# Semisimple symmetric spaces without compact manifolds locally modelled thereon

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**Abstract:** Let $G$ be a real reductive Lie group and $H$ a closed subgroup of $G$ which is reductive in $G$. In our earlier work it was shown that, if the homomorphism $i : H^\bullet(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}; \mathbf{C}) \to H^\bullet(\mathfrak{g}_{\mathbf{C}}, (\mathfrak{k}_H)_{\mathbf{C}}; \mathbf{C})$ is not injective, there does not exist a compact manifold locally modelled on $G/H$. In this paper, we give a classification of the semisimple symmetric spaces $G/H$ for which $i$ is not injective. We also study the case when $G$ cannot be realised as a linear group.

**Key words:** Local model; $(G, X)$-structure; Clifford–Klein form; symmetric space; relative Lie algebra cohomology; invariant polynomial.

**1. Introduction and preliminary results.** Let $G$ be a real reductive Lie group and $H$ a closed subgroup of $G$ which is reductive in $G$. We say that $G/H$ is a homogeneous space of reductive type. A manifold $M$ is said to be locally modelled on $G/H$, or said to admit a $(G, G/H)$-structure, if it is covered by open sets that are diffeomorphic to open sets of $G/H$ and the transition functions are locally given by translations by elements of $G$. We always assume that the transition functions verify the cocycle condition. This assumption is harmless: if $G/H$ is connected and $G$ acts transitively on $G/H$, it is automatically satisfied.

A typical example of a manifold locally modelled on $G/H$ is the following: if $\Gamma$ is a discrete subgroup of $G$ acting properly discontinuously and freely on $G/H$, the projection $\pi : G/H \to \Gamma\backslash G/H$ is a covering map and $\Gamma\backslash G/H$ is locally modelled on $G/H$. $\Gamma\backslash G/H$ is called a Clifford–Klein form of $G/H$ and $\Gamma$ is called a discontinuous group for $G/H$. Since Kobayashi [9] initiated a systematic study of Clifford–Klein forms of homogeneous spaces of reductive type in general setting, the determination of all $G/H$ admitting compact Clifford–Klein forms has been one of the central problems in the theory of discontinuous group. There are various obstructions for the existence of compact Clifford–Klein forms (e.g. [2], [11], [18], [21]), and also for the existence of compact manifolds locally modelled on $G/H$ ([3], [14], [17]). See [7], [12], [13], [15], [16] for surveys on this topic.

**Remark 1.1.** To the best knowledge of the author, it is not known if there is a homogeneous space of reductive type admitting a compact manifold locally modelled thereon, but not admitting a compact Clifford–Klein form. In the following two cases, any compact manifold locally modelled on $G/H$ is automatically a Clifford–Klein form.

- $G$ is an adjoint group and $H$ is a maximal compact subgroup $K_G$ of $G$ (by the Hopf–Rinow theorem).
- $G/H = O(n+1, 1)/O(n, 1), O(n, 2)/O(n, 1)$ [8].

Extending the idea of Kobayashi–Ono [14], we obtained the following obstruction:

**Theorem 1.2** ([19, Theorem 1.3])**.** *Let $G$ be a real reductive Lie group, $H$ a closed subgroup of $G$ which is reductive in $G$, and $K_H$ a maximal compact subgroup of $H$. We write $\mathfrak{g}$ for the Lie algebra of $G$ and $\mathfrak{g}_{\mathbf{C}}$ for its complexification. If the following condition* **(A)** *is satisfied, there does not exist a compact manifold locally modelled on $G/H$:*

**(A)** *The homomorphism of relative Lie algebra cohomology* $i : H^\bullet(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}; \mathbf{C}) \to H^\bullet(\mathfrak{g}_{\mathbf{C}}, (\mathfrak{k}_H)_{\mathbf{C}}; \mathbf{C})$ *is not injective.*

**Remark 1.3.** (1) **(A)** is rewritten as follows:

**(A′)** The homomorphism of relative Lie algebra cohomology $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbf{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbf{R})$ is not injective.

