On the growth rate of ideal Coxeter groups in hyperbolic 3-space

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Abstract: We study the set $\mathcal{G}$ of growth rates of ideal Coxeter groups in hyperbolic 3-space; this set consists of real algebraic integers greater than 1. We show that (1) $\mathcal{G}$ is unbounded above while it has the minimum, (2) any element of $\mathcal{G}$ is a Perron number, and (3) growth rates of ideal Coxeter groups with n generators are located in the closed interval $[n-3, n-1]$.

Key words: Coxeter group; growth function; growth rate; Perron number.

1. Introduction. Let P be a *hyperbolic Coxeter polytope* which is a polytope in hyperbolic space whose dihedral angles are submultiples of π. The set S of reflections with respects to facets of P generates a discrete group Γ which has P as a fundamental domain. We call (Γ, S) the *Coxeter system* associated to P. For $k \in \mathbf{N}$, let a_k be the number of elements of Γ whose word length with respects to S is equal to k. Then (Γ, S) has the exponential growth rate $\tau = \limsup_{k\to\infty} \sqrt[k]{a_k}$ which is a real algebraic integer bigger than 1 ([5]). Recently arithmetic properties of the growth rate of hyperbolic Coxeter groups have attracted considerable attention; for two and three-dimensional cocompact hyperbolic Coxeter groups, Cannon–Wagreich and Parry showed that their growth rates are Salem numbers ([2,12]), where a real algebraic integer $\tau > 1$ is called a *Salem number* if τ^{-1} is an algebraic conjugate of τ and all algebraic conjugates of τ other than τ and τ^{-1} lie on the unit circle. Floyd also proved that the growth rates of two-dimensional cofinite hyperbolic Coxeter groups are Pisot–Vijayaraghavan numbers, where a real algebraic integer $\tau > 1$ is called a *Pisot–Vijayaraghavan number* if all algebraic conjugates of τ other than τ lie in the open unit disk ([3]). Kellerhals and Perren conjectured that the growth rates of hyperbolic Coxeter groups are Perron numbers in general, where a real algebraic integer $\tau > 1$ is called a *Perron number* if all algebraic conjugates of τ other than τ have moduli less than the modulus of τ ([9]). Komori and Umemoto proved their conjecture for three-dimensional cofinite hyperbolic Coxeter simplex groups ([10]). In this paper we consider the growth rate of ideal Coxeter groups in hyperbolic 3-space; a Coxeter polytope P is called *ideal* if all vertices of P are located on the ideal boundary of hyperbolic space. Related to Jakob Steiner's problem on the combinatorial characterization of polytopes inscribed in the two-sphere S^2, ideal polytopes in hyperbolic 3-space has been studied extensively ([4,13]). We consider the distribution of growth rates of three-dimensional hyperbolic ideal Coxeter groups; the set $\mathcal{G}$ of growth rates will be shown to be unbounded above while it has the minimum which is attained by a unique Coxeter group. Kellerhals studied the same problem for two and three-dimensional cofinite hyperbolic Coxeter groups, and Kellerhals and Kolpakov for two and three-dimensional cocompact hyperbolic Coxeter groups ([7,8]). We will also prove that any element of $\mathcal{G}$ is a Perron number, which supports the conjecture of Kellerhals and Perren for three-dimensional hyperbolic ideal Coxeter groups. Moreover we will show that any ideal Coxeter group Γ with n generators has its growth rate τ in the closed interval $[n-3, n-1]$, and Γ is right-angled if and only if $\tau = n-3$. We should remark that Nonaka also detected the minimum growth rate of ideal Coxeter groups, and showed all growth rates to be Perron numbers ([11]). Since we used a criterion for growth rates to be Perron numbers (Proposition 1) and a result of Serre (in the proof of Proposition 3), our arguments are shorter that those of Nonaka.

