A positivity conjecture related first positive rank and crank moments for overpartitions

By Xinhua XIONG

Research Institute for Symbolic Computation, Johannes Kepler University, Altenberger Straße 69, A-4040 Linz, Austria

(Communicated by Masaki KASHIWARA, M.J.A., Oct. 12, 2016)

Abstract: Recently, Andrews, Chan, Kim and Osburn introduced a q-series $h(q)$ for the study of the first positive rank and crank moments for overpartitions. They conjectured that for all integers $m \geq 3$,

$$\frac{1}{(q)_\infty}(h(q) - mh(q^m))$$

has positive power series coefficients for all powers of q. Byungchan Kim, Eunmi Kim and Jeehyeon Seo provided a combinatorial interpretation and proved it is asymptotically true. In this note, we show this conjecture is true if m is any positive power of 2, and we show that in order to prove this conjecture, it is only to prove it for all primes m. Moreover we give a stronger conjecture. Our method is completely different from that of Kim et al.

Key words: Overpartitions; q-series; positivity.

1. Introduction. An overpartition [3] is a partition in which the first occurrence of each distinct number may be overlined. For example, the 14 overpartitions of 4 are

$$4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2,$$
$$\overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1,$$
$$\overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

Let $\overline{N}(n,m)$ denote the number of overpartitions of n whose rank is m and $\overline{M}(n,m)$ the number of overpartitions of n whose first residual crank is m. Andrews, Chan, Kim and Osburn [1] defined the positive rank and crank moments for overpatitions:

$$\overline{N}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{N}(m,n)$$

and

$$\overline{M}_k^+(n) := \sum_{m=1}^{\infty} m^k \overline{M}(m,n).$$

They proved the inequality

$$\overline{M}_1^+(n) > \overline{N}_1^+(n) \tag{1}$$

holds for all $n \geq 1$. They also gave the difference $\overline{M}_1^+(n) - \overline{N}_1^+(n)$ a combinatorial interpretation. In order to prove (1), Andrews, Chan, Kim and Osburn [1] introduced the function

$$h(q) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}}{1-q^n},$$

and conjectured that

$$\frac{1}{(q)_\infty}(h(q) - mh(q^m))$$

has positive coefficients for all $m \geq 3$, where we use the standard q-series notation, $(a)_\infty = (a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1})$. In [4], Byungchan Kim, Eunmi Kim and Jeehyeon Seo provided a combinatorial interpretation for the coefficients of $\frac{1}{(q)_\infty} h(q^m)$. According to their definition, a m-string in an ordinary partition is the parts consisting of $m(1+k), m(3+k), \ldots, m(2j-1+k)$ with a positive integer j and a nonnegative integer $k \leq j$, and a weight of m-string is 1 if $k = 0$ or j, and 2, otherwise. They defined $C_m(n)$ as the weighted sum of the number of m-strings along the partitions of n, i.e.,

$$C_m(n) = \sum_{\lambda \vdash n} \sum_{\substack{\pi \text{ is a} \\ m\text{-string of } \lambda}} \mathrm{wt}(\pi).$$

It is clear that

2010 Mathematics Subject Classification. Primary 11P82, 05A17.

doi: 10.3792/pjaa.92.117

$$\frac{1}{(q)_\infty} h(q^m) = \sum_{n\ge 1} C_m(n) q^n.$$

So the conjecture of Andrews, Chan, Kim and Osburn [1] can be interpreted as there are more (weighted count of) 1-strings than m times of (weighted count of) m-strings along the partitions of n. In the paper [4], Byungchan Kim, Eunmi Kim and Jeehyeon Seo, by using the circle method of Wright [5] and some results from [2], proved the conjecture of Andrews-Chan-Kim-Osburn is asymptotically true. In this note, we will prove this conjecture is true if m is any power of 2. Moreover, we show that in order to prove the conjecture, it is only to prove it is true for all primes m.

Theorem 1.1. *For all integers $m \ge 2$,*

$$\frac{1}{(q)_\infty} (h(q) - 2^m h(q^{2^m}))$$

has positive power series coefficients for all positive powers of q.

Theorem 1.2. *Suppose for a prime p,*

$$\frac{1}{(q)_\infty} (h(q) - p h(q^p))$$

has positive power series coefficients for all positive powers of q. Then for all integers $m \ge 2$,

$$\frac{1}{(q)_\infty} (h(q) - p^m h(q^{p^m}))$$

has positive power series coefficients for all positive powers of q.

2. Proof of Theorem 1.1 and Theorem 1.2. For a prime $p \ge 1$, we define the function $M_p(q) = \frac{1}{(q)_\infty}(h(q) - p h(q^p))$, then we have $M_2(q)$ has positive power series coefficients for all positive powers of q^n for all n except that the coefficients of q^2 and q^4 are zero. This lemma is the Corollary 2.4 of [1].