(2) Following [3] and [12, Notes 3.13] we gen-

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eralised [19, Theorem 1.3] to an arbitrary manifold locally modelled on a homogeneous space. But we do not know if this is an essential generalisation (see Remark 1.1).

(3) [19, Theorem 1.3] was stated in terms of cohomology of the associated compact homogeneous space $H^\bullet(G_U/H_U;\mathbf{C})$. We replaced it by relative Lie algebra cohomology since it is well-defined even if $G$ cannot be realised as a linear group (see Section 4).

We note that, given $G/H$, it is not so easy to verify directly whether **(A)** holds or not. Instead, we use the following proposition to find examples of $G/H$ that satisfies **(A)**.

**Proposition 1.4** ([19, Proposition 3.2]). *We keep the notation of Theorem 1.2. Denote by* $\mathrm{Pol}(\mathfrak{h}_\mathbf{C})^{\mathfrak{h}_\mathbf{C}}$ *the algebra of* $\mathfrak{h}_\mathbf{C}$*-invariant polynomials on* $\mathfrak{h}_\mathbf{C}$ *and define* $J_{(\mathfrak{g}_\mathbf{C},\mathfrak{h}_\mathbf{C})}$ *to be the ideal of* $\mathrm{Pol}(\mathfrak{h}_\mathbf{C})^{\mathfrak{h}_\mathbf{C}}$ *generated by*

$$\{P|_{\mathfrak{h}_\mathbf{C}} : P \in \mathrm{Pol}(\mathfrak{g}_\mathbf{C})^{\mathfrak{g}_\mathbf{C}}, \text{ the constant term of } P \text{ is zero}\}.$$

*Consider the following condition*:

**(B)** *There exists an element* $Q$ *of* $\mathrm{Pol}(\mathfrak{h}_\mathbf{C})^{\mathfrak{h}_\mathbf{C}}$ *such that* $Q \notin J_{(\mathfrak{g}_\mathbf{C},\mathfrak{h}_\mathbf{C})}$ *and* $Q|_{(\mathfrak{k}_H)_\mathbf{C}} = 0$.

*Then we have the implication* **(B)** $\Rightarrow$ **(A)**.

The proof of Proposition 1.4 is based on H. Cartan's algebraisation of Chern–Weil theory and its application to relative Lie algebra cohomology [6, §10].

In [19] we gave some examples of a homogeneous space that satisfies **(A)**. In this paper we give a classification of all semisimple symmetric spaces satisfying **(A)** (see Theorem 2.1). The details of the proof will appear elsewhere.

**2. Main result.** By a semisimple symmetric space we mean a homogeneous space $G/H$ such that $G$ is a real semisimple Lie group and $H$ is an open subgroup of a fixed point set $G^\sigma$ of some involution $\sigma$ on $G$. In this section we assume $G$ to be linear. If $\mathfrak{g}$ is simple or $(\mathfrak{g},\mathfrak{h})$ is isomorphic to $(\mathfrak{l}\oplus\mathfrak{l},\Delta\mathfrak{l})$ ($\mathfrak{l}$: simple Lie algebra), a semisimple symmetric space $G/H$ is called irreducible. At the Lie algebra level $(\mathfrak{g},\mathfrak{h})$, any semisimple symmetric space is uniquely decomposed into irreducible ones. The complete classification of irreducible symmetric spaces up to possibly outer automorpihsms of $\mathfrak{g}$ is given by Berger [4].

