Derivatives of meromorphic functions and sine function

By Pai YANG,[*] Xiaojun LIU[**] and Xuecheng PANG[***]

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Abstract: In the paper, we take up a new method to prove the following result. Let f be a meromorphic function in the complex plane, all of whose zeros have multiplicity at least $k+1$ $(k \geq 2)$ and all of whose poles are multiple. If $T(r, \sin z) = o\{T(r, f(z))\}$ as $n \to \infty$, then $f^{(k)}(z) - \sin z$ has infinitely many zeros.

Key words: Meromorphic function; normal familiy; sine function.

1. Introduction. In his excellent paper [1], W. K. Hayman proved the following result.

Theorem A. *Let f be a transcendental meromorphic function with finitely many zeros in* **C**. *Then $f^{(k)}$ assumes every finite non-zero value infinitely often.*

A natural problem arises: what can we say if "finite non-zero value" in Theorem A is replaced by a small function $\alpha(z)$ with respect to $f(z)$?

In 2008, Theorem A was generalized by the following theorem of Pang, Nevo and Zalcman [2].

Theorem B. *Let f be a transcendental meromorphic function in* **C**, *all but finitely many of whose zeros are multiple, and let $\alpha(\not\equiv 0)$ be a rational function. Then $f' - \alpha$ has infinitely many zeros.*

In 2008, Liu, Nevo and Pang proved the following result [3].

Theorem C. *Let $f(z)$ be a transcendental meromorphic function of finite order in* **C**, *and $\alpha(z) = P(z)\exp Q(z) \not\equiv 0$, where P and Q are polynomials. Let also $k \geq 2$ be an integer. Suppose that*

(a) *all zeros of f have multiplicity at least $k+1$, except possibly finitely many, and*

(b) $\overline{\lim\limits_{r\to\infty}}\left(\frac{T(r,\alpha)}{T(r,f)} + \frac{T(r,f)}{T(r,\alpha)}\right) = \infty$.

Then the function $f^{(k)}(z) - \alpha(z)$ has infinitely many zeros. Moreover, in the case that $\rho(f) \notin \mathbf{N}$, then the result holds with condition (b) *only.*

Clearly, $\alpha(z)$ has only finitely many zeros and poles in Theorem B and Theorem C. Chen, Pang and Yang considered the case that $\alpha(z)$ has infinitely many zeros and poles. In fact, the following result [4] was proved in 2015.

Theorem D. *Let f be a nonconstant meromorphic function in* **C**, *all of whose zeros have multiplicity at least $k+1$ $(k \geq 2)$, except possibly finitely many. Let α be a nonconstant elliptic function such that $T(r, \alpha) = o\{T(r, f)\}$ as $r \to \infty$. Then $f^{(k)} = \alpha$ has infinitely many solutions* (*including the possibility of infinitely many common poles of f and α*).

Noting that $\alpha(z)$ is a certain class of double-periodic function in Theorem D, it is a very interesting work to consider the case $\alpha(z)$ is a certain class of single-periodic function. In this direction, we prove the following results with some new ideas.

Theorem 1.1. *Let f be a meromorphic function of infinite order in* **C**. *Suppose that*

(a) *all zeros of f have multiplicity at least $k+1$ $(k \geq 2)$, except possibly finite many, and*

(b) *all poles of f are multiple, except possibly finite many.*

Then $f^{(k)}(z) - \sin z$ has infinitely many zeros.

Theorem 1.2. *Let f be a meromorphic function of finite order in* **C**. *Suppose that*

(a) *all zeros of f have multiplicity at least $k+1$ $(k \geq 2)$, except possibly finite many, and*

(b) *$T(r, \sin z) = o\{T(r, f(z))\}$ as $n \to \infty$ outside of a possible exceptional set of finite linear measure.*

Then $f^{(k)}(z) - \sin z$ has infinitely many zeros.

Remark. Theorem 1.1 and Theorem 1.2 still hold if $\sin z$ is replaced by $\cos z$.

Notation. Let **C** be the complex plane and D be a domain in **C**. For $z_0 \in \mathbf{C}$ and $r > 0$, we write $\Delta(z_0, r) := \{z \,|\, |z - z_0| < r\}$, $\Delta := \Delta(0, 1)$ and $\Delta'(z_0, r) := \{z | 0 < |z - z_0| < r\}$. Let $V(z_0, \theta_0, A) :=$

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[*] College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, China.
[**] Department of Mathematics, University of Shanghai for Science and Technology, Shanghai 200093, China.
[***] Department of Mathematics, East China Normal University, Shanghai 200062, China.

