Note on spun normal surfaces in 1-efficient ideal triangulations

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(Communicated by Kenji FUKAYA, M.J.A., Sept. 14, 2015)

Abstract: This is an announcement of the results in the paper [4] which is one of the series of papers dealing with the existence of spun normal surfaces in 3-manifolds with ideal triangulations. We give short comments for proofs of the results without details.

Key words: 3-manifold; normal surface; ideal triangulation.

1. Introduction. Essential surfaces in a 3-manifold are important to study the 3-manifold. Normal surfaces are useful way of representing such essential surfaces and a key tool for algorithms in 3-dimensional geometry and topology. Especially spun normal surfaces represent proper essential surfaces using ideal triangulations of 3-manifolds with tori and Klein bottle boundary components. So we studied the existence of spun normal surfaces in 3-manifolds with ideal triangulations in a series of the papers [4], [5], [6]. Thurston gave lectures on this topic and Walsh [8] gave the first detailed account of how to construct a spun normal surface starting from a properly embedded essential surface, which is not a virtual fiber. She established that proper essential surfaces which are not virtual fibers can be put into spun normal form, so long as the triangulations have no edges isotopic into the boundary. In our first paper, we replace the condition about the ideal triangulations having no edges isotopic into the boundary to the 1-efficient ideal triangulations. 1-efficiency is a stronger condition than in [8], but we are able to deal with semi-fiber surface and fiber surfaces with some restriction. In two followup papers, spinning essential surfaces in general ideal triangulations will be considered.

The following are some standard definitions, which will be used throughout the paper. Note that all surfaces considered are embedded.

Definition 1. Suppose that M is a compact 3-manifold with incompressible boundary. A properly embedded surface F is essential if it is incompressible and ∂-incompressible. By this we mean the induced maps $\pi_1(F) \to \pi_1(M)$ and $\pi_1(F, \partial F) \to \pi_1(M, \partial M)$ are both injections.

Definition 2. Suppose that M is a compact 3-manifold with incompressible boundary. Assume also that all the boundary components of M are tori and Klein bottles. M is called P^2-irreducible if any embedded 2-sphere bounds a 3-cell and there are no embedded 2-sided projective planes. M is called anannular if there are no essential properly embedded annuli and atoroidal if there are no incompressible non-peripheral embedded tori or Klein bottles.

Thurston established in the late 1970s that if M is compact, with boundary consisting of incompressible tori and Klein bottles and M is anannular and atoroidal, then M admits a complete hyperbolic metric of finite volume. We are especially interested in the existence of spun normal surfaces in ideal triangulations of this class of hyperbolic 3-manifolds.

2. Survey on normal surface theory. Let M be a compact 3-manifold with a triangulation $\Im$. Normal surfaces are kinds of minimal surfaces defined in a combinatorial point of view. We define 7 elementary disk types in each tetrahedron of $\Im$, 3 quadrilaterals dividing two-two vertices in the tetrahedron and 4 triangular disks dividing one-three vertices (see Fig. 1). A normal surface in M is a properly embedded surface intersecting each tetrahedron of $\Im$ in these 7 elementary disk types. If $\Im$ contains t tetrahedra, we can make a one-to-one correspondence of normal surfaces and $7t$-tuples which satisfy a system of linear equations, called matching equations. From the solution space of the system of matching equations, we can obtain a finite set of normal surfaces spanning all normal surfaces with respect to geometric sums. We call such

1991 Mathematics Subject Classification. Primary 57M99; Secondary 57M10.

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doi: 10.3792/pjaa.91.118

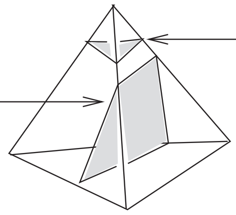

Fig. 1. Elementary disk types.

normal surfaces *fundamental surfaces*. These surfaces play a key role in algorithmic problems since some crucial surfaces must show up in this finite set of spanning surfaces. There is an excellent reference on normal surface theory [7].

Now we extend the theory on 3-manifolds using ideal triangulations. The only difference is embedding a surface with boundary. We introduce the following notations.

Definition 3. Let M be a 3-manifold with an ideal triangulation $\Im$ and let $\hat{M}$ be the compact 3-manifold obtained by deleting regular neighborhoods of ideal vertices of $\Im$ from M with the induced truncated triangulation $\hat{\Im}$.

(1) A spun normal surface S in M is an embedded surface formed from elementary disks in the tetrahedra consisting of finitely many quadrilaterals and infinitely many triangular disks. Disjoint subsets of the triangular disks form a finite collection of disjoint infinite cylinders in S which spiral around ideal vertices. The remainder of S outside these cylinders is compact.