2. Preliminaries. The upper half space $\mathbf{H}^3 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3 > 0\}$ with the metric $|dx|/x_3$ is a model of hyperbolic 3-space, so called

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the *upper half space model*. The Euclidean plane $\mathbf{E}^2 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_3 = 0\}$ and the point at infinity ∞ compose the boundary at infinity $\partial\mathbf{H}^3$ of $\mathbf{H}^3$. A subset $B \subset \mathbf{H}^3$ is called a *hyperplane* of $\mathbf{H}^3$ if it is a Euclidean hemisphere or a half plane orthogonal to $\mathbf{E}^2$. When we restrict the hyperbolic metric $|dx|/x_3$ of $\mathbf{H}^3$ to B, it becomes a model of hyperbolic plane. We define a *polytope* as a closed domain P of $\mathbf{H}^3$ which can be written as the intersection of finitely many closed half spaces H_B bounded by hyperplanes B, say $P = \bigcap H_B$. In this presentation of P, $F_B = P \cap B$ is a hyperbolic polygon of B. F_B is called a *facet* of P, and B is called the *supporting hyperplane* of F_B. If the intersection of two facets F_{B_1} and F_{B_2} of P consists of a geodesic segment, it is called an *edge* of P; the intersection $\bigcap F_B$ of more than two facets is a point, then it is called a *vertex* of P. If F_{B_1} and F_{B_2} intersect only at a point of the boundary $\partial\mathbf{H}^3$ of $\mathbf{H}^3$, it is called an *ideal vertex* of P. A polytope P is called *ideal* if all of its vertices are ideal.

A *horosphere* Σ of $\mathbf{H}^3$ based at $v \in \partial\mathbf{H}^3$ is defined by a Euclidean sphere in $\mathbf{H}^3$ tangent to $\mathbf{E}^2$ at v when $v \in \mathbf{E}^2$, or a Euclidean plane in $\mathbf{H}^3$ parallel to $\mathbf{E}^2$ when $v = \infty$. When we restrict the hyperbolic metric of $\mathbf{H}^3$ to Σ, it becomes a model of Euclidean plane. Let $v \in \partial\mathbf{H}^3$ be an ideal vertex of a polytope P in $\mathbf{H}^3$ and Σ be a horosphere of $\mathbf{H}^3$ based at v such that Σ meets just the facets of P incident to v. Then the *vertex link* $L(v) := P \cap \Sigma$ of v in P is a Euclidean convex polygon in the horosphere Σ. If F_{B_1} and F_{B_2} are adjacent facets of P incident to v, then the Euclidean dihedral angle between $F_{B_1} \cap \Sigma$ and $F_{B_2} \cap \Sigma$ in Σ is equal to the hyperbolic dihedral angle between the supporting hyperplanes B_1 and B_2 in $\mathbf{H}^3$ (cf. [13, Theorem 6.4.5]).

An ideal polytope P is called *Coxeter* if the dihedral angles of edges of P are submultiples of π. Since any Euclidean Coxeter polygon is a rectangle or a triangle with dihedral angles $(\pi/2, \pi/3, \pi/6)$, $(\pi/2, \pi/4, \pi/4)$ or $(\pi/3, \pi/3, \pi/3)$, we see that the dihedral angles of an ideal Coxeter polytope must be $\pi/2$, $\pi/3$, $\pi/4$ or $\pi/6$.

Any Coxeter polytope P is a fundamental domain of the discrete group Γ generated by the set S consisting of the reflections with respects to its facets. We call (Γ, S) the *Coxeter system* associated to P. In this situation we can define the *word length* $\ell_S(x)$ *of* $x \in \Gamma$ *with respect to* S by the smallest integer $k \geq 0$ for which there exist $s_1, s_2, \cdots, s_k \in S$ such that $x = s_1 s_2 \cdots s_k$. The *growth function* $f_S(t)$ of (Γ, S) is the formal power series $\sum_{k=0}^{\infty} a_k t^k$ where a_k is the number of elements $g \in \Gamma$ satisfying $\ell_S(g) = k$. It is known that the *growth rate* of (Γ, S), $\tau = \limsup_{k\to\infty} \sqrt[k]{a_k}$ is bigger than 1 ([5]) and less than or equal to the cardinality $|S|$ of S from the definition. By means of Cauchy-Hadamard formula, the radius of convergence R of $f_S(t)$ is the reciprocal of τ, i.e. $1/|S| \leq R < 1$. In practice the growth function $f_S(t)$ which is analytic on $|t| < R$ extends to a rational function $P(t)/Q(t)$ on $\mathbf{C}$ by analytic continuation where $P(t), Q(t) \in \mathbf{Z}[t]$ are relatively prime. There are formulas due to Solomon and Steinberg to calculate the rational function $P(t)/Q(t)$ from the data of finite Coxeter subgroups of (Γ, S) ([15,16]. See also [6]).