The main result of this paper is the following:

**Theorem 2.1.** *Let* $G/H$ *be a semisimple symmetric space. Then the above conditions* **(A)** *and* **(B)**, *and the following condition* **(C)** *are all equivalent*:

**(C)** $(\mathfrak{g},\mathfrak{h})$ *has an irreducible factor* $(\mathfrak{g}',\mathfrak{h}')$ *satisfying* $\operatorname{rank}\mathfrak{h}' > \operatorname{rank}\mathfrak{k}_{H'}$ *and isomorphic to none of the following*:
- $(\mathfrak{l}\oplus\mathfrak{l},\Delta\mathfrak{l})$ ($l$ : *simple Lie algebra*)
- $(\mathfrak{l}_\mathbf{C},\mathfrak{l})$ ($l$ : *simple Lie algebra*)
- $(\mathfrak{sl}(2n+1,\mathbf{C}),\mathfrak{so}(2n+1,\mathbf{C}))$ $(n \geqslant 1)$
- $(\mathfrak{sl}(2n,\mathbf{C}),\mathfrak{sp}(n,\mathbf{C}))$ $(n \geqslant 1)$
- $(\mathfrak{so}(2n,\mathbf{C}),\mathfrak{so}(2n-1,\mathbf{C}))$ $(n \geqslant 3)$
- $(\mathfrak{e}_{6,\mathbf{C}},\mathfrak{f}_{4,\mathbf{C}})$

**Remark 2.2.** Theorem 2.1 says, in particular, that **(A)** and **(B)** are equivalent for the semisimple symmetric spaces. We do not know whether it is true for any homogeneous space of reductive type. In particular, we do not have a direct proof of the equivalence for the semisimple symmetric spaces. Note that the implication **(B)** $\Rightarrow$ **(A)** holds for any homogeneous space of reductive type (Proposition 1.4).

Among Berger's classification of the irreducible symmetric spaces [4], we list all irreducible symmetric pairs $(\mathfrak{g},\mathfrak{h})$ satisfying **(C)**:

**type *A***

$\star$ $(\mathfrak{sl}(2n,\mathbf{C}),\mathfrak{so}(2n,\mathbf{C}))$ $(n \geqslant 1)$
$\circ$ $(\mathfrak{sl}(p+q,\mathbf{C}),\mathfrak{sl}(p,\mathbf{C})\oplus\mathfrak{sl}(q,\mathbf{C})\oplus\mathbf{C})$ $(p,q \geqslant 1)$
$\star\star$ $(\mathfrak{sl}(p+q,\mathbf{R}),\mathfrak{so}(p,q))$ ($p,q$: odd)
$\star$ $(\mathfrak{su}(p,q),\mathfrak{so}(p,q))$ ($p,q$: odd)
$\circ$ $(\mathfrak{su}(n,n),\mathfrak{sl}(n,\mathbf{C})\oplus\mathbf{R})$ $(n \geqslant 2)$
$\star\star$ $(\mathfrak{sl}(2n,\mathbf{R}),\mathfrak{sl}(n,\mathbf{C})\oplus\mathfrak{u}(1))$ $(n \geqslant 2)$
$\star$ $(\mathfrak{sl}(n,\mathbf{H}),\mathfrak{sl}(n,\mathbf{C})\oplus\mathfrak{u}(1))$ $(n \geqslant 2)$
$\circ$ $(\mathfrak{sl}(p+q,\mathbf{R}),\mathfrak{sl}(p,\mathbf{R})\oplus\mathfrak{sl}(q,\mathbf{R})\oplus\mathbf{R})$ $(p,q \geqslant 1)$
$\circ$ $(\mathfrak{sl}(p+q,\mathbf{H}),\mathfrak{sl}(p,\mathbf{H})\oplus\mathfrak{sl}(q,\mathbf{H})\oplus\mathbf{R})$ $(p,q \geqslant 1)$