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$\{z|\,|\arg(z-z_0)-\theta_0|<A\}$, $\overline{V}(z_0,\theta_0,A):=\{z|\,|\arg(z-z_0)-\theta_0|\le A\}$ and $\Gamma(z_0,r):=\{z|\,|z-z_0|=r\}$. Let $n(r,f)$ denote the number of poles of $f(z)$ in $\Delta(0,r)$ (counting multiplicity). We write $f_n \overset{\chi}{\Rightarrow} f$ in D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ in D if the convergence is in the Euclidean metric.

For f meromorphic in D, set

$$f^{\#}(z):=\frac{|f'(z)|}{1+|f(z)|^2} \text{ and}$$
$$S(D,f):=\frac{1}{\pi}\iint_D [f^{\#}(z)]^2\,\mathrm{d}x\mathrm{d}y.$$

The Ahlfors–Shimizu characteristic is defined by $T_0(r,f)=\int_0^r \frac{S(t,f)}{t}\,\mathrm{d}t$. Let $T(r,f)$ denote the usual Nevanlinna characteristic function. Since $T(r,f)-T_0(r,f)$ is bounded as a function of r, we can replace $T_0(r,f)$ with $T(r,f)$ in the paper.

The order $\rho(f)$ of the meromorphic function f is defined as

$$\rho(f):=\overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log r} \text{ or } \rho(f):=\overline{\lim_{r\to\infty}}\frac{\log T_0(r,f)}{\log r}.$$

2. Auxiliary results for the proof of Theorem 1.1.

Lemma 2.1. *Let $\mathcal{F}$ be a family of functions meromorphic in D, all of whose zeros have multiplicity at least k, and suppose that there exists $A\ge 1$ such that $|f^{(k)}(z)|\le A$ whenever $f(z)=0$. Then if $\mathcal{F}$ is not normal at $z_0\in D$, there exist, for each $0\le\alpha\le k$,*

(a) *points $z_n\in D$, $z_n\to z_0$;*
(b) *functions $f_n\in\mathcal{F}$; and*
(c) *positive numbers $\rho_n\to 0$*

such that $\rho_n^{-\alpha}f_n(z_n+\rho_n\zeta)=g_n(\zeta)\overset{\chi}{\Rightarrow} g(\zeta)$ in $\mathbf{C}$, where g is a nonconstant meromorphic function in $\mathbf{C}$ such that $g^{\#}(\zeta)\le g^{\#}(0)=kA+1$. In particular, g has order at most 2.

This is the local version of [5, Lemma 2] (cf. [6, Lemma 1]; [7, pp. 216–217]). The proof consists of a simple change of variable in the result cited from [5]; cf. [8, pp. 299–300].

Lemma 2.2 ([9, p. 12])**.** *Let $f(z)$ be a meromorphic function of infinite order in $\mathbf{C}$. Then there exist points $a_n\to\infty$ and positive numbers $\delta_n\to 0$ such that $f^{\#}(a_n)\to\infty$ and $S(\Delta(a_n,\delta_n),f)\to\infty$.*

Lemma 2.3 ([10, Theorem 1$'$ on p. 67])**.** *Let $k\ge 2$ be an integer and let $\{f_n\}$ be a family of meromorphic functions in D, all of whose poles are multiple and whose zeros all have multiplicity at least $k+1$. Let $\{h_n\}$ be a sequence of holomorphic functions in D such that $h_n\Rightarrow h$ in D, where $h\not\equiv 0$ in D. Suppose that for each n, h and h_n have the same zeros with the same multiplicity and $f_n^{(k)}(z)\neq h_n(z)$ for $z\in D$. Then $\{f_n\}$ is normal in D.*

Lemma 2.4 ([11, Theorem 1])**.** *Let f be a meromorphic function in Δ, and let a_1, a_2, a_3 be three distinct complex numbers. Assume that the number of zeros of $\prod_{i=1}^3(f(z)-a_i)$ in Δ is $\le n$, where multiple zeros are counted only once. Then*

$$S(r,f)\le n+\frac{A}{1-r},\quad 0\le r<1,$$

where $A>0$ is a constant, which depends on a_1, a_2, a_3 only.

