On dependence of meromorphic functions sharing some finite sets IM

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Abstract: In connection with Nevanlinna's five-value theorem ([2]), the author showed in [3] that two meromorphic functions sharing five one-point or two-point sets IM are Möbius transforms of each other. Now, we consider $n+1$ meromorphic functions sharing some finite sets IM.

Key words: Uniqueness theorem; sharing sets; Nevanlinna theory.

1. Introduction. For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\overline{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S IM (ignoring multiplicities) if $f^{-1}(S) = g^{-1}(S)$. In particular if S is a one-point set $\{a\}$ IM, then we say also that f and g share a IM.

In [2], R. Nevanlinna showed the following theorem:

Theorem A. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing distinct five points in $\overline{\boldsymbol{C}}$ IM, then $f = g$.*

Let n, q be two positive integer such that $q > n + 1 + 2/n$. We can easily see, by the same method as the proof of Theorem A, that if $n+1$ meromorphic functions on $\boldsymbol{C}$ share q pairwise disjoint n-point sets IM, then at least two of them are identical (see, also, Theorem 4).

On the other hand, the author proved in [3]:

Theorem B. *Let $S_1, \cdots, S_5$ be one-point or two-point sets in $\overline{\boldsymbol{C}}$. Assume that $S_1, \cdots, S_5$ are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share $S_1, \cdots, S_5$ IM, then f is a Möbius transform of g.*

In the proof of Theorem B, we can see that there is a Möbius transformation T such that $T(f) + T(g) = 0$ if $f \neq g$, and that the case where the number of two-point sets is one and the case where it is greater than one slightly differ. In this paper we consider $n+1$ meromorphic functions on $\boldsymbol{C}$ sharing some finite sets, and we show the following two theorems:

Theorem 1. *Let n be a positive integer and let $S_1, \cdots, S_{p+q}$ be pairwise disjoint non-empty finite sets in $\overline{\boldsymbol{C}}$ with at most $n+1$ elements, where p and q are non-negative integers with $q \geq 2$. Let $m_j = \sharp S_j$ be the number of elements of S_j. Assume that $m_j \leq n$ for $j = 1, \cdots, p$ and $m_j = n+1$ for $j = p+1, \cdots, p+q$, and assume that $n+1$ mutually distinct nonconstant meromorphic functions $f_1, \cdots, f_{n+1}$ on $\boldsymbol{C}$ share $S_1, \cdots, S_{p+q}$ IM. If $m_1 + \cdots + m_p + \frac{(n+1)q}{2} > n(n+1) + 2$, then there exists a Möbius transformation T such that $T(f_1) + \cdots + T(f_{n+1}) = 0$.*

Theorem 2. *Let n be a positive integer and let $S_1, \cdots, S_5$ be pairwise disjoint non-empty finite sets in $\overline{\boldsymbol{C}}$ such that $\sharp S_1 = \cdots = \sharp S_4 = 1, \sharp S_5 = n+1$. Assume that $n+1$ mutually distinct nonconstant meromorphic functions $f_1, \cdots, f_{n+1}$ on $\boldsymbol{C}$ share $S_1, \cdots, S_5$ IM. Then there exists a Möbius transformation T such that $T(f_1) + \cdots + T(f_{n+1}) = 0$.*

We assume that the reader is familiar with the standard notations and results of the value distribution theory (see, for example, [1]). In particular, we express by $S(r, f)$ quantities such that $\lim_{r\to\infty, r\notin E} S(r, f)/T(r, f) = 0$, where E is a subset of $(0, \infty)$ with finite linear measure and it is variable in each cases.

2. A lemma. Before beginning the proofs of Theorems, we show the following

Lemma 3. *Let $\xi_1, \cdots, \xi_m$ and $\eta_1, \cdots, \eta_n$ be mutually distinct points in $\overline{\boldsymbol{C}}$, where m and n are positive integers with $m + n \geq 3$. Then there exists a Möbius transformation T such that all $T(\xi_j), T(\eta_j)$ are in $\boldsymbol{C}$ and that $\sum_{j=1}^{m} T(\xi_j)/m = \sum_{j=1}^{n} T(\eta_j)/n$.*

Proof. We may assume that all points are in $\boldsymbol{C}$. If $\sum_{j=1}^{m} \xi_j/m = \sum_{j=1}^{n} \eta_j/n$, then let T be the identity.