**type *B* and type *D***

$\star$ $(\mathfrak{so}(p+q,\mathbf{C}),\mathfrak{so}(p,\mathbf{C})\oplus\mathfrak{so}(q,\mathbf{C}))$ ($p,q \geqslant 2$ or $p$: even, $q=1$)
$\circ$ $(\mathfrak{so}(2n,\mathbf{C}),\mathfrak{sl}(n,\mathbf{C})\oplus\mathbf{C})$ $(n \geqslant 2)$
$\star\star$ $(\mathfrak{so}(n,n),\mathfrak{so}(n,\mathbf{C}))$ $(n \geqslant 2)$
$\star$ $(\mathfrak{so}^*(2n),\mathfrak{so}(n,\mathbf{C}))$ $(n \geqslant 2)$
$\star\star$ $(\mathfrak{so}(p+r,q+s),\mathfrak{so}(p,q)\oplus\mathfrak{so}(r,s))$ ($p,q$: odd and $r \geqslant 1$)
$\circ$ $(\mathfrak{so}(n,n),\mathfrak{sl}(n,\mathbf{R})\oplus\mathbf{R})$ $(n \geqslant 2)$
$\circ$ $(\mathfrak{so}^*(4n),\mathfrak{sl}(n,\mathbf{H})\oplus\mathbf{R})$ $(n \geqslant 1)$

**type *C***

$\circ$ $(\mathfrak{sp}(n,\mathbf{C}),\mathfrak{sl}(n,\mathbf{C})\oplus\mathbf{C})$ $(n \geqslant 1)$
$\star$ $(\mathfrak{sp}(p+q,\mathbf{C}),\mathfrak{sp}(p,\mathbf{C})\oplus\mathfrak{sp}(q,\mathbf{C}))$ $(p,q \geqslant 1)$
$\circ$ $(\mathfrak{sp}(2n,\mathbf{R}),\mathfrak{sp}(n,\mathbf{C}))$ $(n \geqslant 1)$
$\circ$ $(\mathfrak{sp}(n,n),\mathfrak{sp}(n,\mathbf{C}))$ $(n \geqslant 1)$

∘ $(\mathfrak{sp}(n,\mathbf{R}), \mathfrak{sl}(n,\mathbf{R}) \oplus \mathbf{R})$ $(n \geqslant 1)$
∘ $(\mathfrak{sp}(n,n), \mathfrak{sl}(n,\mathbf{H}) \oplus \mathbf{R})$ $(n \geqslant 1)$

**type $E_6$**

⋆ $(\mathfrak{e}_{6,\mathbf{C}}, \mathfrak{sp}(4,\mathbf{C}))$
⋆ $(\mathfrak{e}_{6,\mathbf{C}}, \mathfrak{sl}(6,\mathbf{C}) \oplus \mathfrak{sl}(2,\mathbf{C}))$
∘ $(\mathfrak{e}_{6,\mathbf{C}}, \mathfrak{so}(10,\mathbf{C}) \oplus \mathbf{C})$
⋆ $(\mathfrak{e}_{6(6)}, \mathfrak{sl}(6,\mathbf{R}) \oplus \mathfrak{sl}(2,\mathbf{R}))$
⋆⋆ $(\mathfrak{e}_{6(6)}, \mathfrak{sl}(3,\mathbf{H}) \oplus \mathfrak{su}(2))$
⋆ $(\mathfrak{e}_{6(-26)}, \mathfrak{sl}(3,\mathbf{H}) \oplus \mathfrak{su}(2))$
∘ $(\mathfrak{e}_{6(6)}, \mathfrak{so}(5,5) \oplus \mathbf{R})$
∘ $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1) \oplus \mathbf{R})$

**type $E_7$**

⋆ $(\mathfrak{e}_{7,\mathbf{C}}, \mathfrak{sl}(8,\mathbf{C}))$
⋆ $(\mathfrak{e}_{7,\mathbf{C}}, \mathfrak{so}(12,\mathbf{C}) \oplus \mathfrak{sl}(2,\mathbf{C}))$
∘ $(\mathfrak{e}_{7,\mathbf{C}}, \mathfrak{e}_{6,\mathbf{C}} \oplus \mathbf{C})$
∘ $(\mathfrak{e}_{7(7)}, \mathfrak{sl}(8,\mathbf{R}))$
∘ $(\mathfrak{e}_{7(7)}, \mathfrak{sl}(4,\mathbf{H}))$
∘ $(\mathfrak{e}_{7(-25)}, \mathfrak{sl}(4,\mathbf{H}))$
∘ $(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbf{R})$
∘ $(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbf{R})$