Lemma 2.5. *Let $\{f_n\}$ be a family of meromorphic functions in $\Delta(z_0,r)$. Suppose that*

(a) *$f_n\overset{\chi}{\Rightarrow} f$ in $\Delta'(z_0,r)$, where $f(\not\equiv 0)$ may be ∞ identically, and*
(b) *there exists $M_0>0$ such that $n(\Delta(z_0,r),\frac{1}{f_n})\le M_0$ for sufficiently large n.*

Then there exists $M>0$ such that $S(\Delta(z_0,r/4),f_n)<M$ for sufficiently large n.

Proof. Without loss of generality, we may assume that $r=2$ and $z_0=0$.

We consider the following two cases.

Case 1. $f\not\equiv 1$ and $f\not\equiv 2$ in $\Delta'(0,2)$.

Obviously, $\frac{1}{f_n}-1\overset{\chi}{\Rightarrow}\frac{1}{f}-1$ in $\Delta'(0,2)$ and $\frac{1}{f}-1\not\equiv 0,\infty$ in $\Delta'(0,2)$. Thus there exists $s\in(1,2)$ such that $\frac{1}{f}-1$ has no poles and zeros on $\Gamma(0,s)$. For sufficiently large n, we have

$$\begin{aligned} &n\left(s,\frac{1}{f_n-1}\right)-n\left(s,\frac{1}{f_n}\right)\\ &\quad=n\left(s,\frac{1}{\frac{1}{f_n}-1}\right)-n\left(s,\frac{1}{f_n}-1\right)\\ &\quad=\frac{1}{2\pi i}\int_{\Gamma(0,s)}\frac{(\frac{1}{f_n}-1)'}{\frac{1}{f_n}-1}\,\mathrm{d}z\to\frac{1}{2\pi i}\int_{\Gamma(0,s)}\frac{(\frac{1}{f}-1)'}{\frac{1}{f}-1}\,\mathrm{d}z.\end{aligned}$$

Observing that $\frac{1}{2\pi i}\int_{\Gamma(0,s)}\frac{(\frac{1}{f_n}-1)'}{\frac{1}{f_n}-1}\,\mathrm{d}z$ is an integer, we have for sufficiently large n,

$$\frac{1}{2\pi i}\int_{\Gamma(0,s)}\frac{(\frac{1}{f_n}-1)'}{\frac{1}{f_n}-1}\,\mathrm{d}z=\frac{1}{2\pi i}\int_{\Gamma(0,s)}\frac{(\frac{1}{f}-1)'}{\frac{1}{f}-1}\,\mathrm{d}z.$$

Set $M_1:=\frac{1}{2\pi i}\int_{\Gamma(0,s)}\frac{(\frac{1}{f}-1)'}{\frac{1}{f}-1}\,\mathrm{d}z+M_0$. We have for sufficiently large n

$$\left(1,\frac{1}{f_n-1}\right)\le n\left(s,\frac{1}{f_n-1}\right)$$

$$= \frac{1}{2\pi i}\int_{\Gamma(0,s)} \frac{(\frac{1}{f}-1)'}{\frac{1}{f}-1}\,\mathrm{d}z + n\left(s, \frac{1}{f_n}\right) < M_1.$$

Obviously, $\frac{1}{f_n}-\frac{1}{2} \overset{\chi}{\Rightarrow} \frac{1}{f}-\frac{1}{2}$ in $\Delta'(0,2)$ and $\frac{1}{f}-\frac{1}{2} \not\equiv 0, \infty$ in $\Delta'(0,2)$. Thus there exists $t \in (1,2)$ such that $\frac{1}{f}-\frac{1}{2}$ has no poles and zeros on $\Gamma(0,t)$. For sufficiently large n, we have

$$n\left(t, \frac{1}{f_n-2}\right) - n\left(t, \frac{1}{f_n}\right)$$
$$= n\left(t, \frac{1}{\frac{1}{f_n}-\frac{1}{2}}\right) - n\left(t, \frac{1}{f_n}-\frac{1}{2}\right)$$
$$= \frac{1}{2\pi i}\int_{\Gamma(0,t)} \frac{(\frac{1}{f_n}-\frac{1}{2})'}{\frac{1}{f_n}-\frac{1}{2}}\,\mathrm{d}z \to \frac{1}{2\pi i}\int_{\Gamma(0,t)} \frac{(\frac{1}{f}-\frac{1}{2})'}{\frac{1}{f}-\frac{1}{2}}\,\mathrm{d}z.$$

Similarly to the previous paragraph, there exists $M_2 > 0$ such that for sufficiently large n, $n(1, \frac{1}{f_n-2}) < M_2$. By Lemma 2.4, there exists $A > 0$ depending on $0, 1, 2$ only such that for sufficiently large n,

$$S\left(\frac{1}{2}, f_n\right) \le n\left(1, \frac{1}{f_n}\right) + n\left(1, \frac{1}{f_n-1}\right) + n\left(1, \frac{1}{f_n-2}\right) + 2A < M_3,$$

where $M_3 = M_0 + M_1 + M_2 + 2A$.