Proof of Theorem 1.1. For $m \ge 2$,

$$\begin{aligned}
&M_{2^m}(q)\\
&= \frac{1}{(q)_\infty} (h(q) - 2^m h(q^{2^m}))\\
&= \frac{1}{(q)_\infty} (h(q) - 2h(q^2) + 2h(q^2) - 4h(q^4) + \cdots\\
&\quad + 2^{m-1} h(q^{2^{m-1}}) - 2^m h(q^{2^m}))\\
&= \frac{h(q) - 2h(q^2)}{(q)_\infty} + 2\,\frac{h(q^2) - 2h(q^4)}{(q)_\infty} + \cdots\\
&\quad + 2^{m-1}\,\frac{h(q^{2^{m-1}}) - 2h(q^{2^m})}{(q)_\infty}\\
&= \frac{h(q) - 2h(q^2)}{(q)_\infty}\\
&\quad + \frac{2}{(1-q)(1-q^3)(1-q^5)\cdots} \cdot \frac{h(q^2) - 2h(q^4)}{\prod_{n=1}^\infty (1-q^{2n})} + \cdots\\
&\quad + \frac{2^{m-1}}{\prod_{n\ne 2^{m-1}k, k\ge 1}(1-q^n)} \cdot \frac{h(q^{2^{m-1}}) - 2h(q^{2^m})}{\prod_{n=1}(1-q^{2^{m-1}n})}\\
&= M_2(q) + \frac{2}{(1-q)(1-q^3)(1-q^5)\cdots} \cdot M_2(q^2) + \cdots\\
&\quad + \frac{2^{m-1}}{\prod_{n\ne 2^{m-1}k, k\ge 1}(1-q^n)} \cdot M_2(q^{2^{m-1}}).
\end{aligned}$$

By the lemma above, $M_2(q)$ has positive coefficients of q^n for all n with except 2 and 4, therefore for all $m \ge 2$, $M_2(q^{2^{m-1}})$ has positive coefficients except that the coefficients of $q^{2^{m+1}}$ and $q^{2^{m+2}}$, but the sum of the left hand will have positive coefficients of q^n for all $n \ge 1$. □

Proof of Theorem 1.2. For $p \ge 3$ a prime and $m \ge 2$,

$$\begin{aligned}
&M_{p^m}(q)\\
&= \frac{1}{(q)_\infty} (h(q) - p^m h(q^{p^m}))\\
&= \frac{1}{(q)_\infty} (h(q) - p h(q^p) + p h(q^p) - p^2 h(q^{p^2}) + \cdots\\
&\quad + p^{m-1} h(q^{p^{m-1}}) - p^m h(q^{p^m}))\\
&= \frac{h(q) - p h(q^p)}{(q)_\infty} + p\,\frac{h(q^p) - p h(q^{p^2})}{(q)_\infty} + \cdots\\
&\quad + p^{m-1}\,\frac{h(q^{p^{m-1}}) - p h(q^{p^m})}{(q)_\infty}\\
&= \frac{h(q) - p h(q^p)}{(q)_\infty}\\
&\quad + \frac{p}{\prod_{n\ne pk, k\ge 1}(1-q^n)} \cdot \frac{h(q^p) - 2h(q^{p^2})}{\prod_{n=1}^\infty (1-q^{pn})} + \cdots\\
&\quad + \frac{p^{m-1}}{\prod_{n\ne p^{m-1}k, k\ge 1}(1-q^n)} \cdot \frac{h(q^{p^{m-1}}) - p h(q^{p^m})}{\prod_{n=1}(1-q^{p^{m-1}n})}\\
&= M_p(q) + \frac{p}{\prod_{n\ne pk, k\ge 1}(1-q^n)} \cdot M_p(q^p) + \cdots\\
&\quad + \frac{p^{m-1}}{\prod_{n\ne p^{m-1}k, k\ge 1}(1-q^n)} \cdot M_p(q^{p^{m-1}}).
\end{aligned}$$

Since each summand of the right hand side of the above has positive coefficients of q^n for all positive integers n, M_{p^m} will have positive coefficients of q^n for all integers n. $\square$

Remark. By our method, it can be easily seen that if the conjecture is true for all primes $m = p$, then it is true for any other natural number m. For example, consider the case $m = 6$,

$$\begin{aligned}
&\frac{h(q)-6h(q^6)}{(q)_\infty}\\
&= \frac{h(q)-2h(q^2)}{(q)_\infty} + \frac{2h(q)-6h(q^6)}{(q)_\infty}\\
&= M_2(q) + \frac{2}{(1-q)(1-q^2)(1-q^3\cdots)}\\
&\quad \cdot \frac{h(q^2)-3h(q^6)}{(1-q^2)(1-q^4)(1-q^6)\cdots}\\
&= M_2(q) + \frac{2}{(1-q)(1-q^3)(1-q^5)\cdots} \cdot M_3(q^2).
\end{aligned}$$

We see that the positivity of coefficients of the power series $M_2(q)$ and $M_3(q)$ will imply the positivity of the coefficients of the power series of $M_6(q)$.