(2) Suppose that F is a properly embedded incompressible and boundary incompressible surface in $\hat{M}$. F spin normalises (or normally spins) if there is a spun normal surface S in M so that $S \cap \hat{M}$ is isotopic to F.

In ideal triangulations, there are normal surfaces (spun normal surfaces) with infinite number of triangular disks. So we only deal with quadrilateral disk types to match a normal surface to a finite tuple and obtain a $3t$-tuple corresponding to a normal surface. This satisfies the system of Q-matching equations and gives a finite set of spanning surfaces, called Q-fundamental surfaces (or simply fundamental surfaces). In the proof of our results, we use normal surface theory on the truncated triangulation $\hat{\Im}$ of $\hat{M}$, rather than to use ordinary normal surface theory on a general triangulation obtained by decomposing the truncated tetrahedra of $\hat{\Im}$ into general tetrahedra. In the latter case, we cannot specify spinning of ideal vertices of the original triangulation $\Im$. Although there are so many elmentary disk types (actually 245 types) in each truncated tetrahedron, the theory goes on in eactly the same way. See [4] for a detailed argument of normal surface theory on truncated triangulations.

3. Spun normal surfaces in 1-efficient ideal triangulations. Jaco and Rubinstein first introduced 0- and 1-efficient triangulations [1]. The following gives the definition of 1-efficiency for ideal triangulations.

Definition 4. An ideal triangulation $\Im$ is 1-efficient if there are no embedded normal spheres, projective planes, Klein bottles, and tori which are not peripheral. Equivalently, there are no normal surfaces with non-negative Euler characteristic, except for peripheral tori and Klein bottles.

In [1], it was proved that the interior of a compact orientable 3-manifold M which is irreducible, atoroidal, anannular, with tori and Klein bottle boundary components has a 1-efficient ideal triangulation. So, it is reasonable to restrict to 1-efficient triangulations for this class of 3-manifolds.

Let M be a 3-manifold with a complete hyperbolic metric of finite volume with tori or Klein bottle cusps and let $\Im$ be a 1-efficient ideal triangulation of M. For each ideal vertex of $\Im$, we take an open regular neighborhood to obtain a compact manifold $\hat{M}$ with an induced truncated triangulation $\hat{\Im}$. We can develop a normal surface theory for the truncated triangulation $\hat{\Im}$ and obtain a finit set of spanning normal surfaces called *fundamental surfaces*. This finite set of fundamental surfaces plays a key role in our argument proving the existence of a normal surface which spins at the boundary cusps.

We begin by describing a procedure to topologically spin a properly embedded surface F which is incompressible and ∂-incompressible in a compact 3-manifold M with boundary consisting of tori and Klein bottles. Topological spinning of F is to add an annulus winding around a torus or Klein bottle boundary component to F and compute the isotopy class of the resulting surface keeping its boundary fixed. It occurs when the sequence of surfaces formed by attaching longer and longer annuli produces infinitely many isotopy classes. Let F_k be a surface obtained by spinning F k times for each boundary curve, along the boundary components of M. We call this a *spinning sequence* for F. Note that ∂F_k's are all coincident. Let $[S]$ denote the isotopy class of a properly embedded incompressible and ∂-incompressible surface S in M, keeping its boundary fixed. We have the following definition and key results for our main theorem.

Definition 5. We say that F *topologically spins* for some choice of directions of smoothing, if there are infinitely many different classes in the sequence $[F_n]$.

Theorem 6. *Let M be a compact 3-manifold with tori or Klein bottle boundary components and $\Im$ a 1-efficient ideal triangulation. Let F be a properly embedded, incompressible and ∂-incompressible surface in M. If F is a fiber having more than one boundary component or a semi-fiber, then F topologically spins for some choice of directions of smoothing. Specially in a semi-fiber case, F topologically spins in any choice of directions of smoothing.*

Theorem 7. *Let M be a compact 3-manifold with tori or Klein bottle boundary components. Let F be a properly embedded, incompressible and ∂-incompressible surface in M. If F is neither a fiber nor a semi-fiber of M, then F topologically spins with 2^r choices of spinning direction, where r is the number of boundary components of M containing a curve of ∂F. Moreover, if F is one-sided, F always topologically spins.*

The proof of the theorems are discussed in a covering space of M. In the fibered case, we work on an associated infinite cyclic covering of M. If we choose smoothing of F in opposite directions along any two boundary curves, F topologically spins in this choice of spinning direction. For the semi-fibered case, we look at the boundary surface of a small regular neighborhood of F which is a fiber of the associated double covering space of M. Then the argument from the fibered case will be applied. In the case of neither a fiber nor a semi-fiber, we consider the covering space M_F of M corresponding to the subgroup $\pi_1(F)$ of $\pi_1(M)$. We can show that there is a unique compact lift of F to M_F if F is neither a fiber nor a semi-fiber. This plays a crucial role in the proof of Theorem 7. The result for an one-sided surface follows directly from the previous results.