Case 2. $f \equiv 1$ or $f \equiv 2$ in $\Delta'(0,2)$.

Clearly, $f \not\equiv 3$ and $f \not\equiv 4$ in $\Delta'(0,2)$. Then as shown in Case 1, there exists $M_4 > 0$ such that $S(\frac{1}{2}, f_n) \le M_4$ for sufficiently large n.

Set $M := \max\{M_3, M_4\}$. Clearly, $S(\frac{1}{2}, f_n) \le M$ for sufficiently large n. □

3. Proof of Theorem 1.1. We argue by contradiction. Suppose that $f^{(k)}(z) - \sin z$ has at most finitely many zeros.

Set $g(z) := \frac{f(z)}{\sin z}$. Clearly, $f(z)$ and $\sin z$ have finitely many common zeros (otherwise, by the assumptions, $f^{(k)}(z) - \sin z$ has infinitely many zeros), and thus all zeros of $g(z)$ have multiplicity at least $k+1$, except possibly finite many. Since the order of f is infinite, the order of g is also infinite. By Lemma 2.2, there exist points $a_n \to \infty$ and positive numbers $\varepsilon_n \to 0$ such that

(3.1) $$g^{\#}(a_n) \to \infty \text{ and } S(\Delta(a_n, \varepsilon_n), g) \to \infty.$$

We write $a_n = x_n + iy_n$. Taking a subsequence and renumbering, we may assume that $y_n \to y^*$.

We consider the following two cases.

Case 1. $y^* \neq \pm\infty$.

Set $b_n := x_n + iy^*$ and $\tau_n := |b_n - a_n| + \varepsilon_n$. Clearly, $\Delta(a_n, \varepsilon_n) \subset \Delta(b_n, \tau_n)$, $b_n \to \infty$ and $\tau_n \to 0$. By (3.1), we have

(3.2) $$S(\Delta(b_n, \tau_n), g) \to \infty \text{ as } n \to \infty.$$

There exist integers j_n and points $\widehat{x}_n \in (-\pi, \pi]$ such that $\widehat{x}_n = x_n - 2\pi j_n$. Taking a subsequence and renumbering, we may assume that $\widehat{x}_n \to \widehat{x}^*$. Clearly, $\widehat{x}^* \in [-\pi, \pi]$. Set

(3.3) $$f_n(z) := f(z + b_n - \widehat{x}_n) \text{ and } g_n(z) := g(z + b_n - \widehat{x}_n)$$

for $z \in E$, where

$$E := \{z | \operatorname{Re} z \in (-2\pi, 2\pi) \text{ and } \operatorname{Im} z \in (-2\pi, 2\pi)\}.$$

By (3.2) and (3.3), we have

(3.4) $$S(\Delta(\widehat{x}_n, \tau_n), g_n) \to \infty \text{ as } n \to \infty.$$

Set $\tau_n^* := \tau_n + |\widehat{x}_n - \widehat{x}^*|$. Clearly, $\Delta(\widehat{x}_n, \tau_n) \subset \Delta(\widehat{x}^*, \tau_n^*)$ and $\tau_n^* \to 0$. By (3.4),

(3.5) $$S(\Delta(\widehat{x}^*, \tau_n^*), g_n) \to \infty \text{ as } n \to \infty.$$

Now, we have for sufficiently large n,

(a1) all zeros of f_n have multiplicity at least $k+1$ and all poles of f_n are multiple in E,

(a2) $f_n^{(k)}(z) \neq \sin(z + iy^*)$ in E.

In fact, by (a), (b) and (3.3), (a1) holds for sufficiently large n. Since $f^{(k)}(z) - \sin z$ has at most finitely many zeros, (a2) holds for sufficiently large n by (3.3).

By Lemma 2.3, $\{f_n\}$ is normal in E. Taking a subsequence and renumbering, we may assume that $f_n \overset{\chi}{\Rightarrow} f^*$ in E.