Now we assume that $\sum_{j=1}^{m} \xi_j/m \neq \sum_{j=1}^{n} \eta_j/n$. Define

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the polynomials $P(z) = (z - \xi_1)\cdots(z - \xi_m)$ and $Q(z) = (z - \eta_1)\cdots(z - \eta_n)$, and we consider Möbius transformations of the form $T(z) = \frac{1}{z+d}$. Since $P'(z)/P(z) = \sum_{j=1}^{m} \frac{1}{z-\xi_j}$, we see that

$$\sum_{j=1}^{m} T(\xi_j) = -\frac{P'(-d)}{P(-d)},$$

and similarly,

$$\sum_{j=1}^{n} T(\eta_j) = -\frac{Q'(-d)}{Q(-d)}.$$

Hence, $\sum_{j=1}^{m} T(\xi_j)/m = \sum_{j=1}^{n} T(\xi_j)/n$ is equivalent to the condition

$$\frac{1}{m}\frac{P'(-d)}{P(-d)} = \frac{1}{n}\frac{Q'(-d)}{Q(-d)}.$$

Therefore it is enough to show that the equation

$$nP'(z)Q(z) - mP(z)Q'(z) = 0$$

has a solution distinct from ξ_j, η_j. The assumption that $\sum_{j=1}^{m} \xi_j/m \neq \sum_{j=1}^{n} \eta_j/n$ implies that the degree of the left-hand side polynomial is $m + n - 2 (> 0)$, and we see that any of ξ_j and η_j is not solution of the equation since ξ_j, η_j are mutually distinct. Therefore we complete the proof. □

3. Proof of Theorem 1 and Corollaries. For the proof we may assume that any S_j does not contain ∞. Put $N = m_1 + \cdots + m_{p+q}$. Then we have $N \geq 3$ and we can see, by the second fundamental theorem, that there is no need to distinguish $S(r, f_j)$. So we express them by $S(r)$. Put $\Phi = \prod_{1\leq j<k\leq n+1} (f_j - f_k) (\not\equiv 0)$. Now, we consider the reduced counting functions $\overline{N}_D(r, S_j)$ and $\overline{N}_E(r, S_j)$. The former counts the points $z \in f_1^{-1}(S_j)$ such that $f_1(z), \cdots, f_{n+1}(z)$ are all distinct, and the latter counts the points $z \in f_1^{-1}(S_j)$ such that at least two of $f_1(z), \cdots, f_{n+1}(z)$ are equal. Then we have, by the first main theorem,

$$\text{(3.1)} \qquad \sum_{j=1}^{p+q} \overline{N}_E(r, S_j) \leq \overline{N}(r, 1/\Phi) \leq n \sum_{j=1}^{n+1} T(r, f_j) + O(1)$$

and, by this and the second main theorem,

$$\begin{aligned}(N-2)T(r, f_k) &\leq \sum_{j=1}^{p+q} (\overline{N}_D(r, S_j) + \overline{N}_E(r, S_j)) + S(r) \\ &\leq \sum_{j=1}^{p+q} \overline{N}_D(r, S_j) + n\sum_{j=1}^{n+1} T(r, f_j) + S(r)\end{aligned}$$

for $k = 1, \cdots, n+1$. By adding the above inequalities for $k = 1, \cdots, n+1$, we obtain

$$\begin{aligned}\{N - 2 - n(n+1)\} \sum_{k=1}^{n+1} T(r, f_k) &\leq (n+1) \sum_{j=1}^{p+q} \overline{N}_D(r, S_j) + S(r) \\ &= (n+1) \sum_{j=1}^{q} \overline{N}_D(r, S_{p+j}) + S(r).\end{aligned}$$

Then we may assume that there exists a Borel subset I of $[1, +\infty)$ whose measure $|I| = +\infty$ and

$$\text{(3.2)} \qquad \left[\frac{2\{N - 2 - n(n+1)\}}{(n+1)q} + o(1)\right] \sum_{j=1}^{n+1} T(r, f_j) \leq \sum_{j=1}^{2} \overline{N}_D(r, S_{p+j}) \quad (r \in I),$$

by rearranging $S_{p+1}, \cdots, S_{p+q}$, if necessary. By Lemma 3, we can take a Möbius transformation T such that $T(S_{p+1}), T(S_{p+2})$ are subsets in $\boldsymbol{C}$ and the sum of all elements of each $T(S_j)$ is the origin for $j = p+1, p+2$. Put $\Psi = \sum_{j=1}^{n+1} T \circ f_j$. Assume that $\Psi \not\equiv 0$. If $f_1(z), \cdots, f_{n+1}(z)$ are distinct elements of $S_{p+1} \cup S_{p+2}$, then $\Psi(z) = 0$. Hence we have, by (3.2),

$$\left[\frac{2\{N - 2 - n(n+1)\}}{(n+1)q} + o(1)\right] \sum_{j=1}^{n+1} T(r, f_j) \leq \overline{N}(r, 1/\Psi) \leq \sum_{j=1}^{n+1} T(r, f_j) + O(1) \quad (r \in I).$$

