\|) = 0$$

for any $x \in M^*$. So, the previous inequality yields the following one:

$$\lim_{n\to\infty} \frac{1}{nT_0} \sum_{i=0}^{n-1} \log \|P^*_{T_0}(X_{iT_0}(x))\| < -\eta,$$

μ-a.e. $x \in M$. By Poincaré recurrence there is $x \in R(X_{T_0}) \cap \mathrm{supp}(\mu)$ satisfying the latter inequality. From this it follows that the orbit O through x is eventually $(\eta, T_0)^*$-contractible with reference point x. Since $R(X_{T_0}) \subset R(X)$, we get $x \in R(X) \cap \mathrm{supp}(\mu)$ and we are done. $\square$

2.3. Proof of Theorem 1. Let μ be a nonatomic ergodic measure of an n-dimensional flow X with $n \geq 3$. Suppose that the Lyapunov exponents off the flow direction of μ are all negative.

Then, by Lemma 6, there are $\eta, T_0 > 0$ and an orbit O which is eventually $(\eta, T_0)^*$-contractible with reference point $x \in R(X) \cap \mathrm{supp}(\mu)$. By putting such η and T_0 in Lemma 5 we obtain $\xi > 0$ such that $B(x, \xi\|X(x)\|) \subset W^{sta}(x)$. Taking $2\delta = \xi\|X(x)\|$ we get $\delta > 0$ satisfying $B(x, 2\delta) \subset W^{sta}(x)$.

It follows that for every $y \in B[x,\delta]$ there is a continuous monotonic function $h_y : [0,\infty[\to [0,\infty[$ with $n_y(0) = 0$ such that

$$\lim_{t\to\infty} d(X_t(x), X_{h_y(t)}(y)) = 0.$$

Since $x \in R(X)$, we have $x \in \omega_X(x)$ and so there is a sequence $t_k \to \infty$ such that $X_{t_k}(x) \to x$ as $k \to \infty$. Then, by replacing $t = t_k$ in the previous limit we obtain $X_{h_y(t_k)}(y) \to x$ for every $y \in B[x,\delta]$. It follows that for every $y \in B[x,\delta]$ there are $k_y \in \mathbf{N}^+$ and $\delta_y > 0$ such that $d(x, X_{h_y(t_{k_y})}(z)) < \frac{\delta}{2}$ for every $z \in B(y, \delta_y)$. Since h_y is monotonic and $B[x,\delta]$ has no singularities, we can assume that $h_y(t_{k_y}) > 0$ for every $y \in B[x,\delta]$. Notice that $\{B(y,\delta_y) : y \in B[x,\delta]\}$ is an open covering of $B[x,\delta]$. Since $B[x,\delta]$ is compact, there are finitely many points $y_1, \cdots, y_l \in B[x,\delta]$ such that $\{B(y_i, \delta_{y_i}) : i = 1, \cdots, l\}$ is an open covering of $B[x,\delta]$. Take $z \in B[x,\delta]$. Then, $z \in B(y_i, \delta_{y_i})$ for some $i = 1, \cdots, l$ so $X_{h_{y_i}(t_{k_{y_i}})}(z) \in B[x, \frac{\delta}{2}]$. Hence, the numbers $t_i = h_{y_i}(t_{k_{y_i}})$ for $1 \leq i \leq l$ are all positive satisfying

$$B[x,\delta] \subset \bigcup_{i=1}^{l} X_{t_i}\left(B\left[x, \frac{\delta}{2}\right]\right). \tag{3}$$

Now define

$$K = \bigcup_{t\geq 0} X_t(B[x,\delta]).$$

Since

$$X_{-s}(K) = \bigcup_{t\geq 0} X_{t-s}(B[x,\delta]) = \bigcup_{t\geq -s} X_t(B[x,\delta]) \supset K$$

$\forall s \geq 0$, we obtain that K is *positively invariant*, i.e., $X_s(K) \subset K$ for every $s \geq 0$.

We also have that $B(x, \frac{\delta}{2}) \subset \mathrm{Int}(K)$ and so $x \in \mathrm{Int}(K)$. Since $x \in \mathrm{supp}(\mu)$ we conclude that $\mathrm{supp}(\mu) \cap \mathrm{Int}(K) \neq \emptyset$.

We claim that

$$K = \bigcup_{0\leq t\leq \max\{t_1,\cdots,t_l\}} X_t(B[x,\delta]). \tag{4}$$

Indeed, take $z \in K$. Hence there are $t \geq 0$ and $y \in B[x,\delta]$ such that $z = X_t(y)$. Since $y \in B[x,\delta]$, (3) implies that there is a sequence $i_j \in \{1, \cdots, l\}$ such that

$$X_{\sum_{j=1}^r t_{i_j}}(y) \in B\left[x, \frac{\delta}{2}\right], \quad \forall r = 1, 2, \cdots. \tag{5}$$

Since each $t_i > 0$, there is $r \in \mathbf{N}^+$ such that

$$\sum_{i=1}^{r} t_{i_j} \leq t \leq \sum_{i=1}^{r} t_{i_j} + \max\{t_1, \cdots, t_l\}.$$

By taking this r in (5) we get $X_t(y) = X_s(\bar{y})$ where

$$s = t - \sum_{i=1}^{r} t_{i_j} \in [0, \max\{t_1, \cdots, t_l\}]$$

and

$$\bar{y} = X_{\sum_{j=1}^r t_{i_j}}(y) \in B[x,\delta].$$

It follows that

$$z = X_t(y) = X_s(\bar{y}) \in \bigcup_{0\leq t\leq \max\{t_1,\cdots,t_l\}} X_t(B[x,\delta])$$

and so

$$K \subset \bigcup_{0\leq t\leq \max\{t_1,\cdots,t_l\}} X_t(B[x,\delta]).$$

Since the reversed inclusion is obvious, we obtain (4).

It follows from (4) that K is compact. Since μ is ergodic, $\operatorname{supp}(\mu)$ is a transitive set. Since $\operatorname{supp}(\mu) \cap \operatorname{Int}(K) \neq \emptyset$, the positive orbit of q eventually meets $\operatorname{Int}(K)$. Since K is compact and positively invariant, $\operatorname{supp}(\mu) \subset K$.

On the other hand, it is easy to see that $\operatorname{Sing}(X) \cap K = \emptyset$ (otherwise there would be some singularities in $B[x, \delta]$ contradicting that x is regular). Since $\operatorname{supp}(\mu) \subset K$, we obtain $\operatorname{supp}(\mu) \cap \operatorname{Sing}(X) = \emptyset$. Then, μ is supported on an attracting periodic orbit by Corollary 4. This completes the proof. □

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