Subcase 1.1. $f^* \not\equiv 0$.

Clearly, there exists $M_0 > 0$ such that $n(\Delta(\widehat{x}^*, 2), 1/f^*) < M_0$. By Hurwitz' Theorem, $n(\Delta(\widehat{x}^*, 1), 1/f_n) < M_0$ for sufficiently large n. Thus, $n(\Delta(\widehat{x}^*, 1), 1/g_n) < M_0$ for sufficiently large n. Let $\delta \in (0,1)$ such that $\sin(z + iy^*) \neq 0$ in $\Delta'(\widehat{x}^*, \delta)$. Thus, $g_n \overset{\chi}{\Rightarrow} \frac{f^*}{\sin(z+iy^*)}$ in $\Delta'(\widehat{x}^*, \delta)$. By Lemma 2.5, there exists $M > 0$ such that $S(\Delta(\widehat{x}^*, \delta/4), g_n) < M$ for sufficiently large n. This contradicts (3.5).

Subcase 1.2. $f^* \equiv 0$.

We see that for sufficiently large n,

$$0 \neq f_n^{(k)}(z) - \sin(z + iy^*) \Rightarrow -\sin(z + iy^*) \text{ in } E.$$

By Hurwitz' Theorem, $\sin(z + iy^*) \neq 0$ in E. Thus,

$$g_n(z) = \frac{f_n(z)}{\sin(z + iy^*)} \Rightarrow \frac{f^*(z)}{\sin(z + iy^*)} = 0 \text{ in } E.$$

Clearly, $g_n^{\#}(z) \Rightarrow 0$ in E, and hence

$$S(\Delta(\widehat{x}^*,1),g_n)=\frac{1}{\pi}\iint_{\Delta(\widehat{x}^*,1)}[g_n^{\#}(z)]^2dxdy\to 0.$$

This contradicts (3.5)

Case 2. $y^*=\pm\infty$.

We claim that there exists points t_n such that

$$\text{(3.6)}\quad \operatorname{Im}t_n\to\infty,\ \frac{f(t_n)}{\sin t_n}\to 0 \text{ and } \frac{f^{(k)}(t_n)}{\sin t_n}\to\infty.$$

Set

$$\text{(3.7)}\qquad g_n(z):=g(z+a_n) \text{ for } z\in\Delta.$$

Since all zeros of $g(z)$ have multiplicity at least $k+1$ (except possibly finite many), we have for sufficiently large n, all zeros of g_n have multiplicity at least $k+1$ in Δ. By (3.1), we have

$$\text{(3.8)}\qquad g_n^{\#}(0)\to\infty \text{ as } n\to\infty.$$

Thus, no subsequence of $\{g_n\}$ is normal at 0. Using Lemma 2.1 for $\alpha=k-(1/2)$, there exist points $z_n\to 0$, positive numbers $\rho_n\to 0$, and a subsequence of $\{g_n\}$ (still denoted by $\{g_n\}$) such that

$$G_n(\zeta)=\frac{g_n(z_n+\rho_n\zeta)}{\rho_n^{k-(1/2)}}\overset{\chi}{\Rightarrow} G(\zeta) \text{ in } \mathbf{C},$$

where G is a nonconstant meromorphic function in $\mathbf{C}$, all of whose zeros have multiplicity at least $k+1$.

We claim that $G^{(k)}(\zeta)\not\equiv 0$. Otherwise, $G(\zeta)=c_{k-1}\zeta^{k-1}+c_{k-2}\zeta^{k-2}+\cdots+c_0$, where $c_0,c_1,\cdots,c_{k-1}$ are constants. Thus, either $G\equiv 0$, or all zeros of G have multiplicity at most $k-1$. A contradiction.

Let ζ_0 be not a zero or pole of $G^{(k)}(\zeta)$, and set $t_n:=a_n+z_n+\rho_n\zeta_0$. Noting that $G_n^{(i)}(\zeta_0)\to G^{(k)}(\zeta_0)$ as $n\to\infty$, we see that

$$g^{(i)}(t_n)=g_n^{(i)}(z_n+\rho_n\zeta_0)=\rho_n^{k-i-(1/2)}G_n^{(i)}(\zeta_0)\to\begin{cases}0 & \text{for } i=0,1,\cdots,k-1.\\ \infty & \text{for } i=k.\end{cases}$$