**type $E_8$**

⋆ $(\mathfrak{e}_{8,\mathbf{C}}, \mathfrak{so}(16,\mathbf{C}))$
⋆ $(\mathfrak{e}_{8,\mathbf{C}}, \mathfrak{e}_{7,\mathbf{C}} \oplus \mathfrak{sl}(2,\mathbf{C}))$

**type $F_4$**

⋆ $(\mathfrak{f}_{4,\mathbf{C}}, \mathfrak{sp}(3,\mathbf{C}) \oplus \mathfrak{sl}(2,\mathbf{C}))$
⋆ $(\mathfrak{f}_{4,\mathbf{C}}, \mathfrak{so}(9,\mathbf{C}))$

**type $G_2$**

⋆ $(\mathfrak{g}_{2,\mathbf{C}}, \mathfrak{sl}(2,\mathbf{C}) \oplus \mathfrak{sl}(2,\mathbf{C}))$.

Here, the signs ⋆⋆, ⋆ and ∘ signify

⋆⋆: The nonexistence of compact Clifford–Klein forms of $G/H$ is seems to be new.
⋆: The nonexistence of compact Clifford–Klein forms of $G/H$ was known earlier by [19], but not for locally modelled case (see Remark 1.1).
∘: The nonexistence of compact manifolds locally modelled on $G/H$ was known earlier by [19].

Note that we saw the nonexistence of compact Clifford–Klein forms of ⋆⋆ in [19, Corollary 1.4] except $(\mathfrak{e}_{6(6)}, \mathfrak{sl}(3,\mathbf{H}) \oplus \mathfrak{su}(2))$.

**Remark 2.3.** There are many examples of an irreducible symmetric space that does not admit compact Clifford–Klein forms but does not satisfy **(A)**–**(C)** (see [2], [9], [11], [20]). For example, $SU(p,q)/SO(p,q)$ $(p,q \geqslant 1)$ does not admit a Clifford–Klein form by [9, Corollary 4.4].

**3. Outline of proof.** $(\mathfrak{g},\mathfrak{h})$ satisfies **(A)**, **(B)** or **(C)** if and only if $(\mathfrak{g},\mathfrak{h})$ has an irreducible factor satisfying **(A)**, **(B)** or **(C)**, respectively. Hence we may assume that a symmetric space $G/H$ is irreducible. By Proposition 1.4, it is enough to prove **(A)** ⇒ **(C)** and **(C)** ⇒ **(B)**.

<u>**(A)** ⇒ **(C)**</u>: If $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{l} \oplus \mathfrak{l}, \Delta\mathfrak{l})$, **(A)** is not satisfied since a group manifold $(L \times L)/\Delta L$ admits a compact Clifford–Klein form [9, Example 4.8]. If $\operatorname{rank} H = \operatorname{rank} K_H$ or $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{l}_{\mathbf{C}}, \mathfrak{l})$, **(A)** is not satisfied by [19, Proposition 6.2]. Thus suppose $(\mathfrak{g},\mathfrak{h})$ is

- $(\mathfrak{sl}(2n+1,\mathbf{C}), \mathfrak{so}(2n+1,\mathbf{C}))$ $(n \geqslant 1)$,
- $(\mathfrak{sl}(2n,\mathbf{C}), \mathfrak{sp}(n,\mathbf{C}))$ $(n \geqslant 1)$,
- $(\mathfrak{so}(2n,\mathbf{C}), \mathfrak{so}(2n-1,\mathbf{C}))$ $(n \geqslant 3)$, or
- $(\mathfrak{e}_{6,\mathbf{C}}, \mathfrak{f}_{4,\mathbf{C}})$.