Theorem 1 (Solomon's formula)**.** *The growth function* $f_S(t)$ *of an irreducible finite Coxeter group* (Γ, S) *can be written as* $f_S(t) = [m_1 + 1, m_2 + 1, \cdots, m_k + 1]$ *where* $[n] = 1 + t + \cdots + t^{n-1}, [m, n] = [m][n]$, *etc. and* $\{m_1, m_2, \cdots, m_k\}$ *is the set of exponents of* (Γ, S).

Theorem 2 (Steinberg's formula)**.** *Let* (Γ, S) *be a hyperbolic Coxeter group. Let us denote the Coxeter subgroup of* (Γ, S) *generated by the subset* $T \subseteq S$ *by* (Γ_T, T), *and denote its growth function by* $f_T(t)$. *Set* $\mathcal{F} = \{T \subseteq S : \Gamma_T$ *is finite*$\}$. *Then*

$$\frac{1}{f_S(t^{-1})} = \sum_{T\in\mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

In this case, $t = R$ is a pole of $f_S(t) = P(t)/Q(t)$. Hence R is a real zero of the denominator $Q(t)$ closest to the origin $0 \in \mathbf{C}$ of all zeros of $Q(t)$. Solomon's formula implies that $P(0) = 1$. Hence $a_0 = 1$ means that $Q(0) = 1$. Therefore $\tau > 1$, the reciprocal of R, becomes a real algebraic integer whose conjugates have moduli less than or equal to the modulus of τ. If $t = R$ is the unique zero of $Q(t)$ with the smallest modulus, then $\tau > 1$ is a real algebraic integer whose conjugates have moduli less than the modulus of τ: such a real algebraic integer is called a *Perron number*.

The following result is a criterion for growth rates to be Perron numbers.

Proposition 1 ([10], Lemma 1)**.** *Consider the following polynomial of degree* $n \geq 2$

$$g(t) = \sum_{k=1}^{n} a_k t^k - 1,$$

where a_k *is a non-negative integer. We also assume*

Table I

(p,q,r,s)	Denominator polynomial
$(2,2,0,2)$	$(t-1)(3t^5+t^4+t^3+t^2+t-1)$
$(2,0,4,0)$	$(t-1)(3t^3+t^2+t-1)$
$(0,6,0,0)$	$(t-1)(3t^2+t-1)$
$(4,2,0,2)$	$(t-1)(4t^5+t^4+2t^3+t^2+2t-1)$
$(4,0,4,0)$	$(t-1)(4t^3+t^2+2t-1)$
$(2,5,0,2)$	$(t-1)(5t^5+2t^4+t^3+3t^2+2t-1)$

that the greatest common divisor of $\{k \in \mathbf{N} \mid a_k \neq 0\}$ *is* 1*. Then there is a real number* r_0*,* $0 < r_0 < 1$ *which is the unique zero of* $g(t)$ *having the smallest absolute value of all zeros of* $g(t)$*.*

3. Ideal Coxeter polytopes with 4 or 5 facets in $\mathbf{H}^3$. Let p, q, r and s be the number of edges with dihedral angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$ of an ideal Coxeter polytope P in $\mathbf{H}^3$. By Andreev theorem [1], we can classify ideal Coxeter polytopes with 4 or 5 facets, and calculate the growth functions $f_S(t)$ of P by means of Steinberg's formula and also growth rates, see Table I. Every denominator polynomial has a form $(t-1)H(t)$ and all coefficients of $H(t)$ satisfy the condition of Proposition 1, so that the growth rates of ideal Coxeter polytopes with 4 or 5 facets are Perron numbers.

As an application of the data of Table I, we have the following result.