Therefore we obtain the estimate

$$2\{N - 2 - n(n+1)\} \leq (n+1)q,$$

which is equivalent to

$$m_1 + \cdots + m_p + \frac{(n+1)q}{2} \leq n(n+1) + 2.$$

So by assumption we conclude $\Psi \equiv 0$, which implies the conclusion of Theorem 1. □

Remark. If we omit, in (3.1), terms $\overline{N}_E(r, S_j)$ $(j = p+1, \cdots, p+q)$, then by the second main theorem we have

$$(m_1+\cdots+m_p-2)T(r,f_k)\le\sum_{j=1}^{p}\overline{N}_E(r,S_j)+S(r)$$
$$\le\overline{N}(r,1/\Phi)+S(r)\le n\sum_{j=1}^{n+1}T(r,f_j)+S(r)$$

for $k=1,\cdots,n+1$, and hence

$$(m_1+\cdots+m_p-2)\sum_{k=1}^{n+1}T(r,f_k)$$
$$\le(n+1)\overline{N}(r,1/\Phi)+S(r)$$
$$\le n(n+1)\sum_{j=1}^{n+1}T(r,f_j)+S(r).$$

Therefore we obtain the inequality

$$m_1+\cdots+m_p\le n(n+1)+2.$$

In the above remark the last inequality holds under the assumption $\Phi\not\equiv 0$. Therefore we have

Theorem 4. *Let n be a positive integer and let $S_1,\cdots,S_p$ be pairwise disjoint non-empty finite sets in $\overline{\boldsymbol{C}}$ with at most n elements, where p is a positive integer. Let $m_j=\sharp S_j$ be the number of elements of S_j. Assume that $n+1$ nonconstant meromorphic functions $f_1,\cdots,f_{n+1}$ on $\boldsymbol{C}$ share $S_1,\cdots,S_p$ IM. If $m_1+\cdots+m_p>n(n+1)+2$, then at least two of $f_1,\cdots,f_{n+1}$ are identical.*

Also, we get the following corollaries of Theorem 1:

Corollary 5. *Let n be a positive integer and let $S_1,\cdots,S_{p+q}$ be pairwise disjoint finite sets in $\overline{\boldsymbol{C}}$, where p and q are integers with $p\ge 0$ and $q\ge 2$. Assume that $\sharp S_j=n$ for $j=1,\cdots,p$, $\sharp S_{p+j}=n+1$ for $j=1,\cdots,q$ and $np+\frac{(n+1)q}{2}>n(n+1)+2$. If $n+1$ mutually distinct nonconstant meromorphic functions $f_1,\cdots,f_{n+1}$ on $\boldsymbol{C}$ share $S_1,\cdots,S_{p+q}$ IM, then there exists a Möbius transformation T such that $T(f_1)+\cdots+T(f_{n+1})=0$.*

Corollary 6. *Let n be a positive integer and let $S_1,\cdots,S_{p+q}$ be pairwise disjoint finite sets in $\overline{\boldsymbol{C}}$, where p and q are integers with $p\ge 0$ and $q\ge 2$. Assume that $\sharp S_j=1$ for $j=1,\cdots,p$, $\sharp S_{p+j}=n+1$ for $j=1,\cdots,q$ and $p+\frac{(n+1)q}{2}>n(n+1)+2$. If $n+1$ mutually distinct nonconstant meromorphic functions $f_1,\cdots,f_{n+1}$ on $\boldsymbol{C}$ share $S_1,\cdots,S_{p+q}$ IM, then there exists a Möbius transformation T such that $T(f_1)+\cdots+T(f_{n+1})=0$.*

Corollary 7. *Let n be a positive integer and let $S_1,\cdots,S_q$ be pairwise disjoint $(n+1)$-point sets in $\overline{\boldsymbol{C}}$, where q is a positive integer. Assume that $q>2n+\frac{4}{n+1}$. If $n+1$ mutually distinct nonconstant meromorphic functions $f_1,\cdots,f_{n+1}$ on $\boldsymbol{C}$ share $S_1,\cdots,S_q$ IM, then there exists a Möbius transformation T such that $T(f_1)+\cdots+T(f_{n+1})=0$.*

4. Proof of Theorem 2. For the proof we may assume that any S_j does not contain ∞. Let a_j be the unique element of S_j $(j=1,\cdots,4)$. If $1\le k,l\le n+1$ and $k\ne l$, then by the second main theorem and by the first main theorem

$$2T(r,f_k)\le\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f_k-a_j}\right)+S(r,f_k)$$
$$\le\overline{N}\left(r,\frac{1}{f_k-f_l}\right)+S(r,f_k)$$
$$\le T(r,f_k)+T(r,f_l)+S(r,f_k).$$