In these cases, **(A)** is not satisfied by the following proposition:

**Proposition 3.1.** *Let $G/H$ be a semisimple symmetric space. If the restriction map $(S(\mathfrak{g}_{\mathbf{C}})^*)^{\mathfrak{g}_{\mathbf{C}}} \to (S(\mathfrak{h}_{\mathbf{C}})^*)^{\mathfrak{h}_{\mathbf{C}}}$ is surjective (or equivalently, $(S(\mathfrak{g})^*)^{\mathfrak{g}} \to (S(\mathfrak{h})^*)^{\mathfrak{h}}$ is surjective), then $G/H$ does not satisfy* **(A)**.

Proposition 3.1 follows from [6, §10] as Proposition 1.4.

<u>**(C)** ⇒ **(B)**</u>: We give some sufficient condition for **(B)** that do not depend on the embedding of $H$ into $G$, but on $G$ and $H$ themselves only:

**Proposition 3.2.** *Let $G$ be a real semisimple Lie group and $H$ a closed subgroup which is reductive in $G$. If the identity component of the centre of $H$ is noncompact, $G/H$ satisfies* **(B)**.

**Proposition 3.3.** *Let $G$ be a complex reductive Lie group and $H$ a closed complex subgroup which is reductive in $G$. Decompose $\mathfrak{g}$ and $\mathfrak{h}$ into direct sums of their ideals*:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathbf{C}^p, \quad \mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_l \oplus \mathbf{C}^q,$$

*where $\mathfrak{g}_1, \ldots, \mathfrak{g}_k, \mathfrak{h}_1, \ldots, \mathfrak{h}_l$ are complex simple Lie algebras. If one of the following conditions holds, $G/H$ satisfies* **(B)**:

(1) $p < q$.
(2) $k < l$.
(3) $\min_{1 \leqslant i \leqslant k} d_2(\mathfrak{g}_i) > \min_{1 \leqslant j \leqslant l} d_2(\mathfrak{h}_j)$.

*Here, $d_2(\mathfrak{g}_i)$ is the second smallest number in the degrees of generators of $(S(\mathfrak{g}_i)^*)^{\mathfrak{g}_i}$*:

$$d_2(\mathfrak{g}_i) = \begin{cases} 3 & \text{if } \mathfrak{g}_i \simeq \mathfrak{sl}(n,\mathbf{C}) \ (n \geqslant 3), \\ 4 & \text{if } \mathfrak{g}_i \simeq \mathfrak{sp}(n,\mathbf{C}) \ (n \geqslant 2) \text{ or} \\ & \quad \mathfrak{so}(n,\mathbf{C}) \ (n \geqslant 7), \\ 5 & \text{if } \mathfrak{g}_i \simeq \mathfrak{e}_{6,\mathbf{C}}, \\ 6 & \text{if } \mathfrak{g}_i \simeq \mathfrak{e}_{7,\mathbf{C}}, \mathfrak{f}_{4,\mathbf{C}}, \mathfrak{g}_{2,\mathbf{C}}, \\ 8 & \text{if } \mathfrak{g}_i \simeq \mathfrak{e}_{8,\mathbf{C}}, \\ \infty & \text{if } \mathfrak{g}_i \simeq \mathfrak{sl}(2,\mathbf{C}). \end{cases}$$

**Proposition 3.4.** *Let $G$ be a real reductive Lie group and $H$ a closed subgroup of $G$ which is reductive in $G$. Assume the following two conditions*:

- $\mathfrak{h}$ *has a factor of* $\mathfrak{sl}(l,\mathbf{R})$ $(l \geqslant 3)$, $\mathfrak{sl}(l,\mathbf{C})$ $(l \geqslant 3)$, *or* $\mathfrak{sl}(l,\mathbf{H})$ $(l \geqslant 2)$.
- $\mathfrak{g}_{\mathbf{C}}$ *does not have a factor of* $\mathfrak{sl}(k,\mathbf{C})$ *for any* $k \geqslant 3$.

*Then $G/H$ satisfies* **(B)**.