From Theorem 6 and 7, we know that if a properly embedded, incompressible and ∂-incompressible surface F in $\hat{M}$ is non-fibered, then F always topologically spins in any choice of directions of smoothing. We now discuss normal spinnings of such surfaces.

Let F be a properly embedded non-fibered surface in $\hat{M}$ which is incompressible and ∂-incompressible. Assume that F has least weight boundary, i.e., ∂F has fewest intersections with the 1-simplices of the induced triangulation of $\partial\hat{M}$. This allows us to normalize a surface isotopic to F keeping the boundary fixed. We first deal with a 2-sided surface F. Since F is not a fiber, F must topologically spin, i.e., the sequence $\langle [F_k] \rangle$ has an infinite number of different isotopy classes, for any choice of spinning directions. We now normalize each surface F_k keeping its boundary fixed and obtain a normal surface $\hat{F}_k$ in $\hat{M}$. Denote the normal isotopy class of $\hat{F}_k$ by $[\hat{F}_k]_n$, where the subscript n indicates a normal isotopy class. Note that the sequence $\langle [\hat{F}_k]_n \rangle$ also has an infinite number of different classes. Since $\hat{F}_k$ is a normal surface, it can be wirtten as a geometric sum of fundamental surfaces as follows:

$$\hat{F}_k = \sum_i n_{k,i} S_i + \sum_{j=1}^{r} l_{k,j} T_j$$

for nonnegative integers $n_{k,i}$, $l_{k,j}$, where S_i's are fundamental surfaces with negative Euler characteristic in $(\hat{M}, \hat{\Im})$ and T_j's are boundary components of $\hat{M}$ which contains some curves of ∂F. Since $\hat{\Im}$ is 1-efficient, the peripheral tori or Klein bottles are the only normal surfaces with nonnegative Euler characteristic. From the sequence, we will find a normal surface $\hat{F}_t$ which has triangular tails along all its boundary curves. Once we found such a surface, we can attach an infinite normal annulus (with only triangular disks) along each boundary curve of $\hat{F}_t$ to obtain normal spinning of F.

Since $\chi(F) = \chi(\hat{F}_k) = \chi(\sum_i n_{k,i} S_i)$ and $\chi(S_i) < 0$, there is only a finite number of choices for $\sum_i n_{k,i} S_i$ so that we can choose a subsequence $\langle \hat{F}_{k_i} \rangle$ given by

$$\hat{F}_{k_i} = S + \sum_{j=1}^{r} l_{k_i,j} T_j$$

for some fixed $S = \sum_i n_{k,i} S_i$. This subsquence still have an infinite number of normal isotopy classes. We denote the subsequence by $\langle \hat{F}_k \rangle$ again and so

$$\hat{F}_k = S + \sum_{j=1}^{r} l_{k,j} T_j.$$

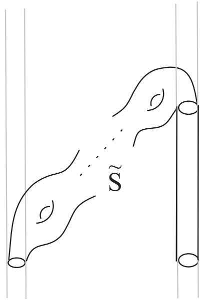

Fig. 2. The distance between boundary curves on $\tilde{S}$ which makes a contradiction to a finite diameter of $\tilde{S}$.

Here if we have some $\hat{F}_k$ which has $l_{k,j} \neq 0$ for all $j = 1, 2, \cdots, r$, then such $\hat{F}_k$ is the desired normal surface $\hat{F}_t$ which has triangular tails along its all boundary curves. Suppose that F does not normally spin. There is some j_1 such that a subsequence $\langle \hat{F}_{k_i} \rangle$ has $l_{k_i,j_1} = 0$ for all $i = 1, 2, \cdots$. Say $j_1 = 1$. We denote the subsequence by $\hat{F}_k$ again. On the other hand, since the sequence $\langle \hat{F}_k \rangle$ has an infinite number of normal isotopy classes, there must be some j_2 such that $l_{k,j_2} \to \infty$ as $k \to \infty$, say $j_2 = 2$, so that $\hat{F}_k$ can be written by

$$\hat{F}_k = S + 0T_1 + l_{k,2}T_2 + \sum_{j=3}^{r} l_{k,j}T_j$$

for $l_{k,2} \to \infty$ as $k \to \infty$. This is the situation that the surface F normally spins along the boundary curves on T_2, but not on T_1. Then we can make a contradiction by using the distance between the lifts of boundary curves of S on T_1 and T_2 in a covering space of M (see Fig. 2). If F is non-separating, we work on the infinite cyclic covering space of M defined by a cohomology class dual to F. In the case of separting F, we work on the covering space corresponding to the subgroup $\pi_1(F)$ of $\pi_1(M)$.