Hence we have $T(r,f_k)\le T(r,f_l)+S(r,f_k)$ and $T(r,f_l)\le T(r,f_k)+S(r,f_k)$. It follows that $S(r,f_k)=S(r,f_l)$ and

$$T(r,f_l)=T(r,f_k)+S(r), \tag{4.1}$$

where $S(r)=S(r,f_k)$ as in the proof of Theorem 1. Also, we have

$$2T(r,f_k)=\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f_k-a_j}\right)+S(r) \tag{4.2}$$
$$=\overline{N}\left(r,\frac{1}{f_k-f_l}\right)+S(r).$$

Put $S_5=\{a_5,\cdots,a_{n+5}\}$, then we have

$$(n+3)T(r,f_k)\le\sum_{j=1}^{n+5}\overline{N}\left(r,\frac{1}{f_k-a_j}\right)+S(r)$$
$$=2T(r,f_k)+\sum_{j=5}^{n+5}\overline{N}\left(r,\frac{1}{f_k-a_j}\right)+S(r)$$
$$\le(n+3)T(r,f_k)+S(r)$$

for $k=1,\cdots,n+1$. It follows from this that

$$\sum_{j=5}^{n+5}\overline{N}\left(r,\frac{1}{f_k-a_j}\right)=(n+1)T(r,f_k)+S(r). \tag{4.3}$$

Take distinct k,l with $1\le k,l\le n+1$. Let $\overline{N}_0(r,\frac{1}{f_k-f_l})$ be the reduced counting function of the zeros of f_k-f_l outside $f_1^{-1}(S_1\cup\cdots\cup S_4)$. Then we get, by (4.2),

$$\overline{N}_0\left(r,\frac{1}{f_k-f_l}\right) \tag{4.4}$$
$$=\overline{N}\left(r,\frac{1}{f_k-f_l}\right)-\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f_k-a_j}\right)$$
$$=S(r).$$

Let $\overline{N}_D(r, S_5)$ be the reduced counting function which counts the points $z \in f_1^{-1}(S_5)$ such that $f_1(z), \cdots, f_{n+1}(z)$ are all distinct. Then, we have, by (4.3),

$$\overline{N}_D(r, S_5) \leq \sum_{j=5}^{n+5} \overline{N}\left(r, \frac{1}{f_k - a_j}\right) = (n+1)T(r, f_k) + S(r)$$

and, by (4.3) and (4.4),

$$\overline{N}_D(r, S_5) \geq \sum_{j=5}^{n+5} \overline{N}\left(r, \frac{1}{f_k - a_j}\right) - \sum_{1 \leq l < m \leq n+1} \overline{N}_0\left(r, \frac{1}{f_l - f_m}\right) = (n+1)T(r, f_k) + S(r).$$

Therefore

$$\overline{N}_D(r, S_5) = (n+1)T(r, f_k) + S(r) \tag{4.5}$$

is obtained. Also, from the second main theorem for f_1 and $a_1, \cdots, a_4$, we may assume that there exists a Borel set $I \subset [1, +\infty)$ whose measure $|I| = +\infty$ and that

$$\overline{N}\left(r, \frac{1}{f_1 - a_1}\right) \geq \frac{1}{2} T(r, f_1) + o(T(r, f_1)) \quad (r \in I) \tag{4.6}$$

by rearranging $a_1, \cdots, a_4$, if necessary. By Lemma 3, we can take a Möbius transformation T such that $T(a_1) = \sum_{j=5}^{n+5} T(a_j) = 0$, and put $\Psi = T(f_1) + \cdots + T(f_{n+1})$. Assume that $\Psi \not\equiv 0$. Then by (4.1) we have

$$T(r, \Psi) \leq \sum_{k=1}^{n+1} T(r, f_k) + O(1) = (n+1)T(r, f_1) + o(T(r, f_1)) \quad (r \in I)$$

and

$$\overline{N}_D(r, S_5) + \overline{N}\left(r, \frac{1}{f_1 - a_1}\right) \leq \overline{N}(r, 1/\Psi).$$

Therefore we obtain, by (4.5), (4.6) and these inequalities,

$$\begin{aligned}(n+1)T(r, f_1) &+ \frac{1}{2} T(r, f_1) + o(T(r, f_1)) \\ &\leq \overline{N}(r, 1/\Psi) + o(T(r, f_1)) \leq T(r, \Psi) + o(T(r, f_1)) \\ &\leq (n+1)T(r, f_1) + o(T(r, f_1)) \quad (r \in I),\end{aligned}$$

which is a contradiction. Hence $\Psi \equiv 0$, which implies the conclusion of Theorem 2. □

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