3. A stronger conjecture. Kim et al. proved the conjecture of Andrews et al. is asymptotically true by using circle method. However, Andrews et al. originally expected to find q-theoretic or combinatorial proofs for this conjecture. Here based on the numerical results, we make the following stronger conjecture. We also expected to find q-theoretic or combinatorial proofs for this conjecture. We can easily see that this conjecture implies the conjecture of Andrews et al.

Conjecture 3.1. Let $p_1 > p_2 \geq 2$ be two primes. Then the function

$$\frac{1}{(q)_\infty}(p_1 h(q^{p_1}) - p_2 h(q^{p_2}))$$

has positive power series coefficients of q^n for all $n \geq p_2$ and has nonnegative power series coefficients of q^n for all $n \geq 1$.

We verified this conjecture for the first 100,000 coefficients of the power series for each prime pair case which are less than 50 by using Mathematica. We provide some coefficients of the power series of $\frac{1}{(q)_\infty}(h(q) - mh(q^m))$ for small primes m, which are also obtained by using Mathematica.

$$\begin{aligned}
&\frac{1}{(q)_\infty}(h(q)-3h(q^3))\\
&= q + 2q^2 + 3q^4 + 3q^5 + 4q^6 + 5q^7 + 9q^8 + 10q^9\\
&\quad + 16q^{10} + 19q^{11} + 26q^{12} + 33q^{13} + 46q^{14}\\
&\quad + 56q^{15} + \cdots
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(q)_\infty}(h(q)-5h(q^5))\\
&= q + 2q^2 + 3q^3 + 6q^4 + 4q^5 + 11q^6 + 13q^7 + 21q^8\\
&\quad + 27q^9 + 36q^{10} + 46q^{11} + 67q^{12} + 82q^{13}\\
&\quad + 111q^{14} + 141q^{15} + \cdots
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(q)_\infty}(h(q)-7h(q^7))\\
&= q + 2q^2 + 3q^3 + 6q^4 + 9q^5\\
&\quad + 16q^6 + 16q^7 + 29q^8 + 38q^9 + 55q^{10}\\
&\quad + 71q^{11} + 103q^{12} + 130q^{13} + 174q^{14}\\
&\quad + 225q^{15} + \cdots
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(q)_\infty}(h(q)-11h(q^{11}))\\
&= q + 2q^2 + 3q^3 + 6q^4 + 9q^5\\
&\quad + 16q^6 + 23q^7 + 36q^8 + 52q^9 + 76q^{10} + 95q^{11}\\
&\quad + 141q^{12} + 185q^{13} + 253q^{14} + 331q^{15} + \cdots
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(q)_\infty}(h(q)-13h(q^{13}))\\
&= q + 2q^2 + 3q^3 + 6q^4 + 9q^5\\
&\quad + 16q^6 + 23q^7 + 36q^8 + 52q^9 + 76q^{10}\\
&\quad + 106q^{11} + 152q^{12} + 192q^{13} + 273q^{14}\\
&\quad + 360q^{15} + \cdots
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(q)_\infty}(h(q)-17h(q^{17}))\\
&= q + 2q^2 + 3q^3 + 6q^4 + 9q^5\\
&\quad + 16q^6 + 23q^7 + 36q^8 + 52q^9 + 76q^{10}\\
&\quad + 106q^{11} + 152q^{12} + 207q^{13} + 286q^{14}\\
&\quad + 386q^{15} + \cdots
\end{aligned}$$

Acknowledgements. The author would like to thank Profs. Peter Paule and George E. Andrews for their valuable comments on an earlier version of this paper and for their encouragements. The author was supported by the Austria Science Foundation (FWF) grant SFB 050-06 (Special Research Programm "Algorithmic and Enumerative Combinatorics").

References

[1] G. E. Andrews, S. H. Chan, B. Kim and R. Osburn, The first positive rank and crank

moments for overpartitions, Ann. Comb. **20** (2016), no. 2, 193–207.

[2] K. Bringmann and K. Mahlburg, Asymptotic inequalities for positive crank and rank moments, Trans. Amer. Math. Soc. **366** (2014), no. 2, 1073–1094.

[3] S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. **356** (2004), no. 4, 1623–1635 (electronic).

[4] B. Kim, E. Kim and J. Seo, On the number of even and odd strings along the overpartitions of n, Arch. Math. (Basel) **102** (2014), no. 4, 357–368.

[5] E. M. Wright, Asymptotic partition formulae: (II) weighted partitions, Proc. London Math. Soc. **S2-36** (1934), no. 1, 117–141.