Clearly, $\frac{f(t_n)}{\sin t_n}=g(t_n)\to 0$. Since $y_n\to\infty$ and $|t_n-a_n|\to 0$, we have $\operatorname{Im}t_n\to\infty$, and hence $1/2<|\frac{\sin^{(k-i)}(t_n)}{\sin t_n}|<2$ for sufficiently large n. Thus we have

$$\frac{f^{(k)}(t_n)}{\sin t_n}=\left.\frac{(g(z)\sin z)^{(k)}}{\sin t_n}\right|_{z=t_n}=\left.\frac{\sum_{i=0}^{i=k}C_k^i g^{(i)}(z)\sin^{(k-i)}(z)}{\sin t_n}\right|_{z=t_n}=\sum_{i=0}^{i=k}C_k^i g^{(i)}(t_n)\frac{\sin^{(k-i)}t_n}{\sin t_n}\to\infty.$$

Without loss of generality, we may assume that $\operatorname{Im}t_n\to+\infty$. Set $F_n(z):=\frac{f(z+t_n)}{\sin t_n}$ for $z\in\Delta$. Now, we have for sufficiently large n,

(b1) all zeros of F_n have multiplicity at least $k+1$ and all poles of F_n are multiple in Δ,

(b2) $F_n^{(k)}(z)\neq\frac{\sin(z+t_n)}{\sin t_n}\Rightarrow\cos z-i\sin z$ in Δ.

In fact, (b1) holds by (a) and (b). Since $f^{(k)}(z)-\sin z$ has at most finitely many zeros, (b2) holds for sufficiently large n.

By Lemma 2.3, $\{F_n\}$ is normal in Δ. However by (3.6), we have

$$F_n(0)=\frac{f(t_n)}{\sin t_n}\to 0 \text{ and } F_n^{(k)}(0)=\frac{f^{(k)}(t_n)}{\sin t_n}\to\infty.$$

Hence, no subsequence of $\{F_n\}$ is normal at $z=0$. This is a contradiction.

4. Auxiliary results for the proof of Theorem 1.2.

Lemma 4.1 ([12, Theorem 1.2])**.** *Let $k\geq 2$ be an integer and f be a meromorphic function of finite order in $\mathbf{C}$. If f has infinitely many poles, then $f^{(k)}$ has infinitely many zeros.*

Lemma 4.2. *Let f be a meromorphic function in $\mathbf{C}$, let $R(\not\equiv 0)$ be a rational function, and let $Q(z)=-z^m+c_{m-1}z^{m-1}+\cdots+c_0$, where $m\geq 2$ is an integer and $c_0,c_1,\cdots,c_{m-1}$ are constants. Suppose that $f^{(k)}(z)=R(z)\exp(Q(z))$, where $k\geq 2$ be an integer. Then for any given constant $\delta\in(0,\frac{3\pi}{2m})$*

$$f^{(k-1)}(z)=(1+r(z))\frac{R(z)\exp(Q(z))}{Q'(z)}+d_0,$$

$$f^{(k-2)}(z)=(1+s(z))\frac{R(z)\exp(Q(z))}{[Q'(z)]^2}+d_1z+d_2$$

in $V(0,0,\frac{3\pi}{2m}-\delta)$, where $r(z)$ and $s(z)$ are meromorphic in $V(0,0,\frac{3\pi}{2m}-\delta)$ and converge uniformly to 0 as $z\to\infty$, d_0, d_1 and d_2 are constants.

Remark. Lemma 4.2 is stated explicitly in [3, pp. 523–528], so we omit the proof.

5. Proof of Theorem 1.2. We consider the following two cases.

Case 1. f has infinitely many poles.

Clearly, $f(z)-\sin(z-k\pi/2)$ has infinitely many poles. Thus by Lemma 4.1, $f^{(k)}(z)-\sin z=(f(z)-\sin(z-k\pi/2))^{(k)}$ has infinitely many zeros.

Case 2. f has finitely many poles.

Suppose that, to the contrary, $f^{(k)}(z)-\sin z$ has only finitely many zeros. Clearly, $f^{(k)}(z)-\sin z$

has finitely many poles, so we have

$$(5.1)\quad (f(z)-\sin(z-k\pi/2))^{(k)} = f^{(k)}(z)-\sin z = T(z)\mathrm{e}^{P(z)},$$

where $T(z)(\not\equiv 0)$ is a rational function and $P(z)$ is a polynomial. By the condition (b) of Theorem 1.2, $P(z)$ is a polynomial of degree ≥ 2.