**Proposition 3.5** ([19, Corollary 6.1])**.** *Let $G$ be a real reductive Lie group and $H$ a closed subgroup of $G$ which is reductive in $G$. Denote by $K_G$, $K_H$ the maximal compact groups of $G$, $H$, respectively. If* $\operatorname{rank} G = \operatorname{rank} H$ *and* $\operatorname{rank} K_G > \operatorname{rank} K_H$, *then $G/H$ satisfies* **(B)**.

**Remark 3.6.** Proposition 3.2 is a generalisation of [10, Corollary 4] and similar to [3, Corollaire 1, Corollaire 3]. Proposition 3.5 is a slight generalisation of [14, Corollary 5] and [9, Proposition 4.10].

By using Propositions 3.2, 3.3, 3.4 and 3.5, we can prove **(C)** $\Rightarrow$ **(B)** for most cases, including all exceptional ones. It remains to check that **(B)** is satisfied when $(\mathfrak{g},\mathfrak{h})$ is one of the following

- $(\mathfrak{sl}(2n,\mathbf{C}),\mathfrak{so}(2n,\mathbf{C}))$ $(n \geqslant 3)$
- $(\mathfrak{sl}(p+q,\mathbf{R}),\mathfrak{so}(p,q))$ $(p,q$: odd$)$
- $(\mathfrak{su}(p,q),\mathfrak{so}(p,q))$ $(p,q$: odd$)$
- $(\mathfrak{sl}(2n,\mathbf{R}),\mathfrak{sl}(n,\mathbf{C})\oplus\mathfrak{u}(1))$ $(n \geqslant 2)$
- $(\mathfrak{sl}(n,\mathbf{H}),\mathfrak{sl}(n,\mathbf{C})\oplus\mathfrak{u}(1))$ $(n \geqslant 2)$
- $(\mathfrak{so}(2n+1,\mathbf{C}),\mathfrak{so}(2n,\mathbf{C}))$ $(n \geqslant 3)$
- $(\mathfrak{so}(2n+1,2n+1),\mathfrak{so}(2n+1,\mathbf{C}))$ $(n \geqslant 1)$
- $(\mathfrak{so}^*(4n+2),\mathfrak{so}(2n+1,\mathbf{C}))$ $(n \geqslant 1)$
- $(\mathfrak{so}(p+r,q+s),\mathfrak{so}(p,q)\oplus\mathfrak{so}(r,s))$ $(p,q$: odd, $r+s$: even and $r \geqslant 1)$.

A direct calculation shows that these indeed satisfy **(B)**.

**4. Nonlinear case.** By A. Borel's theorem [5], for any linear real semisimple Lie group $G$, a Riemannian symmetric space $G/K_G$ of noncompact type admits a compact Clifford–Klein form. Let $\pi : \tilde{G} \to G$ be a covering map and put $\widetilde{K_G} = \pi^{-1}(K_G)$. If $\tilde{G}$ is not linear, the proof of [5] does not work for $\tilde{G}/\widetilde{K_G}$ because of the following two reasons:

- We cannot use Selberg's lemma to control the freeness of the action.
- If $\pi$ is an infinite covering map, $\widetilde{K_G}$ is noncompact. Hence a discrete subgroup of $\tilde{G}$ may not act properly discontinuously on $\tilde{G}/\widetilde{K_G}$.

The following result shows that the compactness of $K_G$ is crucial:

**Corollary 4.1.** *Let $G/K_G$ be a Hermitian symmetric space of noncompact type and $\pi : \tilde{G} \to G$ be a universal covering map. Put $\widetilde{K_G} = \pi^{-1}(K_G)$. Then there does not exist a compact manifold locally modelled on $\tilde{G}/\widetilde{K_G}$.*

*In particular, there does not exist a discrete subgroup of $\tilde{G}$ acting properly discontinuously, freely, and cocompactly on $\tilde{G}/\widetilde{K_G}$.*

Corollary 4.1 follows from Proposition 3.2.

**Remark 4.2.** [1] applied A. Borel's theorem to construct the discrete series representations of a semisimple Lie group with finite centre. In its erratum nonlinear case is discussed. Unfortunately, our method gives no information for finite centre case.

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