The following theorem is our main result dealing with normal spinnings of non-fibered surfaces. For details of the proof, see [4].

Theorem 8. *Let M be an anannular, atoroidal, irreducible and P^2-irreducible 3-manifold with tori or Klein bottle boundary components and $\Im$ be a 1-efficient ideal triangulation of M. If F is a properly embedded, incompressible and ∂-incompressible 2-sided surface in M which is not a fiber, then F can be spun normalized in $(M, \Im)$ with 2^r choices of spinning direction, where r is the number of the boundary components of M containing a curve of ∂F.*

Since two one-sided surfaces which represent the same homology class can never be disjoint, a surface cannot be isotoped a long way to another surface with its boundary fixed. This is just like the case of a non-fibered surface. Hence we obtain the same result for a one-sided surface F.

Theorem 9. *Let M be an anannular, atoroidal, irreducible and P^2-irreducible 3-manifold with tori or Klein bottle boundary components and $\Im$ be a 1-efficient ideal triangulation of M. If a properly embedded, incompressible, ∂-incompressible surface F is one-sided, then F always normally spins in M with 2^r choices of spinning directions, where r is the number of boundary components of M containing a curve of ∂F.*

Since Theorem 8 says that a properly embedded, incompressible and ∂-incompressible 2-sided surface which is not a fiber can be spun normalized in a 3-manifold with a 1-efficient triangulation, the next theorem follows directly from Theorem 8 and Theorem 9.

Theorem 10. *Let M be a semi-bundle over a non-orientable surface K and let $\Im$ be a 1-efficient ideal triangulation of M. If a surface F is a properly embedded, incompressible, ∂-incompressible surface in M, then F can be spun normalized.*

Now we turn to the question of the existence of normal spinnings of a fibered surface. We will find a sufficient condition for spinning of a fiber surface F in a surface bundle M over a circle.

Let $\tilde{M}$ be the infinite cyclic covering of M, given by the cohomology class dual to a choice of fibering. $\tilde{M}$ has the induced ideal triangulation from $\Im$ which is a 1-efficient ideal triangulation of M. If there is a closed normal surface S in $\tilde{M}$, S becomes a barrier (see [1] for details of barrier arguments) and lifts of fiber surface F cannot go through S by normalizing process. This is a similar situation to the case that lifts of a non-fibered surface cannot be isotoped a long way to a surface with its boundary fixed. This implies the existence of normal spinnings of F.

Conversely, suppose that F normally spins in both directions; all positive or all negative. There are lifts of the resulting spun normal surfaces in each direction, which bounds a closed normal surface in the infinite cyclic covering $\tilde{M}$. Actually the geometric sum of the both lifts is defined as the union of a closed normal surface and a pair of infinite annuli parallel to the boundary cylinders of $\tilde{M}$. So this establishes the existence of a closed normal surface in $\tilde{M}$.

Theorem 11. *Let F be a compact surface with $\partial F \neq \emptyset$ and let M be an S^1-bundle over F with a 1-efficient ideal triangulation $\Im$. If F normally spins in either choice of directions, where all spinning along ∂F goes in the same direction in $\tilde{M}$, either all positive or all negative, then there is a closed normal surface in $\tilde{M}$, where $\tilde{M}$ is the infinite cyclic covering of M dual to the fiber with the induced ideal triangulation from $\Im$. Conversely, if $\tilde{M}$ has such a closed normal surface, then F normally spins with all choices of spinning directions.*

If M is a fiber bundle over a circle with fiber a surface F with non-empty boundary, then M admits layered triangulations ([2]). It is easy to see that for such a layered triangulation, there cannot be any closed normal surfaces in $\tilde{M}$ and by Theorem 11, we can obtain the following result.

Theorem 12. *Let F be a surface with $\partial F \neq \emptyset$ and let M be an S^1-bundle over F with a layered triangulation $\Im$. Then F does not normally spin with all spinning in the same direction. Specifically, if F has only one boundary component, then F does not normally spin.*

Theorem 12 confirms the result in [3] that there is no spun normal surface representing a spanning surface in the figure-8 knot complement with the ideal triangulation of two tetrahedra which is a layered triangulation.

Acknowledgement. This study was supported by research funds from Chosun University, 2012.

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