Proposition 2. *The set* $\mathcal{G}$ *of growth rates of three-dimensional hyperbolic ideal Coxeter polytopes is unbounded above.*

Proof. After glueing m copies of the ideal Coxeter pyramid with $p=r=4$ along their sides successively, we can construct a hyperbolic ideal Coxeter polytope P_n with $n=m+4$ facets. In Fig. 1 we are looking at the ideal Coxeter polytope P_8 with 8 facets from the point at infinity ∞, which consists of 4 copies of ideal Coxeter pyramid with $p=r=4$ whose apexes are located at ∞; squares represent bases of pyramids and disks are supporting hyperplanes of these bases. The growth function of P_n has the following denominator polynomial

$$(t-1)H(t) = (t-1)(2(n-3)t^3 + (n-4)t^2 + (n-3)t - 1),$$

from which we see that the growth rate of P_n diverges when n goes to infinity. □

We should remark that all coefficients of $H(t)$ except its constant term are non-negative. There-

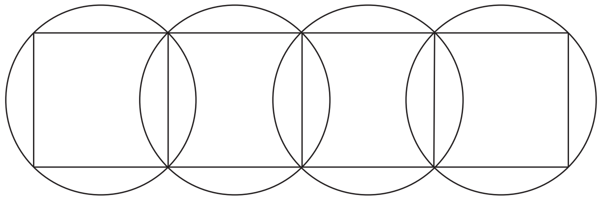

Fig. 1

fore we can apply Proposition 1 to conclude that the growth rate of P_n is a Perron number. Moreover $H(t)$ has a unique zero on the unit interval $[0,1]$ and the following inequalities hold:

$$H\left(\frac{1}{n-3}\right) = \frac{n-2}{(n-3)^2} > 0,$$
$$H\left(\frac{1}{n-1}\right) = \frac{-n^2+n-4}{(n-1)^3} < 0.$$

They imply that the growth rate of P_n satisfies

$$n-3 \leqq \tau \leqq n-1,$$

which will be generalized in the next section.

4. The growth rates of ideal Coxeter polytopes in $\mathbf{H}^3$. Recall that p, q, r and s be the number of edges with dihedral angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$ of an ideal Coxeter polytope P in $\mathbf{H}^3$. By means of Steinberg's formula, we can calculate the growth function $f_S(t)$ of P as

$$1/f_S(1/t) = 1 - n/[2] + p/[2,2] + q/[2,3] + r/[2,4] + s/[2,6],$$

where $[2,3]=[2][3]$, etc. It can be rewritten as

$$1/f_S(t) = 1 - nt/[2] + pt^2/[2,2] + qt^3/[2,3] + rt^4/[2,4] + st^6/[2,6] = \frac{1}{[2,2,3,4,6]}G(t),$$

where

$$G(t) = [2,2,3,4,6] - nt[2,3,4,6] + pt^2[3,4,6] + qt^3[2,4,6] + rt^4[2,3,6] + st^6[2,3,4].$$

Proposition 3. *Put* $a=p/2$, $b=q/3$, $c=r/4$, $d=s/6$*. Then*

$$a+b+c+d = n-2. \tag{1}$$

Proof. By a result of Serre ([14]. See also [6])

$$G(1) = [2,3,4,6](1)(2-n+p/2+q/3+r/4+s/6) = 0$$

□

By using this equality (1) we represent $H(t) = G(t)/(t-1)$ as

$$\begin{aligned} H(t) &= -[2,3,4,6] + at[3,4,6] + bt(2t+1)[2,4,6] \\ &\quad + ct(3t^2+2t+1)[2,3,6] \\ &\quad + dt(5t^4+4t^3+3t^2+2t+1)[2,3,4] \\ &= -1 + (-4+a+b+c+d)t \\ &\quad + (-9+3a+5b+5c+5d)t^2 \\ &\quad + (-15+6a+11b+14c+14d)t^3 \\ &\quad + (-20+9a+17b+25c+29d)t^4 \\ &\quad + (-23+11a+22b+33c+49d)t^5 \\ &\quad + (-23+12a+24b+36c+66d)t^6 \\ &\quad + (-20+11a+23b+35c+71d)t^7 \\ &\quad + (-15+9a+19b+31c+61d)t^8 \\ &\quad + (-9+6a+13b+22c+40d)t^9 \\ &\quad + (-4+3a+7b+11c+19d)t^{10} \\ &\quad + (-1+a+2b+3c+5d)t^{11}. \end{aligned}$$

From this formula we have the following result (see also [11], Theorem 3).