We claim that f has infinitely many zeros. Otherwise, suppose that f has finitely many zeros. Then $f(z)=T_0(z)\mathrm{e}^{P_1(z)}$ and hence $f^{(k)}(z)=T_1(z)\mathrm{e}^{P_1(z)}$, where $T_0(z)(\not\equiv 0)$ and $T_1(z)(\not\equiv 0)$ are rational functions, $P_1(z)$ is a polynomial. By (5.1),

$$(5.2)\quad T(z)\mathrm{e}^{P(z)}+\sin z = T_1(z)\mathrm{e}^{P_1(z)}.$$

Since $P(z)$ is a polynomial of degree ≥ 2, by (5.2), $P_1(z)$ must have the same degree and the leading coefficient as $P(z)$. We write (5.2) in the form

$$(5.3)\quad T(z)+\sin z\,\mathrm{e}^{-P(z)} = T_1(z)\mathrm{e}^{P_1(z)-P(z)}.$$

By standard results in Nevanlinna theory and (5.3), we have

$$\rho(T(z)+\sin z\,\mathrm{e}^{-P(z)}) = \rho(\mathrm{e}^{-P(z)}) = \deg P(z) > \deg(P_1(z)-P(z)) = \rho\big(T_1(z)\mathrm{e}^{P_1(z)-P(z)}\big).$$

This is a contradiction.

Set $\lambda := \sqrt[m]{\frac{-1}{a_m}}$, where a_m is the leading coefficient of $P(z)$. Substituting $z=\lambda\xi$ into (5.1), we obtain that

$$(5.4)\quad (g(\xi)-\sin(\lambda\xi-k\pi/2))^{(k)} = g^{(k)}(\xi)-\lambda^k\sin\lambda\xi = R(\xi)\mathrm{e}^{Q(\xi)},$$

where $g(\xi)=f(\lambda\xi)$, $Q(\xi)=P(\lambda\xi)$ and $R(\xi)=\lambda^k T(\lambda\xi)$. Thus $Q(\xi)$ has the following form

$$Q(\xi) = -\xi^m + c_{m-1}\xi^{m-1}+\cdots+c_0,$$

where $m\geq 2$ is an integer and $c_0, c_1, \cdots, c_{m-1}$ are constants.

Since f has infinitely many zeros, we can assume that g has infinitely many zeros $\{\xi_n\}$, and all of them are of multiplicity at least $k+1$. Thus we get

$$(5.5)\quad g(\xi_n)=g'(\xi_n)=\cdots=g^{(k)}(\xi_n)=0.$$

Let S be a subsequence of $\{\xi_n\}$ (denote it also by $\{\xi_n\}$) such that $\arg(\xi_n)$ converges to α. By (5.4) and (5.5), we have for all n

$$(5.6)\quad g^{(k)}(\xi_n) = R(\xi_n)\exp(Q(\xi_n))+\lambda^k\sin\lambda\xi_n = 0.$$

If $\alpha\notin\bigcup_{j=0}^{j=m-1}[\frac{2\pi j}{m}-\frac{\pi}{2m}, \frac{2\pi j}{m}+\frac{\pi}{2m}]$, then $R(\xi_n)\mathrm{e}^{Q(\xi_n)}+\lambda^k\sin\lambda\xi_n\to\infty$, which contradicts (5.6). Without loss of generality, we may assume that $\alpha\in[-\frac{\pi}{2m}, \frac{\pi}{2m}]$.

By (5.4) and Lemma 4.2,

$$(5.7)\quad g^{(k-1)}(\xi_n) = (1+r(\xi_n))\frac{R(\xi_n)\exp(Q(\xi_n))}{Q'(\xi_n)} + d_1 - \lambda^{k-1}\cos\lambda\xi_n = 0,$$

$$(5.8)\quad g^{(k-2)}(\xi_n) = (1+s(\xi_n))\frac{R(\xi_n)\exp(Q(\xi_n))}{Q'^2(\xi_n)} + d_2\xi_n + d_3 - \lambda^{k-2}\sin\lambda\xi_n = 0,$$

where $r(\xi)$ and $s(\xi)$ are meromorphic in $V(0,0,\frac{\pi}{m})$ and converge uniformly to 0 as $\xi\to\infty$, d_1, d_2 and d_3 are constants. Eliminating $\sin\lambda z_n$ from (5.6) and (5.8), we have for all n

$$(5.9)\quad R(\xi_n)\exp(Q(\xi_n)) = -\frac{\lambda^2(d_2\xi_n+d_3)Q'^2(\xi_n)}{Q'^2(\xi_n)+\lambda^2+t(\xi_n)},$$

where $t(\xi)=\lambda^2 s(\xi)$. Clearly, $t(\xi)$ are meromorphic in $V(0,0,\frac{\pi}{m})$ and converge uniformly to 0 as $\xi\to\infty$. Noting $\sin^2\lambda\xi_n+\cos^2\lambda\xi_n=1$, we have by (5.6) and (5.7),

$$(5.10)\quad \lambda^2\left[(1+r(\xi_n))\frac{R(\xi_n)\exp(Q(\xi_n))}{Q'(z_n)}+d_1\right]^2 + [R(\xi_n)\exp(Q(\xi_n))]^2 = \lambda^{2k}$$

for all n. Eliminating $R(\xi_n)\exp(Q(\xi_n))$ from (5.9) and (5.10), we have for all n

$$(5.11)\quad [\lambda(d_2\xi_n+d_3)Q'^2(\xi_n)]^2 + [\lambda^2(1+r(\xi_n))(d_2\xi_n+d_3)Q'(\xi_n) - d_1(Q'^2(\xi_n)+\lambda^2+t(\xi_n))]^2 - \lambda^{2k-2}[Q'^2(\xi_n)+\lambda^2+t(\xi_n)]^2 = 0.$$

The coefficient of the highest power of ξ_n in (5.11) is $\lambda^2 d_2^2 m^4$, so we have $d_2=0$. Thus (5.11) has been reduced into the following form

$$(5.12)\quad [\lambda d_3 Q'^2(\xi_n)]^2 + [\lambda^2 d_3(1+r(\xi_n))Q'(\xi_n) - d_1(Q'^2(\xi_n)+\lambda^2+t(\xi_n))]^2 - \lambda^{2k-2}[Q'^2(\xi_n)+\lambda^2+t(\xi_n)]^2 = 0.$$

The coefficient of the highest power of ξ_n in (5.12) is $(d_1^2+\lambda^2 d_3^2-\lambda^{2k-2})m^4$, so we have

$$(5.13)\quad d_1^2+\lambda^2 d_3^2-\lambda^{2k-2}=0.$$

Thus we have for all n

$$(5.14)\quad -2\lambda^2 d_1 d_3(1+r(\xi_n))Q'^3(\xi_n) + [\lambda^4 d_3^2(1+r(\xi_n))^2+2d_1^2(\lambda^2+t(\xi_n))$$

$$- 2\lambda^{2k-2}(\lambda^2 + t(\xi_n))]Q'^2(\xi_n)$$
$$- 2\lambda^2 d_1 d_3(1 + r(\xi_n))(\lambda^2 + t(\xi_n))Q'(\xi_n)$$
$$+ (d_1^2 - \lambda^{2k-2})(\lambda^2 + t(\xi_n))^2 = 0.$$

The coefficient of the highest power of ξ_n in (5.14) is $-2\lambda^2 d_1 d_3(1 + r(\xi_n))$, so we have

$$d_1 d_3(1 + r(\xi_n)) = 0 \text{ for all } n. \tag{5.15}$$

Noting that $d_2 = 0$ and $R(\xi_n)\exp(Q(\xi_n)) \neq 0$ for sufficiently large n, we have $d_3 \neq 0$ by (5.9). Since $1 + r(\xi_n) \to 1$ as $n \to 0$, we get $d_1 = 0$ by (5.15). Thus (5.14) has been reduced into the following form

$$[\lambda^4 d_3^2(1 + r(\xi_n))^2 - 2\lambda^{2k-2}(\lambda^2 + t(\xi_n))]Q'^2(\xi_n) - \lambda^{2k-2}(\lambda^2 + t(\xi_n))^2 = 0. \tag{5.16}$$

Clearly, we must have

$$\lambda^4 d_3^2(1 + r(\xi_n))^2 - 2\lambda^{2k-2}(\lambda^2 + t(\xi_n)) \to \lambda^4 d_3^2 - 2\lambda^{2k} = 0. \tag{5.17}$$

Thus $d_3^2 = 2\lambda^{2k-4}$ and then $d_1^2 + \lambda^2 d_3^2 - \lambda^{2k-2} = \lambda^{2k-2} \neq 0$, which contradicts (5.13).

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