Theorem 3. *The growth rates of ideal Coxeter polytopes in* $\mathbf{H}^3$ *are Perron numbers.*

Proof. When n the number of facets satisfies $n \geqq 6$, the equality (1) of Proposition 3 implies $a+b+c+d = n-2 \geqq 4$. Then all coefficients of $H(t)$ except its constant term are non-negative. Hence Proposition 1 implies the assertion. For $n = 4, 5$, this claim was already proved in the previous section. □

Moreover the equality (1) induces the following two functions $H_1(t)$ and $H_2(t)$ satisfying $H_1(t) \leqq H(t) \leqq H_2(t)$ for any $t > 0$:

$$\begin{aligned} H_1(t) &= -1 + (-4+(n-2))t + (-9+3(n-2))t^2 \\ &\quad + (-15+6(n-2))t^3 + (-20+9(n-2))t^4 \\ &\quad + (-23+11(n-2))t^5 + (-23+12(n-2))t^6 \\ &\quad + (-20+11(n-2))t^7 + (-15+9(n-2))t^8 \\ &\quad + (-9+6(n-2))t^9 + (-4+3(n-2))t^{10} \\ &\quad + (-1+(n-2))t^{11} = (1+t)^2(-1-3t+nt) \\ &\qquad (1+t^2)(1-t+t^2)(1+t+t^2)^2, \\ H_2(t) &= -1 + (-4+(n-2))t + (-9+5(n-2))t^2 \\ &\quad + (-15+14(n-2))t^3 + (-20+29(n-2))t^4 \\ &\quad + (-23+49(n-2))t^5 + (-23+66(n-2))t^6 \\ &\quad + (-20+71(n-2))t^7 + (-15+61(n-2))t^8 \\ &\quad + (-9+40(n-2))t^9 + (-4+19(n-2))t^{10} \\ &\quad + (-1+5(n-2))t^{11} \\ &= (1+t)^2(1+t^2)(1+t+t^2)(-1-3t+nt-5t^2 \\ &\quad + 2nt^2 - 7t^3 + 3nt^3 - 9t^4 + 4nt^4 - 11t^5 + 5nt^5). \end{aligned}$$

Now we assume that $n \geqq 6$. Then all coefficients of $H_1(t)$ and $H_2(t)$ except their constant terms are non-negative so that each of them has a unique zero in $(0, \infty)$. The following inequalities

$$H_1\left(\frac{1}{n-3}\right) = 0, \quad H_2\left(\frac{1}{n-1}\right) = -\frac{6}{(n-1)^5} < 0$$

guarantee that the zero of $H(t)$ is located in $[\frac{1}{n-1}, \frac{1}{n-3}]$. Combining with the similar result for $n = 4, 5$ in the previous section, we have the following theorem which is our main result.

Theorem 4. *The growth rate* τ *of an ideal Coxeter polytope with* n *facets in* $\mathbf{H}^3$ *satisfies*

$$n-3 \leqq \tau \leqq n-1. \tag{2}$$

Corollary 1. *An ideal Coxeter polytope* P *with* n *facets in* $\mathbf{H}^3$ *is right-angled if and only if its growth rate* τ *is equal to* $n-3$.

Proof. The factor $H(t)$ of the denominator polynomial $G(t) = (t-1)H(t)$ of the growth function of P is equal to $H_1(t)$ if and only if $b = c = d = 0$, which means that all dihedral angles are $\pi/2$. □

From the inequality (2), we see that the growth rate τ of an ideal Coxeter polytope with n facets with $n \geqq 6$ satisfies $\tau \geqq 3$. Therefore combining with the result of growth rates for $n = 4, 5$ shown in the previous section, we also have the following corollary (see also [11], Theorem 4).

Corollary 2. *The minimum of the growth rates of three-dimensional hyperbolic ideal Coxeter polytopes is* $0.492432^{-1} = 2.03074$*, which is uniquely realized by the ideal Coxeter simplex with* $p = q = s = 2$.

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